ANNALS OF ECONOMICS AND FINANCE **9-1**, 39–75 (2008)

Is Volatility Priced?

Yueh-Neng Lin

Department of Finance National Chung Hsing University
E-mail: ynlin@dragon.nchu.edu.tw

and

Ken Hung*

Department of Finance National Dong Hwa University
E-mail: kenhung@mail.ndhu.edu.tw

The asymmetric nature of the volatility response to return shocks could simply reflect the existence of time-varying risk premiums. This study proposes a stochastic volatility process allowing for time-varying correlation with underlying returns, in which the market price of volatility risk is naturally taken into account. Historical S&P 500 returns over the period January 1969—December 2004 are investigated under Kalman filtration. We successfully identify and isolate the volatility risk premium in the pricing process, and thereafter demonstrate the relative contributions of price premiums and volatility premiums to underlying returns. The market price of volatility risk is found to be positive and increases with investment horizons. The existence of volatility risk premium may help solve the pricing puzzle in CAPM that empirically underprices low-beta assets but overprices high-beta assets, which reasons the importance of this study.

Key Words: Volatility risk premium; Stochastic volatility; Kalman filter; Quasi-maximum likelihood.

JEL Classification Numbers:

* The authors wish to thank the referees of the thirty-fourth annual meeting of Western Decision Sciences Institute held in Vancouver, British Columbia, for their insightful comments that have helped to improve the exposition of this article in significant ways. We would also like to thank the participants of the 2005 FMA European Conference for various comments and suggestions in Siena, Italy. Yueh-Neng Lin gratefully acknowledges research support from Taiwan National Science Council (grant number: NSC 93-2416-H-005-029).

1529-7373/2008

1. INTRODUCTION

Past studies about price risk premiums usually assume that the asset return each follows a normal distribution with constant volatility.[1] In this case asset returns in different time-points will contribute equal weights to historical volatility and thus cannot explain the stylized phenomenon of volatility cluster.[2] More recent generalizations of the classic CAPM are three- or four-factor models. Bollerslev et al. (1988) implement a conditional CAPM with time-varying covariances structure, and ultimately time-varying market betas. Mandelbrot (1963), Fama (1965) and Diebold et al. (2000) empirically find that asset returns are usually fat-tailed and non-normal distributed. Ghysels et al. (1996) point out three stylized characteristics of financial asset returns including dependence structure, volatility clustering and volatility randomness. Taylor (1986) and Shephard (1995) propose the concept of stochastic volatility to help capture the features of high peaks, fat tails and volatility clustering in underlying returns. Black (1976) and Christie (1982) discover a negative correlation between current returns and future volatilities, which increases the skewness of the return distribution. The effect may (in part) be attributed to a chain of events according to which a negative return causes an increase in the debt-to-equity ratio, in turn resulting in an increase in the future volatility of the return to equity, so-called "leverage effect". On the other hand, the volatility feedback effect, along with the well-documented persistent volatility dynamics, also implies an observationally equivalent negative correlation between current returns and future volatility, as a shock to the volatility will require an immediate return adjustment to compensate for the increased future risk. Empirical evidence along these lines generally confirms that aggregate market volatility responds asymmetrically to negative and positive returns,[3] but the economic magnitude is often small and not always statistically significant (e.g., Schwert, 1989; Nelson, 1991; Gallant et al., 1992; Glosten et al., 1993; Engle and Ng, 1993; Duffee, 1995; Bekaert and Wu, 2000). Moreover, the evidence tends to be weaker for individual stocks (e.g., Tauchen et al., 1996; Andersen et al., 2001). Importantly, the magnitude also depends on the volatility proxy employed in the estimation, with options implied volatilities generally ex-

[1]The fundamental CAPM and APT pricing theories, for example, usually assume that return volatility is constant over time. If investors are risk-averse, the CAPM developed by Sharpe (1964) and Lintner (1965) implies a positive, linear relationship between the expected market risk premium and conditional market variance.

[2]The phenomenon of volatility clustering indicates that the current movement of asset returns is positively correlated to past price shocks.

