ANNALS OF ECONOMICS AND FINANCE **8-1**, 95–112 (2007)

Production and Inventory Behavior of Capital*

Yi Wen

Research Department, Federal Reserve Bank of St. Louis
E-mail: yi.wen@stls.frb.org

This paper provides a dynamic optimization model of durable goods inventories to study the interactions between investment demand and the production of capital goods. There are three major findings: first, capital suppliers' inventory behavior makes investment demand more volatile in equilibrium; second, equilibrium price of capital is characterized by downward stickiness; and third, the responses of the capital market to interest rate and other environmental changes are asymmetric. All are the result of equilibrium interactions between demand and supply.

Key Words: Investment; Capital theory; Capital supply; Inventory; Durable goods.
JEL Classification Numbers: E22, E23, E32.

1. INTRODUCTION

Inventory investment as a component of aggregate spending accounts for less than one percent of GDP, yet the drop in inventory investment accounts for 87 percent of the drop in GDP during the average postwar recession (Blinder and Maccini, 1991). Among inventories, durable goods inventories are the most volatile – nearly five times as volatile as non-durable goods inventories in terms of variance (see, e.g., Blinder, 1986, table 1). Hence, understanding the production and inventory behavior of the durable goods industry is essential for understanding the business cycle.

This paper focuses on one particular type of durable good: capital. In the U.S., about half of the output produced by the durable goods sector is sold to producers as capital equipment. Unfortunately, the literature on firms' optimal behavior of production and inventory investment with regard to capital goods is remarkably thin. Most of the literature on capital

* The views expressed are those of the author and do not reflect official positions of the Federal Reserve System.

1529-7373/2007

deals with capital demand (i.e., investment) instead of supply.[1] This may be attributable to the fact that there are no theoretical models available for dealing with durable goods inventories in general and capital goods inventories in particular. The difficulty involved is that, on the one hand, capital is a durable good, and durability is a user's measure, not a producer's measure, so modeling the production and inventory behavior of capital requires consideration for capacity demand from the view point of capital buyers; and on the other hand, production and inventory accumulation of capital goods is a supply-side problem, thus requiring simultaneous handling of upstream firms which produce, store, and sell capital equipment to downstream firms. The traditional (S,s) approach for inventories, for example, is inadequate for this task. It would assume that there exists a fixed cost of ordering capital goods, so firms have the incentive to order more capital equipment than needed in an (S,s) style, in order to reduce the average fixed cost of capital purchases.[2] This demand-side approach is quite limited for understanding capital good inventories because few firms would order excess capital equipment simply because of fixed costs of ordering or delivery, especially considering the fact that most fixed costs of capital investment are either variable fixed costs or disproportionately small relative to the price of capital. Even if firms do order excess capital in order to reduce the average fixed costs of purchases, the excess capital installed is treated as excess capacity instead of as inventories in accounting books.[3]

According to textbook theories, national savings are the chief source of domestic investment. Yet in reality how savings get translated into investment is a subtle issue. If investment demand is defined as demand for financial capital, then it is rather easy to imagine how household savings (the supply of funds) provide the source of financial investment. But if investment demand is defined as demand for tangible capital goods (i.e., machinery), then how aggregate savings end up meeting investment demand is not that simple. For one thing, capital goods must be produced, and production of capital goods takes time. Thus, national savings have to come from production determined in the past. Since only productive capital (or finished capital goods which are ready for use) are purchased by firms, the time-to-built factor is on the supply side, not on the demand side. For this reason, the demand for capital may not be satisfied unless

[1]The most influential paper on this subject is Tobin's (1969) q theory. For the more recent literature, see Abel, Dixit, Eberly and Pindyck (1996), Able and Eberly (1994), Hayashi (1982), and Lucas and Prescott (1971), among many others.

[2]For the recent literature on the (S,s) inventory model, see, e.g., Caballero and Engel (1999), Fisher and Hornstein (2000), and Kahn and Thomas (2002a).

[3]The literature on the lumpiness of investment behavior deals with volatility of capital from the demand side. This literature has left out the issue of capital supply with respect to capital goods production and its associated inventory behavior. See for example, Thomas (2002) and Kahn and Thomas (2002b) and the references therein.

the suppliers of capital can anticipate this demand many periods in advance. This time dimension on the supply side of capital is hidden behind the national income accounting. The issue is further complicated by inventories. In national income accounting, inventories are treated as part of aggregate demand. But in reality inventories may be related more closely to the supply side than to the demand side. For example, to enhance the flexibility of supply and to avoid opportunity costs of losing sales, capital suppliers may have incentive to accumulate inventories of capital goods by producing above the expected capital demand from capital buyers. Such inventory behavior would certainty affect the supply capacity of capital and hence national savings. Thus, while it is easy to determine how an increase in the interest rate affects investment demand from capital buyers (at least according to textbook theory), it is not clear how this should affect the production and inventory behavior of capital (i.e., the supply of capital). A simple textbook-style upward-sloping savings curve is clearly inadequate and may be misleading in drawing conclusions about the determination of equilibrium investment.

