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Testing The Existence of Multiple Cycles in Financial and Economic Time Series

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In this article we show that multiple cycles can occur in financial and economic time series. We model these cycles by means of Gegenbauer processes, using a procedure that permits us to test multiple roots at fixed frequencies over time and thus, it permits us to approximate the length of each cycle. This procedure is applied to one economic time series (US monthly unemployment rate) and a financial one (US Federal Funds rate of interest), and the results show that both series can be specified in terms of a multiple cyclical fractional model.

Key Words: Fractional integration; Long memory; Gegenbauer processes.
JEL Classification Number: C22.

1. INTRODUCTION

The existence of cycles in macroeconomic and financial time series is a well-known stylised fact. However, its appropriate modelling is a matter that still remains controversial. Numerous authors have tried to describe them and to consider their stability over time. In the context of economic time series, Romer (1986, 1994), Diebold and Rudebusch (1992) and Watson (1994) have, for example, explored data to know if fluctuations have been smoother (lower amplitude and longer duration) after Second World War. Also, Neftci (1984), Hamilton (1989), Beaudry and Koop (1993) investigated new business cycles features, showing that cycles exhibit an asymmetry in their phases: recessions being deeper and shorter than expansions. Other authors (e.g., Candelon and Henin, 1995; Hess and Iwata, 1997; Candelon and Gil-Alana, 2004; etc.) characterised the distributions of these features via bootstrapped simulations, using them as benchmarks

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to gauge the adequacy of macroeconomic models. All these authors, however, only consider the existence of a single cycle underlying the series. Using financial time series, Peters (1994) analysed the Dow Jones industrial index over the period 1888 to 1991 and found evidence of a two month and four year cycles. McKinzey (2001), using Australian stock prices, finds evidence of long memory in the return generating process and cycles of approximately 3, 6 and 12 years in average duration. Following this line of research, we look in this paper at the possibility of more than one single cycle in the series and consider multiple cyclical structures that might be contaminating the results based on one single cycle.

Starting from an empirical data-based approach, it appears that many economic time series present a persistent periodic behaviour, which cannot be caught by the classical ARIMA (or even ARFIMA) models. Therefore, recent years have witnessed the publication of several papers dealing with long memory models able to take into account a possible harmonic component in the data. Gray et al. (1989) proposed a new class of long memory models, which generalises the class of ARFIMA models, insofar as the spectral density function is not necessarily unbounded at the origin, but anywhere in the interval $[0, \pi]$. Giraitis and Leipus (1995) and, later, Woodward et al. (1998) give an extension of the model in Gray et al. (1989), for which the spectral density is unbounded for a finite number of k frequencies, denoted Gegenbauer frequencies, on the interval $[0, \pi]$. This k-factor extension was first suggested in the concluding remarks of Gray et al. (1989) and is used by Robinson (1994) in a hypothesis testing context.[1] Related estimators for the k-factor generalized fractional integration model have also been developed by Arteche and Robinson (2000), Ferrara and Guégan (2001), Smallwood and Beaumont (2001), Arteche (2002) and Smallwood and Beaumont (2003).

The outline of the article is as follows: Section 2 describes the statistical model and its implications in terms of economic policy and planning inference. In Section 3 we describe the procedure employed in the paper for testing this type of model. Section 4 contains a small Monte Carlo simulation study in order to find a plausible strategy to determine the appropriate number of cyclical structures. Section 5 contains the empirical application, while Section 6 concludes.

[1] Porter-Hudak (1990) and Hassler et al. (1994) proposed two different seasonal long memory models, which are in fact special cases of the k-factor model, insofar as the frequencies are the seasonal frequencies.

2. A MODEL WITH MULTIPLE CYCLICAL COMPONENTS

Starting with a single cycle, Gray et al. (1989, 1994) consider a model of the form:

$$(1 - 2\cos\, w_r L + L^2)^d x_t = u_t, \quad t = 1, 2, \cdots, \tag{1}$$

with $x_t = 0$ for $t \leq 0$, and where L is the lag operator (i.e., $Lx_t = x_{t-1}$); d can be any real number, and where u_t is an $I(0)$ process, defined as a covariance stationary process, with spectral density function that is positive and finite at any frequency on the spectrum; $w_r = 2\pi r/n, r = n/j$, and j indicates the number of time periods within the cycle. Clearly, when $d = 0$ in (1), $x_t = u_t$, and a "weakly autocorrelated" x_t is allowed for, as opposed to the case of $d > 0$ when the process is said to be "strongly autocorrelated" or also called "strongly dependent", so-named because of the strong association (in the cyclical structure) between observations widely separated in time. Gray et al. (1989) showed that x_t in (1) is stationary if $d < 0.50$. They also showed that the polynomial in (1) can be expressed in terms of the Gegenbauer polynomial $C_{j,d}$ such that, calling $\mu = \cos\, w$,

$$(1 - 2\mu L + L^2)^{-d} = \sum_{j=0}^{\infty} C_{j,d}(\mu) L^j, \tag{2}$$

for all $d \neq 0$, where

$$C_{j,d}(\mu) = \sum_{k=0}^{[j/2]} \frac{(-1)^k (d)_{j-k} (2\mu)^{j-2k}}{k!(j-2k)!}; \quad (d)_j = \frac{\Gamma(d+j)}{\Gamma(d)},$$