[3]Guedhami and Sy (2004) find evidence of a positive (negative) and significant relationship between the market risk premium and conditional market variance in bull (bear) market context.

hibiting much more pronounced asymmetry (e.g., Bates, 2000; Wu and Xiao, 2002; Eraker, 2004). Traditional conditional heteroskedasticity models such as ARCH and GARCH cannot fully capture the asymmetry in volatility shocks. Jacquier et al. (1994) and Harvey and Shephard (1996) propose a stochastic volatility model allowing for correlation with underlying returns to capture the distribution skewness. Schwert (1990), Nelson (1991), Campbell and Kyle (1993) and Engle and Ng (1993) find that the correlation between stochastic volatility and underlying returns plays an important role in capturing the volatility asymmetry. Nandi (1998) addresses the significance of time-varying correlation for pricing S&P 500 options under stochastic volatility. The asymmetric nature of the volatility response to return shocks could simply reflect the existence of time-varying risk premiums (Pindyck, 1984; French et al., 1987; Campbell and Hentschel, 1992). If volatility is priced, an anticipated increase in volatility raises the required return on equity, leading to an immediate stock price decline. Hence the causality is different: the leverage hypothesis claims that return shocks lead to changes in conditional volatility, whereas the time-varying volatility risk premium theory contends that return shocks are caused by changes in conditional volatility. The volatility feedback effect underlies the ARCH-M model originally, on one hand, developed by Engle et al. (1987). GARCH perspectives also produce approaches to empirical asset pricing, as with the empirical pricing kernels of Engle and Rosenberg (2002). On the other hand, limited studies are found to investigate the existence of volatility risk premiums in asset returns though stochastic volatility has been characterized to return dynamics. Melino and Turnbull (1990) find that stochastic volatility allowing for non-positive volatility risk premiums can provide a better pricing fit for options than a constant volatility. Lamoureu and Lastrapes (1993) suggest a time-varying volatility risk premium. While using Heston's (1993) option pricing formula to price currency options, Guo (1996) finds that volatility risk premiums are significantly different from zero. He points out that the assumption of zero correlation between underlying asset and volatility in Hull and White's (1987) option pricing formula causes a failure in capturing volatility risk premiums and thus biases volatility forecasting of implied volatility. Guo also finds that constant or deterministic historical volatilities do not contain volatility risk premiums and thus cause a poor forecast of realized volatility. Empirical applications of the ARCH-M, and related stochastic volatility models, have met with mixed success. Some studies (see, e.g., French et al., 1987; Chou, 1988; Campbell and Hentschel, 1992; Ghysels et al., 2002) have reported consistently positive and significant estimates of the risk premium, while others (see, e.g., Campbell, 1987; Turner et al., 1989; Breen et al., 1989; Nelson, 1991; Chou et al., 1992; Glosten et al., 1993) document negative values, unstable signs, or otherwise insignificant estimates. Moreover, the contemporaneous risk-return tradeoff appears sensitive to the use of ARCH as opposed to stochastic volatility formulations (Koopman and Uspensky, 1999), the length of the return horizon (Harrison and Zhang, 1999), along with the instruments and conditioning information used in empirically estimating the relationship (Harvey, 2001; Brandt and Kang, 2004). These conflicting results are not necessarily inconsistent with the basic Intertemporal CAPM (ICAPM) (Merton, 1980),[4] in that the risk-return tradeoff relationship depends importantly on the particular volatility measure employed in the empirical investigations.[5] Moreover, whereas such a relationship for the market portfolio would be consistent with the CAPM or ICAPM, it only holds in general equilibrium settings under restrictive assumptions (Backus and Gregory, 1993; Campbell, 1993; Glosten et al., 1993). As evinced by the stylized features in underlying returns and previous literature, this study proposes a stochastic volatility process allowing for time-varying correlation with underlying returns, in which the market price of volatility risk is naturally taken into account.