This paper takes a first step towards addressing the supply-side issues of capital by providing a canonical model for the production and inventory behavior of capital. In the model buyers order capital goods from suppliers to produce output, and suppliers produce and sell capital goods to the buyers. As an initial step in this literature, to facilitate analysis, perfect competition in the capital goods market is assumed. Hence both the buyers and the suppliers of capital are price takers in the capital goods market. The production of capital is assumed to take at least one period of time, thus production plans need to be committed to before demand is known.[4] Due to the uncertainty in investment demand from the buyers (e.g., due to profit or demand shocks to downstream firms), the suppliers may incur inventories of capital goods produced when demand for capital is below expectation. The supplier, however, has the option either to sell inventories at lower price in order to reduce the cost of holding inventories, or to accumulate inventories, anticipating higher demand in the next period.[5] Optimal production and inventory investment decisions as well as the equilibrium price of capital are characterized. Comparative statics are

[4]This reflects the important concept of time-to-built (see Kydland and Prescott, 1982).

[5]There are two types of capital: equipment and structures. Since structures are much less divisible and hence far more costly both in terms of price and inventory storage, they are mostly produced according to orders. Hence inventories of structures are less common than inventories of equipment. However, according to the U.S. housing data (houses are a form of structures), suppliers will often start construction on a house before an order comes in, which suggests that there are also inventories in structures.

conducted to study the effects of changes in the interest rate and in demand uncertainty on the supply-demand behavior of capital in equilibrium.[6]

It is found that a competitive capital supplier's optimal behavior is characterized by an inventory target policy that specifies the optimal level of production based on expected investment demand from capital buyers. Such inventory holding behavior of the capital supplier can dramatically change the dynamics of equilibrium investment demand. Without inventories, the demand for capital is met completely by capital production. Due to time-to-built, production plans are determined by past information about expected future demand. Thus investment demand cannot be adjusted *ex post* to reflect news about its current profitability. This leads to less volatile investment demand. With inventories, however, the supply of capital effectively becomes perfectly elastic up to the point of a stockout, which enables capital buyers to re-adjust investment demand according to new information about the returns to capital. Hence, investment demand becomes more volatile in equilibrium. It is also shown that the response of the capital market to policy changes is asymmetric due to the capital suppliers' production and inventory behavior. For example, an increase in the interest rate has a larger effect on equilibrium investment when the market is thick (i.e., there are more firms with high demand) than when it is thin, despite the fact that individual firm's investment demand is always a function of the interest rate.[7] Another interesting implication of the model is that the price of capital appears to be sticky downward, but flexible upward. This is also a consequence of the inventory behavior of the capital suppliers.[8]

The rest of the paper is organized as follows. The model is described in Section 2. Closed-form policies for optimal demand, supply, inventory investment, and the equilibrium price of capital are derived and characterized in Section 3 under the assumption of $i.i.d$ shocks to downstream firms' profitability. Section 4 shows that the main results of the model still hold when shocks are serially correlated. Finally, section 5 concludes the paper.

[6] Although equilibrium capital prices are endogenously determined in the model by demand and supply, the model is still a partial equilibrium model in the sense that there are no consumers to make consumption and saving decisions to determine an equilibrium interest rate. Extending the current framework to a general equilibrium model is a topic worth pursuing in the future.

[7] This paper uses a representative firm model. hence a thick market is equivalent to high demand and a thin market is equivalent to low demand of the representative firm.

[8] The downward-sticky price behavior has also been shown by Amihud and Mendelson (1983), Blinder (1982) and Reagan (1982) in models with non-durable goods inventories. Aso see Wen (2005).

2. THE MODEL

Downstream Firms: A representative buyer purchases capital goods as capacity investment to produce output. The revenue function of the downstream firm is given by

$$f(k_t, \theta_t),$$

where k represents capital stock, θ is an $i.i.d$ random variable representing shocks to the firm's revenue, and $f(\cdot)$ satisfies $f_k > 0, f_k(0, \theta) = \infty, f_{kk} < 0$, and $f_{k\theta} > 0$, where the last assumption indicates that θ shifts the firm's capital demand curve upwards. Since there is no need to impose perfect competition on the buyer's supply behavior, $f(k, \theta)$ can be interpreted either as a revenue function with a downward-sloping price curve, or as a production function with the output price normalized to one whenever perfect competition in the buyer's output market is imposed. However, with respect to the buyer's input market, perfect competition is assumed. The market price of capital input (cost of investment) is denoted λ_t, which the firm takes as given. Assuming full capacity utilization, the firm chooses a sequence of either the capital stock, $\{k_t\}_{t=1}^{\infty}$, or the rate of investment, $\{I_t\}_{t=0}^{\infty}$, to maximize the discounted expected profit,

$$\max E \sum_{t=0}^{\infty} \beta^t \left[f(k_t, \theta_t) - \lambda_t I_t \right],$$

subject to

$$k_{t+1} = I_t + (1 - \delta) k_{t-1},$$

where $k_0 > 0$ is given, $\beta \in (0, 1)$ is the inverse of the interest rate (discount factor), and δ is the rate of capital depreciation.

Upstream Firms: A representative supplier produces capital goods (y_t) using labor according to a linear production technology. This implies that the cost function is linear in output, ay_t, where a is a positive constant. Assuming a one period production lag between the commitment of input and the availability of output for sale (i.e., the firm must make production plans one period in advance before demand for capital in period t is known), total output (capital goods) available for sale in period t is the existing stock of inventories carried over from last period (s_{t-1}) plus the current output that was committed to last period (y_t). This assumption of production lags reflects the concept of time-to-built (e.g., see Kydland and Prescott, 1982). Without loss of generality, it is assumed that the depreciation rate of inventories is zero, and that there are no other costs of holding inventories except those associated with time discounting, β. Under the assumption of perfect competition in the capital market, the upstream firm takes the output (capital) price (λ_t) and investment demand from buyers (I_t) as given. The firm's revenue is thus given by $\lambda_t I_t$. The firm chooses sequences of production plans (y_t) and inventory investment ($s_t - s_{t-1}$) to maximize a discounted sum of expected profits,

$$\max_{\{y_t, s_t\}} E \sum_{t=0}^{\infty} \beta^t \left[\lambda_t I_t - a y_t\right],$$

subject to

$$I_t + s_t = s_{t-1} + y_t,$$

and

$$\begin{aligned} s_t &\geq 0, \\ y_t &\geq 0, \end{aligned}$$

where $s_{-1} \geq 0$ is given.