$\Gamma(x)$ represents the Gamma function and a truncation will be required in (2) to make the polynomial operational. Thus, the process in (1) becomes:

$$x_t = \sum_{j=0}^{t-1} C_{j,d}(\mu) u_{t-j}, \quad t = 1, 2, \cdots, \tag{3}$$

and when $d = 1$, we have

$$x_t = 2\mu x_{t-1} - x_{t-2} + u_t, \quad t = 1, 2, \cdots, \tag{4}$$

which is a cyclical $I(1)$ process with the periodicity determined by μ. Tests of (4) based on autoregressive (AR) alternatives were proposed amongst others by Ahtola and Tiao (1987). Their tests are embedded in an $AR(2)$ process of form:

$$x_t = \phi_1 x_{t-1} + \phi_2 x_{t-2} + u_t, \tag{5}$$

which, under the null hypothesis:

$$H_0 : |\phi_1| < 2 \ \text{ and } \ \phi_2 = -1, \tag{6}$$

becomes the cyclical $I(1)$ model (4). Unit root cycles were also examined by Chan and Wie (1988) and Gregoir (1999a, b) who derive the limiting distribution of least squares estimates of AR processes with complex-conjugate unit roots, with inference based on parametric estimates.[2]

In this article, we make use of the fractional structure (1), testing cyclical roots with integer or fractional orders of integration in raw time series. However, as mentioned in the introduction, this paper is concerned with the k-factor Gegenbauer processes and thus, we consider processes of form:

$$\prod_{j=1}^{h} 1 - 2\mu(j)L + L^2)^{d_j} x_t = u_t, \quad t = 1, 2, \cdots, \tag{7}$$

where h is a finite integer; $|\mu(j)| \leq 1$ for $j = 1, 2, \cdots, h$, and d_j is a fractional number for $j = 1, 2, \cdots, h$. These processes, for which the common point is to have a spectral density with a finite number of peaks on the interval $[0, \pi]$, have been extensively investigated concerning the parameter estimation problem. See, for instance, Gray et al. (1989), Giraitis and Leipus (1995), Chung (1996a, b), Yajima (1996), Hosoya (1997) and Ferrara and Gugan (1999, 2001).

The use of the parametric approach described in (7) to investigate the long run behaviour of time series consists of testing a parametric model for the series and relying on the long run implications of the estimated model. The primary advantage is the precision gained by focusing the information in the series through the parameter estimates. A drawback is that the parameter estimates are sensitive to the class of models considered and may be misleading because of misspecification. However, the possibility of misspecification with parametric models can never be settled conclusively, and the problem can be addressed by considering a large class of models. This is the approach of the present paper. For this purpose, we employ a version of the tests of Robinson (1994) that permits us to test long memory multiple cyclical models. The main advantage of this procedure is that it is based on the Lagrange Multiplier principle and thus, it does not require efficient estimates of the fractional (cyclical) differencing parameters, as is the case with other procedures.

[2]Bierens (2001) also uses a model of this sort in the context of the UK monthly unemployment series.

3. THE TESTING PROCEDURE

Following Bhargava (1986), Schmidt and Phillips (1992) and others in the parameterization of unit-root models, Robinson (1994) considers the regression model:

$$y_t = \beta' z_t + x_t \quad t = 1, 2, \cdots, \tag{8}$$

where y_t is a given time series; z_t is a $(kx1)$ vector of exogenous variables; β is a $(kx1)$ vector of unknown parameters; and the regression errors x_t are such that:

$$\rho(L;\theta) x_t = u_t \quad t = 1, 2, \cdots, \tag{9}$$

where ρ is a given function, which depends on L and the $(px1)$ parameter vector θ, adopting the form:

$$\rho(L;\theta) = (1-L)^{d_L+\theta_L}(1-L^4)^{d_S+\theta_S} \prod_{j=1}^{h} (1 - 2\cos\, w_{r_S}^{j} L + L^2)^{d_j+\theta_j}, \tag{10}$$

for real given numbers $d_L, d_S, d_1, \cdots, d_h$, and integer h.[3] Under the null hypothesis, defined by:

$$H_0 : \theta = 0, \tag{11}$$

(10) becomes:

$$\rho(L;\theta = 0) = \rho(L) = (1-L)^{d_L}(1-L^4)^{d_S} \prod_{j=1}^{h} (1 - 2\cos\, w_{r_S}^{j} L + L^2)^{d_j}. \tag{12}$$

This is a very general specification that permits us to consider different models under the null. For example, if $d_L = 1$ and $d_S = d_j = 0$ for all j, we have the classical unit-root model (Dickey and Fuller, 1979; Phillips and Perron, 1988; or the alternative in Kwiatkowski et al., 1992, etc.) and, if d_L is a real value, the fractional models examined in Diebold and Rudebusch (1989), Baillie (1996) and others. Similarly, imposing $d_S = 1$ and $d_L, d_j = 0$, we have the seasonal unit root model (Dickey, Hasza and Fuller, 1984, Hyllerberg et al., 1990, etc.) and if d_S is real, the seasonal fractional model analysed in Porter-Hudak (1990) and Gil-Alana (2002). Finally, if $d_L = d_S = 0$ and $d_1 = 1$, we have the unit root cycles of Ahtola and Tiao (1987); if d_1 is real, the model in Gray et al. (1989) and, if $j > 1$,

[3] Equation (10) is not exactly the same equation as in Robinson (1994). In his model, the second polynomial in the right hand side of (10) is $(1 + L)^d$, though it can be easily shown that (10) satisfies the same properties as Robinson (1994) since $(1 - L^4) = (1 - L)(1 + L)(1 + L^2)$.

the k-factor Gegenbauer processes studied in Ferrara and Guégan (2001) and others.