Our theoretical results are based on a discrete time analogue of square root diffusion models used by Cox et al. (1985), Heston (1993) and many others. Nelson (1990), Nelson and Foster (1994), Corradi (2000) and Heston and Nandi (2000) previously develop the technique to derive various closed-form expressions for discrete-time diffusion approximations of the one-factor continuous-time stochastic volatility model. The same basic idea could in principle be generalized to other more complicated model structures, including multiple volatility factors and jumps, at the expense of notational and computational complexity (see, e.g. Andersen et al., 2002; Eraker et al., 2003; and Chernov et al., 2003). Nonetheless, the relatively simple one-factor affine Heston model is rich enough to explain our empirical findings in regards to the return-volatility regressions for the S&P 500 market index. Similar to Engle and Ishida's (2002) work, we parameterize the volatility of volatility so that the variance of the variance is linear in the variance. The main difference is that the past information set can no longer be summarized by a conditional variance, since this is no longer defined by our model. Instead, a more general conditioning data set is introduced. Our empirical results reveal the existence of volatility clustering and volatility asymmetry caused by a negative correlation between return and volatility shocks. The market price per unit volatility risk is found to be significantly

[4]The classical Intertemporal CAPM (ICAPM) model of Merton (1980) implies that the excess return on the aggregate market portfolio should be positively and directly proportionally related to the volatility of the market (see also Pindyck, 1984).

[5]More general multifactor models also complicate the risk-return tradeoff relationship, as the projection of the returns on the volatility must now control for other state variables (see, e.g., Abel, 1988; Tauchen and Hussey, 1991; Backus and Gregory, 1993; Scruggs, 1998).

positive and time-varying, indicating the importance of volatility risk premiums in S&P 500 index returns. The non-zero volatility risk premium in a non-traded volatility asset may contribute to explain the pricing puzzle in CAPM that cannot fully interpret the behavior of expected returns. Since the S&P 500 index is generally used as a proxy of the market portfolio, the volatility shocks negatively correlated to S&P 500 returns may also be negatively correlated to aggregate consumption growth, and thus results in a negative volatility risk premium. However, the negative impact of volatility shocks on the total expected returns is offset and dominated by the volatility asymmetry. The net effect of volatility shocks on the total return rate will turn out to be positive,[6] indicating that investors will demand positive volatility risk premiums to counter with volatility shocks. This study also demonstrates how to calculate the premiums for price risk and volatility risk. The justification for the importance of this approach arises from the argument that the risk premiums are important determinants in asset pricing. For comparison purposes, a constant-volatility model that converges to the geometric Brownian motion in the continuous-time limit is also considered. Further, empirical results from out-of-sample forecast in mean returns suggest that volatility risk premium under stochastic volatility framework has richer information content about asset returns than the constant-volatility model.

This paper is organized as follows. In Section 2 we address the question whether S&P 500 index returns exhibit some degree of volatility randomness, after describing our dataset and providing summary statistics. The models and methodology employed to describe the patterns followed by S&P 500 returns are explained in Section 3. Section 4 presents the main empirical results of our analysis and Section V contains our concluding comments.

[6]If the volatility shock (represented by ε_t) is positive (i.e., $\varepsilon_t > 0$), then volatility (denoted by h_t) will increase (i.e., $h_t \uparrow$) and the return shock (represented by x_t) negatively correlated to the volatility shock will be negative (i.e., $x_t < 0$) and thus reduces return rates ($R_t \downarrow$). However, due to the increase in volatility ($h_t \uparrow$), the magnitude of reduction in R_t becomes even more pronounced (i.e., $R_t \downarrow\downarrow$). Therefore, the relationship between return rate (R_t) and the term ($\sqrt{h_t}x_t$) containing the volatility risk premium appears to be positive. In contrast, if the volatility shock is negative ($\varepsilon_t < 0$), then volatility will decrease (i.e., $h_t \downarrow$) and the return shock will be positive ($x_t > 0$) due to volatility asymmetry. The positive return shock ($x_t > 0$) will cause an increase in the return rate ($R_t \uparrow$). However, the decrease in volatility will shrink the magnitude of return increase. The net effect will result in a positive relationship between the return rate (R_t) and the term ($\sqrt{h_t}x_t$) containing the volatility risk premium.