Competitive Equilibrium: A competitive equilibrium is a set of decision rules for capital sales (I_t), capital production (y_t), inventory holdings (s_t), and the price of capital (λ_t) such that both up- and down-stream firms' profits are maximized. The first order conditions are given by:

$$f_k(k_t, \theta_t) = \lambda_t - \beta(1-\delta) E_t \lambda_{t+1} \tag{1}$$

$$a = E_{t-1}\lambda_t + \mu_t \tag{2}$$

$$\lambda_t = \beta E_t \lambda_{t+1} + \pi_t \tag{3}$$

$$[k_t - (1-\delta)k_{t-1}] + s_t = s_{t-1} + y_t \tag{4}$$

$$\pi_t s_t = 0 \tag{5}$$

$$\mu_t y_t = 0 \tag{6}$$

where equation (1) determines the buyer's optimal demand for capital, equation (2) determines the supplier's optimal production of capital, equation (3) determines the supplier's optimal inventory holdings, equation (4) is the capital goods market clearing condition, and equations (5) and (6) are Kuhn-Tucker conditions for the nonnegativity constraints on the supplier's inventories and output level (hence π and μ are the complementary slackness multipliers).

Equation (1) shows that the optimal demand for capital decreases when δ increases (i.e., when the durability of goods decreases) or when the interest rate increases (i.e., when β decreases), holding capital prices constant.

This is the familiar user's cost effect of durability and the interest rate on demand. Equation (2) shows that the optimal supply of capital goods is chosen at the point where the marginal cost of production (a) equals the expected value of capital in the goods market (λ_t), adjusted by a slackness multiplier μ (which is zero if $y_t > 0$). Equation (3) shows that the optimal level of inventories held by the supplier is determined at the point where the marginal cost of increasing inventories (λ_t, which is also the opportunity cost for not selling one unit of capital goods) equals the discounted expected benefit of having one unit of capital goods available for sale next period (λ_{t+1}) plus the benefit of relaxing the slackness constraint by one unit (π_t, which is zero if the constraint does not bind). Since capital is durable, there is an intertemporal substitution effect of durability on future demand of capital, which can be seen from the equation

$$I_t = k_t - (1 - \delta)k_{t-1},$$

where purchase of the capital stock last period reduces the current investment demand for capital. The more durable is the good, the larger such effect is.

3. OPTIMAL SUPPLY OF CAPITAL

The source of uncertainty in the model (θ) stems from the capital buyer. A high θ implies a high demand for capital. Since production of capital takes one period, the supplier needs to forecast future investment demand and determine the optimal level of inventories. The following analysis shows that the optimal decision rules of the capital supplier are characterized by a threshold strategy that specifies a target level of inventories such that the inventory constraint binds if θ is above the threshold and does not bind if θ is below the threshold.

Assume that $y_t > 0$ to begin with. Hence $\mu_t = 0$. Consider two possible cases:

Case A: θ is below a threshold, which suggests that the investment demand for capital is low. In this case, the nonnegativity constraint on the supplier's inventories does not bind. Hence $\pi_t = 0$ and $s_t \geq 0$. Equations (2) and (3) imply that the competitive price of capital is constant,

$$\lambda_t = \beta a.$$

Thus equation (1) implies

$$f_k(k_t, \theta_t) = \beta\delta a,$$

which gives the equilibrium capital demand under case A as an increasing function of θ,

$$k_t = k^*(\theta_t), \quad \text{where } \frac{\partial k^*(\theta)}{\partial \theta} > 0.$$

The market clearing condition (4) then implies

$$s_t = y_t + s_{t-1} + (1-\delta)k_{t-1} - k^*(\theta_t).$$

The threshold value for θ is determined by the constraint $s_t \geq 0$, which implies

$$k_t^*(\theta_t) \leq y_t + s_{t-1} + (1-\delta)k_{t-1}, \tag{6}$$

or

$$\begin{aligned} \theta_t &\leq (k^*)^{-1}(y_t + s_{t-1} + (1-\delta)k_{t-1}) \\ &\equiv z_t, \end{aligned} \tag{7}$$

where z denotes the optimal threshold value of θ such that there is a stockout if $\theta > z$. Namely, z is defined as

$$k^*(z_t) \equiv y_t + s_{t-1} + (1-\delta)k_{t-1}. \tag{8}$$

Since $k^*(\theta)$ is a monotonically increasing function, we have

$$\frac{\partial k^*(z)}{\partial z} > 0 \text{ and } \frac{\partial z(y)}{\partial y} > 0. \tag{8$'$}$$

Case B: θ is below the threshold, which suggests that investment demand is high. In this case, the supplier's nonnegativity constraint on inventories binds. Hence $\pi_t > 0$ and $s_t = 0$. The market-clearing condition (4) implies that equilibrium investment demand is met with the supplier's entire existing stock of capital goods,

$$k_t - (1-\delta)k_{t-1} = y_t + s_{t-1}. \tag{9}$$

Thus we have

$$k_t = \begin{cases} k^*(\theta_t) & , \text{ if } \theta_t \leq z \\ y_t + s_{t-1} + (1-\delta)k_{t-1} & , \text{ if } \theta_t > z \end{cases}.$$