In this paper we consider multiple cyclical structures and thus, we take $d_L = d_S = 0$. In such a situation, (10) becomes:

$$\rho(L;\theta) = \prod_{j=1}^{h} (1 - 2\cos\, w_{r_S}^{j} L + L^2)^{d_j + \theta_j}, \tag{13}$$

and similarly (12):

$$\prod_{j=1}^{h} (1 - 2\cos\, w_{r_S}^{j} L + L^2)^{d_j}. \tag{14}$$

Plugging then (13) in (8) and (9), y_t follows under the null, a multiple cyclical $I(d)$ model of the form advocated by Ferrara and Guégan (2001) and Smallwood and Beaumont (2001, 2003).

We next describe the test statistic. We observe $\{(y_t, z_t), t = 1, 2, \cdots, n\}$, and suppose that the $I(0) u_t$ in (9) have spectral density given by:

$$f(\lambda;\tau) = \frac{\sigma^2}{2\pi} g(\lambda;\tau), \quad -\pi < \lambda \leq \pi,$$

where the scalar σ^2 is known and g is a function of known form, which depends on frequency λ and the unknown $(qx1)$ vector τ. Based on H_0 (11), the residuals in (8), (9) and (13) are

$$\hat{u}_t = \prod_{j=1}^{h} (1 - 2\cos\, w_{r_S}^{j} L + L^2)^{d_j} y_t - \hat{\beta}' w_t, \tag{15}$$

where $\hat{\beta} = \left(\sum_{t=1}^{n} s_t s_t'\right)^{-1} \sum_{t=1}^{n} S_t \prod_{j=1}^{h} (1 - 2\cos\, w_{r_S}^{j} L + L^2)^{d_j} y_t$, and $s_t = \prod_{j=1}^{h} (1 - 2\cos\, w_{r_S}^{j} L + L^2)^{d_j} z_t$.

Unless g is a completely known function (e.g., $g \equiv 1$, as when u_t is white noise), we have to estimate the nuisance parameter τ, for example by $\hat{\tau} = \arg\min_{\tau \in T} \sigma^2(\tau)$, where T is a suitable subset of R^q Euclidean space, and

$$\sigma^2(\tau) = \frac{2\pi}{n} \sum_{s=1}^{n-1} g(\lambda_s;\tau)^{-1} I_{\hat{u}}(\lambda_s),$$

with $I_{\hat{u}}(\lambda_s) = |(2\pi n)^{-1/2} \sum_{t=1}^{n} \hat{u}_t e^{i\lambda_s t}|^2$; $\lambda_s = \frac{2\pi s}{n}$.[4]

[4]Note that $\hat{\beta}$. is an OLS estimate. In case of autocorrelated disturbances, it may be improved via GLS.

The test statistic, which is derived via Lagrange Multiplier (LM) principle, takes the form:

$$\hat{R} = \hat{r}'\hat{r}; \quad \hat{r} = \left(\frac{n}{\hat{A}}\right)^{1/2} \frac{\hat{a}}{\hat{\sigma}^2}, \tag{16}$$

where

$$\hat{a} = \frac{-2\pi}{n} \sum_{s=1}^{*} \Psi(\lambda_s) g(\lambda_s; \hat{\tau})^{-1} I_{\hat{u}}(\lambda_s); \quad \hat{\sigma}^2(\tau) = \sigma^2(\hat{\tau});$$

$$\hat{A} = \frac{2}{n} \left(\sum_{s=1}^{*} \Psi(\lambda_s)^2 - \sum_{s=1}^{*} \Psi(\lambda_s)\hat{\varepsilon}(\lambda_s)' \left(\sum_{s=1}^{*} \hat{\varepsilon}(\lambda_s)\hat{\varepsilon}(\lambda_s)' \right)^{-1} \sum_{s=1}^{*} \hat{\varepsilon}(\lambda_s)\Psi(\lambda_s) \right),$$

$\Psi(\lambda_s) = [\Psi_1(\lambda_s); \cdots; \Psi_j(\lambda_s); \cdots; \Psi_h(\lambda_j)]$; $\Psi_j(\lambda_s) = \log|2(\cos\lambda_s - \cos\, w^j_{r_s})|$; $\hat{\varepsilon}(\lambda_s) = \frac{\partial}{\partial\tau} \log\, g(\lambda_s; \hat{\tau})$. and the summation on $*$ in the above expressions refers to the discrete bounded frequencies λ_s.