2. STYLIZED FACTS ABOUT S&P 500 RETURN VOLATILITY

This study uses daily 1-, 30-, 100- and 300-day return data on the S&P 500 index, over the period January 1969 to December 2004, representing 9083, 9054, 8984 and 8784 observations, respectively. We take the log-difference of the value of the index, so as to convert the data into continuously compounded returns. The raw data are presented in Figure 1 where prices are shown on the left axis. The rather smooth curve shows what has happened to this index over the last 36 years. The great growth of equity prices over the period and the subsequent decline after the new millennium. The 1- and 30-day return series as shown on the right axis is centered around zero, while a small positive average return is found for 100- and 300-day returns. The most dramatic event is the crash of October 1987 which dwarfs short-term returns in the size of the decline and subsequent partial recovery. However, such occasional jumps and crashes diminish at long time horizons. It is apparent that the amplitude of the returns is changing—the magnitude of the changes is sometimes large and sometimes small, and what is so-called volatility clustering. It is also clear that the volatility is higher when prices are falling. Volatility tends to be higher in bear markets. This is the asymmetric volatility effect that Nelson (1991) described with his EGARCH model. The period 1990 through the end of 1996 is recorded as an era of low volatility that was accompanied by a slow and steady growth of equity prices, reflecting investor confidence, while 1997 to the present has been a time of higher volatility. Table 1 shows some statistics on the data that illustrate the stylized features mentioned above: fat tails, volatility asymmetry and volatility clustering. The standard deviations of 1-, 30-, 100- and 300-day returns correspond to annualized volatilities of 1.60%, 1.57%, 1.56% and 1.64%, respectively. The skewness coefficient indicates that the returns distributions are negatively skewed. Finally, the kurtosis which measures the thickness of the tails of the distribution shows a strong evidence that extremes are more substantial for short-term returns than for long-term returns. Similar evidence is seen graphically in Figure 2, which is a quantile plot and histogram with superimposed normal density. The correlogram of the squared returns, presented in Figure 3, indicates substantial dependence in the volatility of returns, i.e. volatility clustering, especially for longer returns. The autocorrelation in squared returns was studied by, for example, Liesenfeld and Jung (2000), Loudon, et al. (2000), Engle and Patton (2001) and Carnero et al. (2004). The squared returns exhibit positive autocorrelation, indicating a tendency to mean-revert. The first-order autocorrelation coefficients of 1-, 30-, 100- and 300-day returns are, respectively, 0.1174, 0.9256, 0.9749 and 0.9900. Such a mean-reverting process for squared returns implies an autoregressive process for conditional variances, for example, a GARCH process. As shown in Table 1, the correlation coefficients between mean returns and historical volatility based on rolling standard deviations of returns of various horizons are −0.3121, −0.7734, −0.7953 and −0.3592. Black (1976), Christie (1982), Nelson (1991), Glosten et al. (1993) and Engle and Ng (1993) all find evidence of volatility being negatively related to equity returns. The asymmetric structure of volatility generates skewed distributions of forecast prices and this gives a skewed option implied volatility surface across moneyness. This study attempts to propose a volatility model that is able to capture and reflect these stylized facts.

TABLE 1.

S&P 500 Index Returns Summary Statistics

Sample period: January 1969—December 2004

Horizon	1 day	30 days	100 days	300 days
M	0.0003	0.0081	0.0268	0.0836
	(0.0756)	(0.0680)	(0.0675)	(0.0702)
Mdn	0.0003	0.0114	0.0304	0.1095
$Std.Err.(\%)$	1.0079	5.4081	9.8279	17.8398
	(1.6000)	(1.5674)	(1.5601)	(1.6350)
$Skewness$	−1.3397	−0.7694	−0.4191	−0.6085
$Kurtosis$	36.0157	5.8028	3.7669	2.9946
$Corr(R_t, \sigma_t)$	−0.3121	−0.7734	−0.7953	−0.3592

The values in the parentheses within M and $Std.Err.(\%)$ categories correspond to annualized means and volatilities, respectively. The log-difference of the value of the index is taken so as to convert the data into continuously compounded returns, R_t. σ_t is historical volatility based on rolling standard deviations of returns.