Clearly, the probability (likelihood) of cases A and B depends on the threshold value chosen by the firm, z_t, which in turn is determined by the production level committed to last period according to (7). Thus choosing the threshold value is achieved by choosing the level of production given the firm's existing inventory stock and the capital buyer's existing capital stock, both of which affect the demand and supply of capital in the current period. Under the assumption that θ is *i.i.d*, it turns out that the firm's optimal strategy is to adjust the production level such that the threshold is a constant. The intuition is that the firm wants to maintain a constant inventory target level when the shocks are not forecastable.[9]

To determine the optimal production policy, we can use equation (2). Denote $\phi(\cdot)$ as the probability density function of θ with non-negative support $[A, B]$. Then equation (2) can be expanded as

$$\begin{aligned} a =& E_{t-1}\lambda_t \\ =& \int_A^z \beta a \phi(\theta) d\theta + \int_z^B \left[f_k\left(k_t, \theta_t\right) + \beta(1-\delta)a \right] \phi(\theta) d\theta, \end{aligned} \tag{10}$$

where the threshold, z, is defined in (7) and (8).

The left-hand side of (10), a, is the cost of producing one extra unit of capital goods today, the right-hand side of (10) is the benefit of having one extra unit of capital goods available for sale next period, which is the competitive price of capital goods, λ. The competitive price of capital takes two possible values the next period depending on the level of demand. $\lambda = \beta a$ if the demand for capital is low due to a low realization of θ; likewise $\lambda = f_k(k, \theta) + \beta(1-\delta)a$ if the demand for capital is high due to a high realization of θ. In the latter case the level of capital demand (k_t) is determined by (9). Thus, the competitive price of capital is characterized by asymmetry, or downward stickiness:

$$\frac{\partial \lambda_t}{\partial \theta_t} = \begin{cases} 0 & , \text{ if } \theta_t \le z \\ f_{k\theta} > 0 & , \text{ if } \theta_t > z \end{cases} .$$

The reason for this kinked price behavior of capital is due to the capital supplier's inventory behavior. Namely, in the event of no stockout, the competitive price of capital is simply the discounted marginal cost of production; this event happens with probability $\int_A^z \phi(\theta)d\theta$. In the event of a stockout, the competitive price of capital is determined by the buyer's marginal revenue, $f_k(k, \theta)$. This event happens with probability $\int_z^B \phi(\theta)d\theta$.

PROPOSITION 1. *An optimal threshold, $\bar{z} > 0$, exists and is unique. Furthermore, $\bar{z}$ depends positively on the variance of θ.*

---

[9]If the shocks are serially correlated, then the optimal inventory target may depend on the forecastable components of the shocks.

Proof. Rewrite (10) (by substituting out k_t using equation 9) as:

$$\begin{aligned} a &= \int_A^z \beta a\phi(\theta)d\theta + \int_z^B \left[f_k(k_t,\theta_t) + \beta(1-\delta)a\right]\phi(\theta)d\theta \\ &= \int_A^z \beta a\phi(\theta)d\theta + \int_z^B \left[f_k\left((y_t + s_{t-1} + (1-\delta)k_{t-1}),\theta_t\right) + \beta(1-\delta)a\right]\phi(\theta)d\theta \\ &= \int_A^z \beta a\phi(\theta)d\theta + \int_z^B \left[f_k\left(k^*(z_t),\theta_t\right) + \beta(1-\delta)a\right]\phi(\theta)d\theta, \end{aligned}$$

where the last equality used the definition of z in (8). The above equation can be simplified (after rearranging terms) to:

$$\begin{aligned} (1-\beta)a &= \int_z^B \left[f_k'\left(k^*(z_t),\theta_t\right) - \beta\delta a\right]\phi(\theta)d\theta \qquad (11) \\ &\equiv \int_z^B g(z_t,\theta_t)\phi(\theta)d\theta. \end{aligned}$$

Notice that $k^*(z)$ is an increasing function of z (see equation (8′)), hence f_k is a decreasing function of z. Thus, $g_z = f_{kk}\frac{\partial k^*(z)}{\partial z} < 0$. Since $g > 0$ (by equation (1), $f_k > \beta\delta a$ under case B)[10], then clearly the right-hand side of (11) is monotonically decreasing in z :

$$\frac{\partial \int_{z_t}^B g(z_t,\theta_t)\phi(\theta)d\theta}{\partial z} = -g(z)\phi(z) + \int_z^B g_z\phi(\theta)d\theta < 0.$$

It is easy to see that the minimum of the right-hand side of (11) is zero when $z = B$ and the maximum is greater than $(1-\beta)\,a$ when $z = 0$ (since $f_k(0,\theta_t) = \infty$). Hence a unique positive solution for z_t exists. Furthermore, since θ is *i.i.d*, the right-hand side of (11) after integration is an implicit function of the form $G(z_t,\Omega) = 0$, where Ω is a set of constant parameters. Hence, z_t is a constant, $z_t = \bar{z}$, which solves $G(\bar{z},\Omega) = 0$ or

$$(1-\beta)a = \int_{\bar{z}}^B g(\bar{z},\theta_t)\phi(\theta)d\theta. \qquad (12)$$

Denote the unconditional mean of θ by $\bar{\theta} \equiv E(\theta_t)$, and notice that in the steady state (i.e., in the absence of uncertainty), $s_t = 0$ for all t. Hence the optimal cut-off point $\bar{z} \geq \bar{\theta}$ because under uncertainty (off the steady state) the optimal level of inventories cannot be less than that in the steady state (which corresponds to $\bar{\theta}$) due to the positive probability of a stockout.