Robinson (1994) showed that under certain very mild regularity conditions,[5]

$$\hat{R} \rightarrow_d \chi^2_h \quad \text{as} \quad n \rightarrow \infty \tag{17}$$

where "$\rightarrow_d$" means convergence in distribution. Thus, we are in a classical large-sample testing situation, by reasons described in Robinson (1994). Moreover, he shows that the above test is efficient in the Pitman sense against local departures from the null. This version of Robinson's (1994) tests (with $h = 1$) was examined in Gil-Alana (2001), and its performance in the context of unit root cycles was compared with Ahtola and Tiao's (1987) tests, the results showing that Robinson's (1994) tests outperform Ahtola and Tiao (1987) in a number of cases. Other versions of his tests have been applied to time series in Gil-Alana and Robinson (1997, 2001), testing for $I(d)$ processes with the roots occurring respectively at zero and the seasonal frequencies. However, testing multiple fractional cyclical models with the tests of Robinson (1994), this is one of the few empirical applications, and one by-product of this work is its emergence as a credible alternative to the usual ARIMA (ARFIMA) specifications, which have become conventional in parametric modelling of macroeconomic and financial time series.

There exist other procedures for estimating and testing the fractionally cyclical differencing parameters, some of them also based on the likelihood function. We believe that as in other standard large-sample testing situations, Wald and LR test statistics against fractional alternatives will have

[5]These conditions are very mild, and concern technical assumptions to be satisfied by $\Psi(\lambda)$.

the same null and local limit theory as the LM tests of Robinson (1994). However, these procedures require efficient estimates of the differencing parameters, and while such estimates can be obtained, no closed-form formulae are available and so the LM procedure of Robinson (1994) seems computationally more attractive.

4. A MONTE CARLO SIMULATION STUDY

In this section we examine the possibility of misspecification in the context of multiple fractional cyclical models, using the version of the tests of Robinson (1994) described in Section 3. For the ease of presentation, we only display the results for the case of unit root cycles, though similar conclusions were obtained when fractional orders of integration were entertained.

In all cases, we assume that the number of cyclical structures in the true model is equal to or higher than the number of cycles tested with the procedure. First, we assume a model of form:

$$(1 - 2\cos\, w_r L + L^2)x_t = u_t,$$

with $r = 2\pi/10$, and test $H_0 : d = d_0$ for values $d_0 = 0, (0.25), 2$. Thus, the rejection probabilities corresponding to $d_0 = 1$ will indicate the size of the test. The nominal size is 5% and $T = 100$ and 500.[6]

TABLE 1.

True model: $(1 - 2\cos w_{n/10} L + L^2)x_t = u_t$									
Alternative: $(1 - 2\cos w_{T/10} L + L^2)^d x_t = u_t$									
n/d_0	0	0.25	0.50	0.75	1.00	1.25	1.50	1.75	2.00
100	0.998	0.916	0.893	0.695	0.087	0.619	0.910	0.971	0.984
500	0.996	1.000	1.000	1.000	0.056	0.997	1.000	1.000	1.000

We see that the size is too large with $T = 100(8.7\%)$, though it considerably improves as we increase the sample size. The rejection probabilities are relatively high in all cases, and the local efficiency of the test seems to assert itself in view of the fact that the rejection values are practically 1 for the alternatives $d = 0.75$ and 1.25 with $T = 500$.

In Table 2, the true model contains two cyclical structures with 10 and 20 periods of length,

$$(1 - 2\cos\, w^1_{s=10} L + L^2)(1 - 2\cos\, w^2_{s=20} L + L^2)x_t = u_t,$$

[6]We generate Gaussian series using the routines of GASDEV and RAN3 of Press et al. (1986) and 10,000 replications were used in each case.

and test H_0 for the same d_0-values as in the previous case, assuming that $h = 1$ and 2. In the former case, the model will be misspecified since we are not taking into account the second cyclical structure. Starting with $h = 1$, we see that the rejection frequencies are close to 1 even for $T = 100$. If $h = 2$, the size is again too large for $T = 100(11.2\%)$, though if $T = 500$ it reduces to 5.7%. In the latter case, the rejection probabilities are 1 in all cases. Finally, in Table 3, the true model contains three cyclical structures with $h = 10, 20$ and 30, and perform the tests with $h = 1, 2$ and 3. Thus, only in the case with $h = 3$ the model will be correctly specified. If $h = 1$

TABLE 2.

True model: $(1 - 2\cos w_{n/10}L + L^2)(1 - 2\cos w_{n/20}L + L^2)x_t = u_t$										
Alternative: $\hat{R}(h = 1)$ and $\hat{R}(h = 2)$										
n	$\hat{R}/d_0$	0	0.25	0.50	0.75	1.00	1.25	1.50	1.75	2.00
100	$h = 1(10)$	0.990	0.999	1.000	1.000	1.000	1.000	1.000	0.999	0.998
	$h = 1(20)$	0.684	0.937	0.997	1.000	1.000	1.000	1.000	1.000	0.978
	$h = 2$	1.000	1.000	1.000	1.000	0.112	0.967	0.998	0.999	0.999
500	$h = 1(10)$	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	$h = 1(20)$	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	$h = 2$	1.000	1.000	1.000	1.000	0.057	1.000	1.000	1.000	1.000

TABLE 3.