3. DATA GENERATING DYNAMICS

In contrast to ARCH and GARCH,[7] the stochastic volatility model assumes that the volatility itself is a random variable. Since the volatility is non-traded and assets perfectly correlated with volatility do not exist in the market,[8] a mechanism of volatility filtering is required. Nelson (1988)

[7]Since ARCH (Engle, 1982) and GARCH (Bollerslev, 1986) assume only one uncertainty source from underlying returns, the volatilities themselves do not have a variation coefficient to additionally capture excess kurtosis. Ghysels et al. (1996) provide detailed comparisons between ARCH/GARCH and stochastic volatility models.

[8]CBOE's VIX and VXN are the weighted implied volatilities of short-term S&P 500 index options and Nasdaq-100 index options, respectively, which are not the quotes with respect to the volatilities themselves.

FIG. 1. The S&P 500 index and its returns of various horizons, January 1969 to December 2004.

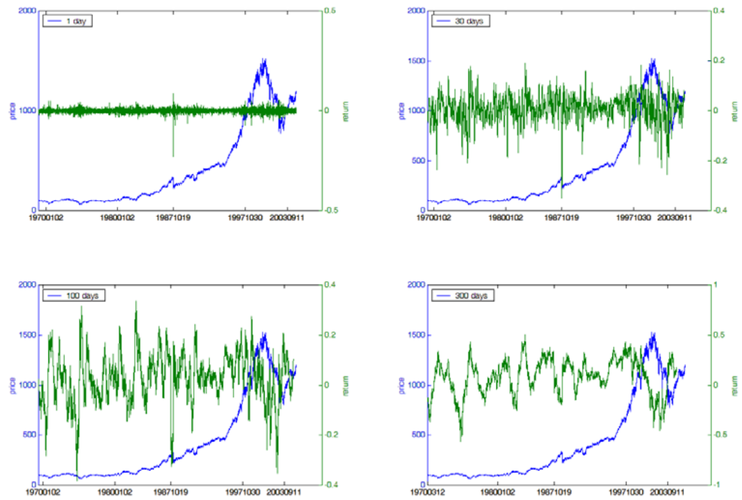
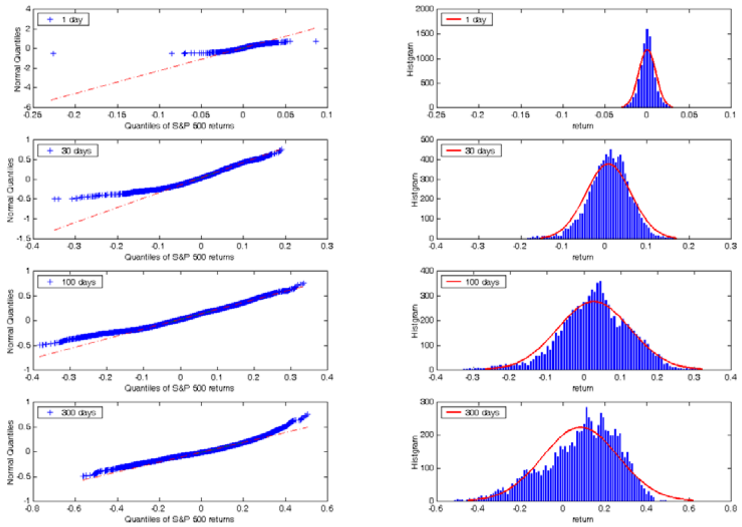
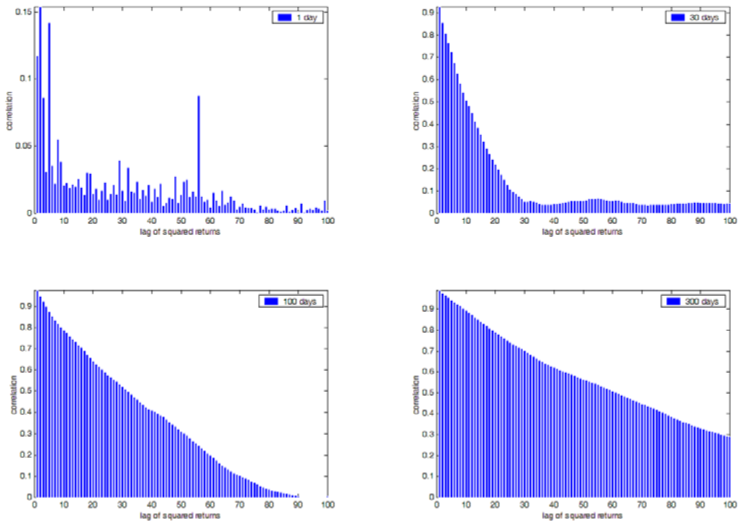


FIG. 2. Quantile plot and histogram with superimposed normal density of S&P 500 index returns.