[10]Note that $E_t\lambda_{t+1} = a$ by equation (2).

Now, consider an increase in the variance of θ that preserves the mean. A mean-preserving spread increases the weight of the tail of the distribution, so the right hand side of (12) increases, which indicates that $\bar{z}$ must also increase in order to maintain (12) since the right hand side is decreasing in $\bar{z}$. ▮

PROPOSITION 2. *$y_t = 0$ is not an equilibrium.*

Proof. Suppose $y_t = 0$ is optimal. Then we must have $\mu > 0$. Following very similar arguments as those discussed above under cases A and B, equation (11) is now replaced by

$$\begin{aligned}(1-\beta)a =&\mu_t + \int_{\hat{z}}^{B} \left[f_k\left(k^{**}(\hat{z}_t),\theta_t\right) - \beta\delta a\right]\phi(\theta)d\theta \\ >& \int_{\hat{z}}^{B} \left[f_k\left(k^{**}(\hat{z}_t),\theta_t\right) - \beta\delta a\right]\phi(\theta)d\theta,\end{aligned} \tag{11'}$$

where $k^{**} = s_{t-1} + (1-\delta)k_{t-1}$ is the optimal capital demand when $y_t = 0$ in the case of high demand (case B), $\theta > \hat{z}$; where the threshold, $\hat{z}$, is similarly defined as in (7) and (8) except under the assumption $y_t = 0$. Comparing (11′) to (11), we have $k^{**}(\hat{z}) > k^*(z)$ since f_k is a decreasing function of k. This implies

$$k^{**} = s_{t-1} + (1-\delta)k_{t-1} > y_t + s_{t-1} + (1-\delta)k = k^*,$$

which contradicts the condition $y_t \geq 0$. ▮

PROPOSITION 3. *The equilibrium decision rules for demand, supply, inventory investment and market price of capital are given by*

$$k_t = \begin{cases} k^*(\theta_t) & , \text{ if } \theta_t \leq \bar{z} \\ k^*(\bar{z}) & , \text{ if } \theta_t > \bar{z} \end{cases}$$

$$I_t = \begin{cases} k^*(\theta_t) - (1-\delta)k^*(\theta_{t-1}) & , \text{ if } \theta_t \leq \bar{z} \ \&\ \theta_{t-1} \leq \bar{z} \\ k^*(\theta_t) - (1-\delta)k^*(\bar{z}) & , \text{ if } \theta_t \leq \bar{z} \ \&\ \theta_{t-1} > \bar{z} \\ k^*(\bar{z}) - (1-\delta)k^*(\theta_{t-1}) & , \text{ if } \theta_t > \bar{z} \ \&\ \theta_{t-1} \leq \bar{z} \\ \delta k^*(\bar{z}) & , \text{ if } \theta_t > \bar{z} \ \&\ \theta_{t-1} > \bar{z} \end{cases}$$

$$y_t = \begin{cases} \delta k^*(\theta_{t-1}) & , \text{ if } \theta_{t-1} \leq \bar{z} \\ \delta k^*(\bar{z}) & , \text{ if } \theta_{t-1} > \bar{z} \end{cases}$$

$$s_t = \begin{cases} k^*(\bar{z}) - k^*(\theta_t) & , \text{ if } \theta_t \leq \bar{z} \\ 0 & , \text{ if } \theta_t > \bar{z} \end{cases}$$

$$\lambda_t = \begin{cases} \beta a & , \ \textit{if } \theta_t \leq \bar{z} \\ [f'_k(k^*(\bar{z}), \theta_t) + \beta(1-\delta)a] & , \ \textit{if } \theta_t > \bar{z} \end{cases} .$$

Proof. By proposition (1) and equation (8), the optimal production policy is given by

$$y_t = k^*(\bar{z}) - s_{t-1} - (1-\delta)k_{t-1}.$$

Substituting this into the values of inventory (s_t) discussed above under cases A and B, respectively, gives

$$s_t = \begin{cases} k^*(\bar{z}) - k^*(\theta_t) & \text{if } \theta_t \leq \bar{z} \\ 0 & \text{if } \theta_t > \bar{z} \end{cases} .$$

Similarly, we have

$$k_t = \begin{cases} k^*(\theta_t) & \text{if } \theta_t \leq \bar{z} \\ k^*(\bar{z}) & \text{if } \theta_t > \bar{z} \end{cases} .$$

Shifting the time subscript back by one period for s_t and k_t, and then substituting them into the production policy, gives

$$y_t = \begin{cases} \delta k^*(\theta_{t-1}) & \text{if } \theta_{t-1} \leq \bar{z} \\ \delta k^*(\bar{z}) & \text{if } \theta_{t-1} > \bar{z} \end{cases} .$$