True model: $(1 - 2\cos w_{n/10}L + L^2)(1 - 2\cos w_{n/20}L + L^2)(1 - 2\cos w_{n/30}L + L^2)x_t = u_t$										
Alternative: $\hat{R}(h = 1)$, $\hat{R}(h = 2)$ and $\hat{R}(h = 3)$										
n	$\hat{R}/d_0$	0	0.25	0.50	0.75	1.00	1.25	1.50	1.75	2.00
100	$h = 1(10)$	0.999	0.999	0.999	0.999	0.999	0.999	0.999	0.999	0.999
	$h = 1(20)$	0.821	0.937	0.982	0.992	0.994	0.996	0.997	0.998	0.999
	$h = 1(30)$	0.375	0.761	0.980	0.999	1.000	1.000	1.000	1.000	1.000
	$h = 2(10, 20)$	0.999	0.999	0.999	0.998	0.998	0.995	0.998	0.695	0.918
	$h = 2(10, 30)$	0.999	0.999	0.999	1.000	1.000	1.000	0.962	0.617	0.998
	$h = 2(20, 30)$	0.989	0.995	1.000	1.000	1.000	1.000	0.992	0.479	0.935
	$h = 3$	0.999	0.999	0.999	1.000	0.112	0.991	0.999	1.000	1.000
500	$h = 1(10)$	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	$h = 1(20)$	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	$h = 1(30)$	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	$h = 2(10, 20)$	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	$h = 2(10, 30)$	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	$h = 2(20, 30)$	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	$h = 3$	1.000	1.000	1.000	1.000	0.058	1.000	1.000	1.000	1.000

or 2, the rejection frequencies are very high in all cases, and if $h = 3$, the sizes are 11.2% with $T = 100$ and 5.8% with $T = 500$.

5. THE EMPIRICAL WORK

We analyse in this section two different datasets, one corresponding to an economic time series (unemployment rate) and the other being a financial variable (Federal Funds rate of interest). The unemployment series is the US unemployment rate, monthly, for the time period 1948m1-2004m1, obtained from the US Bureau of Labour Statistics. The Federal Funds rate of interest is the cost of borrowing immediately available funds, primarily for one day. It is also monthly, running from 1954m7 to 2004m1, and obtained from the Federal Reserve Bank of St. Louis database.

Figure 1 displays plots of the two time series, with their corresponding correlograms and periodograms. We observe that both series may have a stationary appearance though the correlograms show a very slow decay and significant values at lags far away from zero. Moreover, the periodograms show a peak at the smallest frequency indicating the possibility of long memory at the zero frequency in both cases.

Denoting each of the time series by y_t, we employ throughout the model given by (8), (9) and (13), with $z_t = (1,t)', t \geq 1, z_t = (0,0)'$. Thus, under H_0 (11),

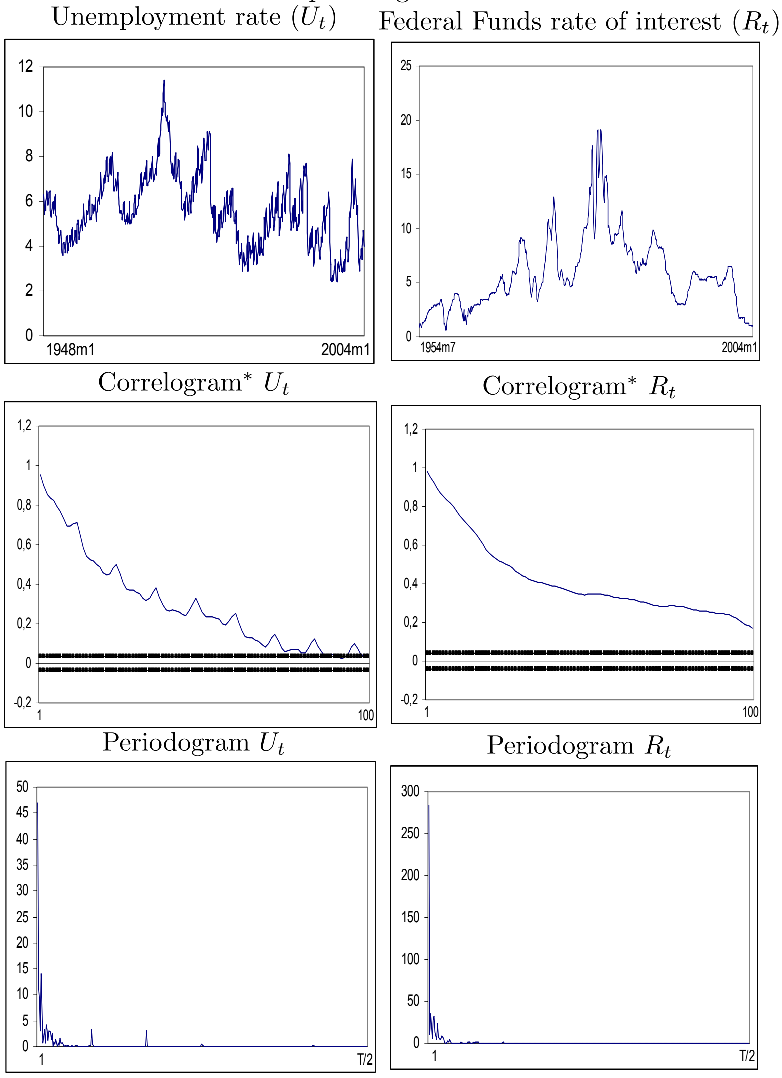
$$y_t = \beta_0 + \beta_1 t + x_t, \qquad t = 1, 2, \cdots \tag{18}$$