The quantile plot has the sample data displayed with the plot symbol '+'. This is designed to be a straight line if returns are normally distributed with sample mean and sample variance, and will have an s-shape if there are more extremes.

FIG. 3. Correlogram of squared returns.



The squared returns exhibit positive autocorrelation, indicating a tendency to mean-revert. The first-order autocorrelation coefficients of 1-, 30-, 100- and 300-day returns are, respectively, 0.1174, 0.9256, 0.9749 and 0.9900. Such a mean-reverting process for squared returns implies an autoregressive process for conditional variances, for example, a GARCH process. The clustering of volatility can be concisely shown by significant autocorrelations in squared returns.

and Harvey et al. (1994) suggest a Kalman filter that nonlinearly transforms the square of returns and volatility to linear state variables. Due to non-linearity of transformation, however, it is possible to obtain an inefficient filtered volatility. Kitagawa (1987) and Watanabe (1993) instead use numerical multi-integration to calculate a filtered volatility, which may be computation expensive. Nelson (1992) and Nelson and Foster (1994) suggest an unspecified ARCH model to filter volatility. The unspecified model, however, does not correspond to an existed model in the ARCH and GARCH literature. In light of unaccomplished volatility filtration theory, this study simply adopts Kalman filter to estimate volatility. The spirit of using Kalman filter to daily update volatility is analogue to the implied diffusion theory (see Derman and Kani's (1994) implied volatility tree, Rubinstein's (1994) implied binomial tree, Dupire (1994) and Dumas et al.'s (1998) implied volatility function), which allows for time-varying local volatilities. Numerous estimation strategies have been proposed in the literature for dealing with the discrete-time stochastic-volatility models. Important contributions include the quasi-maximum likelihood (QML) estimator for the discrete-time stochastic volatility model in Ruiz (1994), Harvey et al. (1994) and Harvey and Shephard (1996); the Bayesian Markov Chain Monte Carlo (MCMC) methods advanced by Jacquier et al. (1994), Eraker (2001), and Kim et al. (1998); the simulated methods of moments approach in Duffie and Singleton (1993); the indirect inference procedure of Gourieroux et al. (1993); the efficient methods of moments (EMM) developed by Gallant and Tauchen (1996) and Gallant and Long (1997); the infinitesimal moment generator underlying the GMM procedure in Hansen (1982), Hansen and Scheinkman (1995) and Conley et al. (1997); the approximation method to the likelihood function building on the Kolmogorov forward equations in Lo (1988) and Aït-Sahalia (2002); and the spectral GMM estimator utilizing the empirical characteristic function in Chacko and Viceira (2003), Jiang and Knight (2002), and Singleton (2001). While all of these procedures yield consistent, and in many cases also asymptotically efficient, parameter estimates for the various model specifications, most of them are computationally demanding and cumbersome to implement in practice (Bollerslev and Zhou, 2002). In contrast, QML is relatively simpler and can be easily accompanied with Kalman filter. In the present paper we adopt QML for our estimation procedure. The basic idea is straight forward. Instead of integrating out the squared returns, the strategy proposed here utilizes Kalman filter for explicitly measuring the latent volatility.

3.1. Stochastic Volatility

Hull and White (1987) build their option model on a stochastic volatility process uncorrelated with underlying returns, equivalently, excluding the possibility of the volatility risk premium, and thus restrict the model ability to describe the asset distribution. Heston (1993) relaxes the assumption of zero correlation and allows for the existence of systematic volatility risk premiums. He assumes that instantaneous volatility v_t follows a mean-reverting square root volatility process, which has been widely used in option pricing and continuous-time finance. The time-t logarithmic index price $\ln S_t$ follows,