The rest are derived by straightforward substitutions. ∎

The decision rules show that, because of the existence of inventories of capital, the variances of investment demand are increased. Without inventories, investment demand is equal to a pre-determined level of production, hence the optimal demand of capital is determined by

$$E_{t-1}f(k_t, \theta_t) = (1 - \beta(1-\delta))a,$$

which suggests that investment demand is not responsive to new information (θ) about the capital buyer's revenue or profits. With inventories, however, the optimal demand of capital is determined (using equations (1)-(3)) by:

$$f_k(k_t, \theta_t) = \delta\lambda_t + (1-\delta)\pi_t,$$

which suggests a higher elasticity of capital with respect to news (θ). In the case where investment demand is low, inventories can be used to absorb the excess supply; in the case where investment demand is high, inventories can be used to fulfill the excess demand until a stockout occurs. Thus, with probability $P = \Pr[\theta \leq \bar{z}]$, we have $\pi_t = 0$ and $\lambda_t = \beta a$, implying that k_t is perfectly correlated with θ_t. An interesting consequence of this is that the competitive market price of capital, λ_t, has the property described by Amihud and Mendelson (1983) and Reagan (1982). Namely, it is downward sticky when demand is low (i.e., $\lambda_t = \beta a$) because firms, speculating that demand may be stronger in the future, opt to hold inventories rather than sell them at a price below marginal cost. Such rational behavior attenuates downward pressure on price. When realized demand is high, on the other hand, the firm draws down its inventories until a stockout occurs and price rises to clear the market ($\lambda_t = [f_k(k^*(\bar{z}), \theta) - \beta\delta a] + \beta a > \beta a$ and in this case λ_t is an increasing function of θ).

Proposition 4. *The target inventory level is decreasing in the interest rate:* $\frac{\partial \bar{z}}{\partial r} < 0$.

Proof. The interest rate is the inverse of β: $r = \frac{1}{\beta}$. A decrease in the interest rate is the same as an increase in β. According to equation (11),

$$(1-\beta)a = \int_{\bar{z}}^{B} \underbrace{[f_k\left(k^*(\bar{z}), \theta_t\right) - \beta\delta a]}_{(+)} \phi(\theta) d\theta,$$

which can also be expressed as

$$(1-\beta)a + \beta\delta a\left(1 - \Phi(\bar{z})\right) = \int_{\bar{z}}^{B} f_k\left(k^*(\bar{z}), \theta_t\right) \phi(\theta) d\theta,$$

where $\Phi()$ denotes the cumulative density function of θ. Differentiating both sides with respect to β gives

$$-a\left[1 - \delta(1 - \Phi(\bar{z}))\right] - \beta\delta a\phi(\bar{z})\frac{d\bar{z}}{d\beta} = -f_k(k^*(\bar{z}), \bar{z})\phi(\bar{z})\frac{d\bar{z}}{d\beta} + \int_{\bar{z}}^{B} f_{kk}\frac{dk}{d\bar{z}}\frac{d\bar{z}}{d\beta}\phi(\theta) d\theta,$$

which can be rearranged to

$$-a\left[1 - \delta(1 - \Phi(\bar{z}))\right] = \left\{ -\underbrace{[f_k(k^*(\bar{z}), \bar{z}) - \beta\delta a]}_{(+)}\phi(\bar{z}) + \int_{\bar{z}}^{B} \underbrace{f_{kk}\frac{dk^*}{d\bar{z}}}_{(-)}\phi(\theta) d\theta \right\} \frac{d\bar{z}}{d\beta}. \tag{13}$$

Notice that $k^*(z)$ is increasing in z (see equation ($8'$)) and $f_{kk} < 0$. Hence the term in front of $\frac{d\bar{z}}{d\beta}$ on the right-hand side is negative. Given that $\Phi(\bar{z}) < 1$, the left-hand side of (13) is negative. Hence $\frac{d\bar{z}}{d\beta}$ must be positive in order for the right-hand side of (13) to be negative as well. ∎

This proposition says that a higher interest rate implies a lower target inventory level. The intuition is that a higher interest rate implies not only a higher cost to the user of capital (thus a lower expected investment demand), but also a higher opportunity cost for holding inventories (i.e., a higher discounting of the future), hence the target inventory level falls.

The equilibrium decision rules show that the economy's response to changes in the interest rate is asymmetric. For example, output level is sensitive to the interest rate only when the market is thick (i.e., when demand is high). In particular, production decreases as the interest rate increases if $\theta > \bar{z}$. Similarly, a change in the interest rate affects the demand for capital only when the market is thick. If the market is thin (low demand), a change in the interest rate has no effect on demand and production of capital. Furthermore, capital price is more sensitive to an interest rate change when the market is thin than when it is thick. This can be seen from the derivative of the price of capital with respect to β:

$$\frac{\partial \lambda}{\partial \beta} = \begin{cases} a & , \text{ if } \theta_t \leq \bar{z} \\ f_{kk} \frac{\partial k^*(\bar{z})}{\partial \bar{z}} \frac{\partial \bar{z}}{\partial \beta} + (1-\delta)a & , \text{ if } \theta_t > \bar{z} \end{cases}$$

where $f_{kk} \frac{\partial k^*(\bar{z})}{\partial \bar{z}} \frac{\partial \bar{z}}{\partial \beta} + (1-\delta)a < a$ since $f_{kk} < 0$ and $\frac{\partial k^*(\bar{z})}{\partial \bar{z}} \frac{\partial \bar{z}}{\partial \beta} > 0$. Also, the volatility of capital price increases as the interest rate rises. This implication stems also from the fact that the inventory target level decreases with the interest rate, hence the non-negativity constraint on inventories binds easier under a high interest rate than under a low interest rate, raising the probability of a thick market (case B).