$$\prod_{j=1}^{h} (1 - 2\cos w_{r_s}^{j} L + L^2)^{d_j} x_t = u_t, \tag{19}$$

We initially impose $h = 4$, but in the case of the interest rate, H_0 was rejected in all cases for non-zero d_j-values. Thus, we try for this series $h = 3$ with $w_{r_s}^{j} = \frac{2\pi r_s}{n}$, $r_s = \frac{n}{s}$, $j = 1, 2, 3$, $s = 6, (6), n/2$. That is, we consider cycles with a periodicity of 6 or multiples of 6 periods (months). Also, the plots in Figure 1 show that the two series may have a component of long memory at the long run or zero frequency. Because of that, we include the case of $r = 0 (s = \infty)$ as the first cyclical (long run) structure. Note that in this case the cyclical polynomial becomes $(1 - 2L + L^2)^{d_1} = (1 - L)^{2d_1}$, so that d_1 refers to half the order of integration at the long run frequency. Moreover, we treat separately the cases $\beta_0 = \beta_1 = 0$ a priori; β_0 unknown and $\beta_1 = 0$ a priori; and both β_0 and β_1 unknown, i.e., we consider respectively the cases of no regressors in the undifferenced regression (18), an intercept, and an intercept and a linear time trend, and model u_t as a white noise process.

FIG. 1.

Original time series with their corresponding correlograms and periodograms



*: The large sample standard error under the null hypothesis of no autocorrelation is $1/\sqrt{n}$ or roughly 0.068.

Tables 4 and 5 reports the combinations of (s_1, s_2, s_3) and (d_1, d_2, d_3) values[7] where H_0 (11) cannot be rejected at the 5% level, respectively for

[7] s_1, s_2 and s_3 refers respectively to the values of s (the number of periods per cycle) for $w_r^j, j = 1, 2$ and 3. In case of $j = 1$, we adopt the notation $s_1 = 0$ to refer to the long run or zero frequency.

unemployment and interest rates. We only display the results for the case of an intercept, the reason being that the coefficients corresponding to the time trend were found to be insignificant in all cases where the null cannot be rejected.[8]

Starting with the Federal Funds rate of interest, we observe that all the non-rejection values take place when $s_1 = 0$ and $s_2 = 12$, with s_3 widely ranging from 24 to 120 periods. Thus, we clearly observe a long run component, an annual cyclical structure (probably due to the monthly nature of the series) and another cyclical structure constrained between 2 and 10 years. With respect to the orders of integration at each of these frequencies, we see that d_1 (long run effect) is in all cases 0.10 and 0.20; d_2 (the annual structure) ranges between 0.30 and 0.60, while d_3 lies between 0.10 and 0.40. Thus, it is the annual structure the most important one and the closest to nonstationarity. On the other hand, the long run and the purely cyclical components are clearly stationary with values strictly below 0.5. Finally, the fact that all the values are smaller than 1 implies that the series is mean reverting with respect to all these components, suggesting that shocks affecting them will disappear in the long run, though it will take longer time in case of those shocks affecting the annual (monthly) structure.

The results for the US unemployment rate are displayed in Table 5. We see that, apart from the long run or zero frequency, there exist three cyclical structures, two of them affecting the annual and semi-annual frequencies (j_2 and j_3) and one purely cyclical with the periodicity constrained between 2 and 10 years. The order of integration at the zero frequency ($2d_1$) is 0.20 in practically all cases. The highest order of integration corresponds to the semi-annual frequency, with d_2 ranging between 0.20 and 0.60; d_3 (annual) lies between 0.10 and 0.30, while the purely cyclical one (d_4) is in all cases 0.10 or 0.20.

We have marked in the two tables in bold the values corresponding to the model that produces the lowest statistic across the values of the s's and the d's. In doing so, the residuals of these selected models should be the closest to white noise and their orders of integration should be approximations to the maximum likelihood estimates. The resulting models are:

$$y_t = \underset{(0.107)}{0.339} + x_t;$$

$$(1-L)^{0.20}(1-2\cos w_{s=12}L+L^2)^{0.50}(1-2\cos w_{s=84}L+L^2)^{0.20}x_t = \varepsilon_t$$

---

[8]Note that the test statistic is evaluated under the null differenced model, which is supposed to be short memory and thus, standard t-tests apply.

TABLE 4.

Combination of values where $H_0(11)$ cannot be rejected for the Federal Funds rate of interest

s_1	s_2	s_3	d_1	d_2	d_3
0	12	24	0.10	0.30	0.40
0	12	24	0.10	0.40	0.40
0	12	24	0.10	0.50	0.30
0	12	24	0.10	0.60	0.20
0	12	24	0.20	0.30	0.30
0	12	24	0.20	0.40	0.20
0	12	24	0.20	0.50	0.10
0	12	36	0.10	0.30	0.40
0	12	36	0.10	0.40	0.30
0	12	36	0.10	0.60	0.20
0	12	36	0.20	0.40	0.20
0	12	36	0.20	0.50	0.10
0	12	48	0.10	0.40	0.30
0	12	48	0.10	0.50	0.2
0	12	48	0.10	0.60	0.2
0	12	48	0.10	0.70	0.1
0	12	48	0.20	0.50	0.10
0	12	60	0.10	0.40	0.30
0	12	60	0.10	0.50	0.20
0	12	60	0.10	0.70	0.10
0	12	60	0.20	0.50	0.10
0	12	72	0.10	0.50	0.20
0	12	72	0.10	0.70	0.10
0	12	72	0.20	0.50	0.10
0	**12**	**84**	**0.10**	**0.50**	**0.20**
0	12	84	0.10	0.70	0.10
0	12	84	0.20	0.50	0.10
0	12	96	0.10	0.50	0.20
0	12	96	0.10	0.70	0.10
0	12	96	0.20	0.50	0.10
0	12	108	0.10	0.50	0.20
0	12	108	0.10	0.70	0.10
0	12	108	0.20	0.50	0.10
0	12	120	0.10	0.50	0.20
0	12	120	0.10	0.70	0.10
0	12	120	0.20	0.50	0.10