4. ROBUSTNESS

The above analysis is based on the simplifying assumption of *i.i.d* shocks. It can be shown that the main results do not depend on this simplifying assumption. Let $\theta_t = \rho\theta_{t-1} + \varepsilon_t$, where $\rho \in (0, 1)$ measures the degree of serial dependence and ε_t is *i.i.d.* Denote the p.d.f. of θ by $\phi(\theta_t, \theta_{t-1})$. With this change in notation, equation (10) becomes

$$\begin{aligned} a &= E_{t-1}\lambda_t \\ &= \int_{\theta \leq z} \beta a \phi(\theta_t, \theta_{t-1}) d\theta_t + \int_{\theta > z} \left[f_k\left(k_t, \theta_t\right) + \beta(1-\delta)a \right] \phi(\theta_t, \theta_{t-1}) d\theta_t. \end{aligned}$$

Similarly, equation (14) becomes:

$$\begin{aligned} a &= \int_{\theta \le z} \beta a \phi(\theta_t, \theta_{t-1}) d\theta_t \\ &+ \int_{\theta > z} \left[f_k\left((y_t + s_{t-1} + (1-\delta)k_{t-1}), \theta_t\right) + \beta(1-\delta)a \right] \phi(\theta_t, \theta_{t-1}) d\theta_t \\ &= \int_{\theta \le z} \beta a \phi(\theta_t, \theta_{t-1}) d\theta_t + \int_{\theta > z} \left[f_k\left(k^*(z_t), \theta_t\right) + \beta(1-\delta)a \right] \phi(\theta_t, \theta_{t-1}) d\theta_t, \end{aligned}$$

which can be simplified to:

$$(1-\beta)a = \int_{\theta > z} \left[f_k\left(k^*(z_t), \theta_t\right) - \beta\delta a \right] \phi(\theta_t, \theta_{t-1}) d\theta_t. \qquad (15)$$

Following an argument similar to that behind equation (11), since the right-hand side of (15) is monotonically decreasing in z, a unique positive solution for z_t therefore exists. However, in this case, since θ_t depends on θ_{t-1}, the right-hand side of (15) after integration is an implicit function of the form $G(z_t, \theta_{t-1}, \Omega) = 0$. Hence, the optimal cut-off value z_t is no longer a constant but a function of $\theta_{t-1} : \bar{z} = z(\theta_{t-1})$, which solves $G(\bar{z}, \theta_{t-1}, \Omega) = 0$.

The equilibrium decision rules take the same form as before except the threshold value ($\bar{z}$) now depends on θ_{t-1}. Hence, the economy's response to changes in the interest rate is still asymmetric. To show that the target inventory level, $\bar{z}(\theta_{t-1})$, is decreasing in the interest rate, $\frac{\partial \bar{z}}{\partial r} < 0$, we can express equation (15) as

$$(1-\beta)a + \beta\delta a\left(1 - \Phi(\bar{z}, \theta_{t-1})\right) = \int_{\bar{z}}^{B} f_k\left(k^*(\bar{z}), \theta_t\right) \phi(\theta_t, \theta_{t-1}) d\theta_t,$$

where $\Phi(\bar{z}, \theta_{t-1})$ denotes the cumulative density function of θ_t. Differentiating both sides with respect to β gives

$$\begin{aligned} & -a\left[1 - \delta(1 - \Phi(\bar{z}, \theta_{t-1}))\right] - \beta\delta a \phi(\bar{z}, \theta_{t-1}) \frac{d\bar{z}}{d\beta} \\ = & -f_k(k^*(\bar{z}), \bar{z}) \phi(\bar{z}, \theta_{t-1}) \frac{d\bar{z}}{d\beta} + \int_{\bar{z}}^{B} f_{kk} \frac{dk}{d\bar{z}} \frac{d\bar{z}}{d\beta} \phi(\theta_t, \theta_{t-1}) d\theta_t, \end{aligned}$$

which can be rearranged to

$$
\begin{aligned}
&- a\left[1-\delta(1-\Phi(\bar{z},\theta_{t-1}))\right] \qquad (16)\\
&= \left\{ -\underbrace{[f_k(k^*(\bar{z}),\bar{z})-\beta\delta a]}_{(+)}\phi(\bar{z},\theta_{t-1}) + \int_{\bar{z}}^{B} \underbrace{f_{kk}\frac{dk^*}{d\bar{z}}}_{(-)}\phi(\theta_t,\theta_{t-1})d\theta_t \right\} \frac{d\bar{z}}{d\beta}.
\end{aligned}
$$

Notice that $k^*(z)$ is increasing in z and $f_{kk} < 0$. Hence the term before $\frac{d\bar{z}}{d\beta}$ on the right-hand side is negative. Given that $\Phi(\bar{z},\theta_{t-1}) < 1$, the left-hand side of (16) is negative. Hence $\frac{d\bar{z}}{d\beta}$ must be positive in order for the right-hand side of (16) to be negative as well.

The intuition for the robustness is that serial correlation in θ does not change the fact that the optimal threshold ($\bar{z}$) is independent of any endogenous variables. Consequently, except for the addition of a new state variable (θ_{t-1}) into the decision rules, all the arguments presented in the previous section remain intact. The only difference it makes is that the threshold ($\bar{z}$) now depends on the expected value of the shock ($E_{t-1}\theta_t$) because production decisions of the capital supplier are made one period in advance.

5. CONCLUSION

The demand side of the capital market has been intensively studied in the literature and is hence relatively well understood, but the supply side of the capital market has been largely neglected. Since, in equilibrium, demand equals supply, understanding the supply side of the capital market is no less important than understanding the demand side. Capital is a special type of durable good (the reproductive force of the economy), and the production of capital takes time (according to Kydland and Prescott (1982), the average time period for capital production is about 4 quarters). Thus to understand how investment demand, one of the most volatile economic variables over the business cycle, is satisfied by national savings in equilibrium, it is essential to understand the production and inventory behavior of capital. This paper shows that the production and inventory behavior of capital suppliers can dramatically alter the equilibrium dynamics of the capital market. In particular, due to capital suppliers' strategic production and inventory behavior in coping with demand uncertainty from capital buyers, investment demand of downstream firms becomes more volatile in equilibrium, equilibrium capital prices become sticky downward, and the responses of the capital market towards policy shocks become asymmetric. In particular, a change in the interest rate has a smaller effect on the capital market when market demand is low. In other words, policy tends to be less effective at influencing equilibrium investment when the market is thin. Whether these implications of the model are validated by the data is an interesting empirical research topic worth pursuing in the future.

REFERENCES

Able, A., A. Dixit, J. Eberly, and R. Pindyck, 1996, Options, the value of capital, and investment. *The Quarterly Journal of Economics* **111 (no. 3)**, 753-777.

Able, A. and J. Eberly, 1994, A unified model of investment under uncertainty. *American Economic Review* **84 (no. 5)**, 1369-1384.

Amihud, Y. and H. Mendelson, 1983, Price smoothing and inventory. *Review of Economic Studies* **50 (no. 1)**, 87-98.

Blanchard, O., 1983, The production and inventory behavior of the American automobile industry. *Journal of Political Economy* **91 (no. 3)**, 365-400.

Blinder, A., 1982, Inventories and sticky prices: More on the microfoundations of macroeconomics. *American Economic Review* **72 (no. 3)**, 334-348.

Blinder, A., 1986, Can the production smoothing model of inventory behavior be saved? *The Quarterly Journal of Economics* **101 (no. 3)**, 431-454.

Blinder, A. and L. Maccini, 1991, Taking stock: A critical assessment of recent research on inventories. *Journal of Economic Perspectives* **5 (no. 1)**, 73-96.

Caballero, R. and E. Engel, 1999, Explaining investment dynamics in U.S. manufacturing: A generalized (S,s) approach. *Econometrica* **67 (no. 4)**, 783-826.

Christiano, L., 1988, Why does inventory investment fluctuate so much? *Journal of Monetary Economics* **21**, 247-280.

Eichenbaum, M., 1989, Some empirical evidence on the production level and production cost smoothing models of inventory investment. *American Economic Review* **79 (no. 4)**, 853-64.

Feldstein, M. and A. Auerbach, 1976, Inventory behavior in durable-goods manufacturing: The target-adjustment model. *Brooking Papers on Economic Activity* **vol. 1976 (no. 2)**, 351-408.

Fisher, J. and A. Hornstein, 2000, (S,s) inventory policies in general equilibrium. *Review of Economic Studies* **67 (no. 1)**, 117-145.

Hayashi, F., 1982, Tobin's marginal q and average q : A neoclassical interpretation. *Econometrica* **50 (no. 1)**, 213-224.

Kahn, J., 1987, Inventories and the volatility of production. *American Economic Review* **77 (no. 4)**, 667-679.

Kahn, J., 1992, Why is production more volatile than sales? Theory and evidence on the stockout-avoidance motive for inventory-holding. *The Quarterly Journal of Economics* **107 (No. 2)**, 481-510.

Kahn, A. and J. Thomas, 2002a, Inventories and the business cycle: An equilibrium analysis of (S,s) policies, Working Paper, University of Minnesota.

Kahn, A. and J. Thomas, 2002b, Nonconvex Factor Adjustments in Equilibrium Business Cycle Models: Do Nonlinearities Matter? *Journal of Monetary Economics* **50 (no. 2)**, 331-60.

Kydland, F. and E. Prescott, 1982, Time to build and aggregate fluctuations. *Econometrica* **50 (No. 6)**, 1345-70.

Lucas, R. E., Jr. and E. C. Prescott, 1971, Investment under uncertainty. *Econometrica* **39 (no. 5)**, 659-681.

Maccini, L. and E. Zabel, 1996, Serial correlation in demand, backlogging and production volatility. *International Economic Review* **37 (no. 2)**, 423-452.

Ramey, V., 1991, Nonconvex costs and the behavior of inventories. *Journal of Political Economy* **99 (no. 2)**, 306-334.

Ramey, V. and K. West, 1999, Inventories, chapter 13 in *Handbook of Macroeconomics*, Vol. 1B, North-Holland.

Reagan, P., 1982, Inventory and price behavior. *Review of Economic Studies* **49 (no. 1)**, 137-142.

Thomas, J., 2002, Is Lumpy Investment Relevant for the Business Cycle? Federal Reserve Bank of Minneapolis Staff Report #302.

Tobin, J., 1969, A general equilibrium approach to monetary theory. *Journal of Money, Credit and Banking* **1 (no. 1)**, 15-29.

Wen, Y., 2003, Durable good inventories and the volatility of production: Explaining the less volatile U.S. economy, Department of Economics, Cornell University.

Wen, Y., 2005, Understanding the Inventory Cycle. *Journal of Monetary Economics* **52(8)**, 1533-55.

West, K., 1986, A variance bounds test of the linear quadratic inventory model. *Journal of Political Economy* **94 (no. 2)**, 374-401.