In bold the values corresponding to the model that produces the lowest statistic.

TABLE 5.

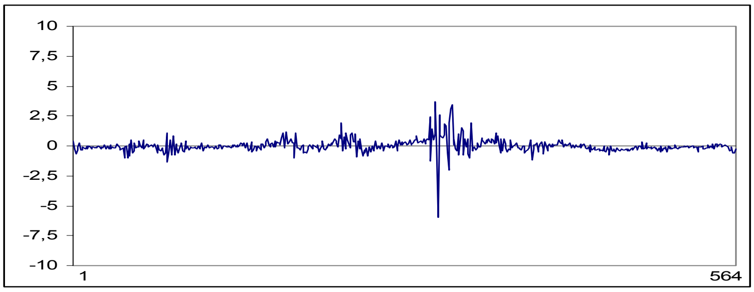
Combination of values where $H_0(11)$ cannot be rejected for the US unemployment rate

s_1	s_2	s_3	s_4	d_1	d_2	d_3	d_4
0	6	12	24	0.10	0.20	0.30	0.10
0	6	12	24	0.10	0.30	0.10	0.20
0	6	12	24	0.10	0.30	0.20	0.20
0	6	12	24	0.10	0.30	0.30	0.10
0	6	12	24	0.20	0.30	0.30	0.10
0	6	12	24	0.10	0.40	0.10	0.20
0	6	12	24	0.10	0.40	0.30	0.10
0	6	12	24	0.10	0.50	0.10	0.20
0	6	12	24	0.10	0.50	0.30	0.10
0	6	12	24	0.20	0.50	0.30	0.10
0	6	12	24	0.10	0.60	0.10	0.20
0	6	12	36	0.10	0.20	0.30	0.10
0	6	12	36	0.10	0.30	0.10	0.20
0	6	12	35	0.10	0.30	0.30	0.10
0	6	12	36	0.10	0.40	0.10	0.20
0	6	12	36	0.10	0.40	0.30	0.10
0	6	12	36	0.10	0.50	0.10	0.20
0	6	12	48	0.10	0.30	0.20	0.10
0	6	12	48	0.10	0.30	0.30	0.10
0	6	12	48	0.10	0.40	0.20	0.10
0	6	12	48	0.10	0.40	0.30	0.10
0	6	12	48	0.10	0.50	0.20	0.10
0	6	12	48	0.10	0.60	0.20	0.10
0	6	12	60	0.10	0.30	0.20	0.10
0	6	12	60	0.10	0.40	0.20	0.10
0	6	12	60	0.10	0.50	0.20	0.10
0	6	12	60	0.10	0.60	0.20	0.10
0	6	12	72	0.10	0.30	0.20	0.10
0	6	12	72	0.10	0.40	0.20	0.10
0	6	12	72	0.10	0.50	0.20	0.10
0	6	12	72	0.10	0.60	0.20	0.10
0	6	12	84	0.10	0.30	0.20	0.10
0	6	12	84	0.10	0.40	0.20	0.10
0	6	12	84	0.10	0.50	0.20	0.10
0	6	12	84	0.20	0.50	0.20	0.10
0	6	12	96	0.10	0.30	0.20	0.10
0	**6**	**12**	**96**	**0.10**	**0.40**	**0.20**	**0.10**
0	6	12	96	0.10	0.50	0.20	0.10
0	6	12	108	0.10	0.30	0.20	0.10
0	6	12	108	0.10	0.40	0.20	0.10
0	6	12	108	0.10	0.50	0.20	0.10
0	6	12	120	0.10	0.30	0.20	0.10
0	6	12	120	0.10	0.40	0.20	0.10
0	6	12	120	0.10	0.50	0.20	0.10

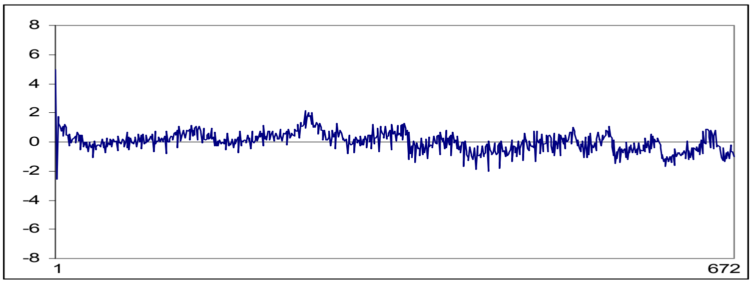
In bold the values corresponding to the model that produces the lowest statistic.

FIG. 2.

Residuals from the selected model for the Federal Funds rate of interest



Residuals from the selected model for the US unemployment rate



for the Federal Funds rate of interest, and

$$y_t = \underset{(0.212)}{1.335} + x_t;$$

$$(1-L)^{0.20}(1-2\cos\, w_{s=6}L+L^2)^{0.40}(1-2\cos\, w_{s=12}L+L^2)^{0.20}$$
$$\times\ (1-2\cos\, w_{s=96}L+L^2)^{0.10}x_t=\varepsilon_t$$

for the unemployment rate, with the standard errors in parenthesis.

Figure 2 displays the residuals of these estimated models. We observe that both have the appearance of white noise, though further refinement can be done by using short memory (autoregressive, moving average, etc.) models. However, we performed several diagnostic tests on these estimated residuals and both pass the diagnostics of homoscedasticity and no serial correlation.[9] A visual inspection at Figure 2 suggests that a structural break may have occurred around World War II, adopting the form of an impact change for the interest rate and of a mean shift for the unemployment rate. Robinson's (1994) testing procedure described in Section 3 permits us to incorporate dummy variables to take into account breaks, with no effect on its standard limit distribution, but this is out of the scope of the present work and it will be examined in future papers.

6. CONCLUDING COMMENTS AND EXTENSIONS

In this article we have proposed the use of a new statistical model for economic and financial time series, which is based on the idea of multiple cycles. We model the cycles by means of Gegenbauer processes, using a procedure that permits us to test multiple unit and fractional roots at fixed frequencies over time and thus, it permits us to approximate the length of each cycle. This procedure is due to Robinson (1994) and it has several distinguishing features that make it especially relevant compared with other methods. In particular, it has standard null and local limit distributions, and this standard behaviour holds whether or not we include deterministic components in the model. Moreover, it does not require Gaussianity for the asymptotic distribution, a feature that it is rarely satisfied in financial time series, with a moment condition only of order 2 required. The procedure is applied to one economic time series (US monthly unemployment rate) and a financial one (Federal Funds rate of interest), and the results show that both series can be specified in terms of a multiple cyclical fractional model.

Starting with the Federal Funds rate of interest, the results show the existence of three cyclical structures: one corresponding to the long run or zero frequency, one annual cycle (probably due to the monthly nature of the series), and a purely cyclical one with length of approximately 7 years. The orders of integration range between 0.1 and 0.4 for the long run and the cyclical structures, and lies between 0.3 and 0.6 for the annual frequency. For the unemployment rate, four cycles are observed: at the long run, the semi-annual and annual frequencies and a cyclical one of 8 years of length. The orders of integration are here around 0.20 for all frequencies except the semi-annual one, with a value constrained between 0.2 and 0.6. The fact that all the values are in the two series smaller than 1 implies that the series are mean reverting with respect to all these components, suggesting that shocks affecting them will disappear in the long run, though it will take longer time in case of those affecting the annual (monthly) structure.

[9] In particular, we use tests of Durbin (1970) and Godfrey (1987a, b) for no serial correlation, and Koenker (1981) for homoscedasticity, using Microfit.

An argument that can be employed against this type of model is that, contrary to seasonal cycles, business cycles are typically weak and irregular and are spread evenly over a range of frequencies rather than peaked at a specific value. However, contrary to that argument, we can explain that, in spite of the fixed frequencies used in this specification, the flexibility can be achieved throughout the first differencing polynomial (long run effect), the cyclical components, the interaction between them and the error term. In that respect, the results presented here lead us to unambiguous conclusions and, with respect to the cyclical component, they are completely in line with the literature on business cycle duration that says that cycles have a duration constrained between 3 and 10 years. Previous researchers on business cycles use the Hodrick-Prescott (1997) filter (HP-filter) or Baxter and King's (1999) band-pass filter, and most authors conclude that business cycles have duration of about six years. The HP filter has been interpreted as an approximation to an ideal high pass filter, eliminating frequencies of 32 quarters or greater. (See, e.g., Prescott, 1986, and King and Rebelo, 1993). Researchers relying on other methods often share this view about the duration of the business cycle component. Baxter and King (1999) construct a band-pass filter designed to extract cycles with duration between 1.5 and 8 years. Englund et al. (1992) and Hassler et al. (1994) use a band-pass filter in the frequency domain to extract cycles with duration between 3 and 8 years. Similar conclusions are obtained in Canova (1998), Burnside (1998), King and Rebelo (1999) and others. The results in the present work concerning the length of the cycle are completely in line with all these previous works.

It would also be worthwhile proceeding to get point estimates of the fractional differencing parameters in this context of multiple cyclical models. For the long run component the literature is extent. (See, e.g., Fox and Taqqu, 1986; Dahlhaus, 1989; Sowell, 1992; Robinson, 1995a,b; etc.). For the purely cyclical part, some attempts have been made by Arteche and Robinson (2000) and Arteche (2002). However, the goal of this paper is to show that fractional models with the roots simultaneously occurring at various frequencies can be credible alternatives to other more classical approaches when modelling many time series and, in that respect, the results presented in this paper leads us to some unambiguous conclusions, with the periodicity of the cycle constrained between 4 and 7 years and the orders of integration being slightly positive at the zero frequency, and ranging between 0 and 0.5 for the seasonal/cyclical components.

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