$$
\begin{aligned}
d\ln S_t &= \left(\mu - \frac{1}{2}v_t\right)dt + \sqrt{v_t}d\omega_{S,t} \\
&= \left(r + \eta_S v_t - \frac{1}{2}v_t\right)dt + \sqrt{v_t}d\omega_{S,t} \\
dv_t &= \kappa_\gamma(\theta_\gamma - v_t)dt + \sigma_\gamma\sqrt{v_t}d\omega_{\gamma,t} \\
&= [\kappa^*_\gamma(\theta^*_\gamma - v_t) + \eta_\gamma v_t]dt + \sigma_\gamma\sqrt{v_t}d\omega_{\gamma,t} \\
& \quad corr(d\omega_{S,t}, d\omega_{\gamma,t}) = \rho dt
\end{aligned}
\tag{1}
$$

where μ is the instantaneous expected rate of percentage changes in index prices, attributed by the risk-free interest rate, r, and the price premium, $\eta_S v_t$. The volatility process takes into account the pricing of volatility risk, $\eta_v v_t$,[9] which additionally contributes to the total expected rate of index returns by the magnitude of $\sqrt{v_t}\omega_{S,t}$ via its correlation with the return shock, ρ. $\omega_{S,t}$ and $\omega_{v,t}$ are standard Wiener processes; $\kappa_\gamma(\kappa_\gamma^*), \theta_\gamma(\theta_\gamma^*)$ and σ_γ are adjusted (risk-neutral) speed, long-run mean and variation coefficient of the volatility process, respectively. In light of this, this study follows Nelson's (1990) and Taylor and Xu's (1993) volatility approximation approaches to propose a discrete-time stochastic volatility process (as shown in equations (2) and (3)) whose continuous-time limit is Heston's (1993) volatility process. We also adopt the methodology provided by Harvey and Shephard (1996) to deal with the correlation between underlying returns and stochastic volatilities.

$$\begin{aligned} R_t &= \mu_S - \frac{1}{2}\mu_{h,t|t-1} + \sqrt{h_t}(x_t + \lambda) && (2) \\ h_t &= \varpi + \beta h_{t-1} + \alpha\sqrt{h_{t-1}}\varepsilon_t && (3) \\ x_t &\sim ID(0,1), \varepsilon_t \sim ID(0,1), corr(x_t, \varepsilon_t) = \rho, \end{aligned}$$

where the log-difference of the value of the index is taken so as to convert the data into continuously compounded returns, R_t. μ_S is the expected percentage rate of return compensated for bearing price risk; h_t is time-t variance of index returns. Given the information set available at time $t-1$, i.e. $I_{t-1}, \mu_{h,t|t-1} = E(h_t|I_{t-1})$ is the conditional expectation of time-t variance h_t. Similarly, $\mu_{h,t-1|t-1|} = E(h_{t-1}|I_{t-1})$ is the expectation of variance h_{t-1} conditional on the information set I_{t-1}. Since the unconditional expectation of variance is computed by $E(h_t) = \omega/(1-\beta)$, one of necessary conditions to guarantee a stationary volatility process is $\beta < 1$. The variation coefficient of the variance process is α contributable to kurtosis of the return distribution. The return innovation is expressed by $x_t \sim ID(0,1)$, whereas the volatility shock is denoted by $\varepsilon_t \sim ID(0,1)$ correlated with contemporaneous x_t by a correlation coefficient ρ. The phenomenon of volatility being negatively related to equity returns is associated with correlation between the shock to returns and the shock to volatility, which is partly responsible for the distribution skewness. By construction the total expected return of R_t comprises the price risk premium, $\mu_S - \mu_{h,t|t-1}/2$, and the volatility risk premium, $E[\sqrt{h_t}(x_t + \lambda)|I_{t-1}]$. Thus, the influence of volatility changes on the R_t is observed through the correlation between

[9]Strict linearity of the volatility risk premium can be supported under log utility when index volatility and market risk have a common component of a particular form. Bates (2000) and Pan (2002) use a similar approximation when modeling the risk premium on equities.

return shocks and volatility shocks and the market price of volatility risk, λ. The closed form for the conditional expectation of $\sqrt{h_t}(x_t+\lambda)$ is expressed by[10]

$$\begin{aligned} E[\sqrt{h_t}(x_t+\lambda)|I_{t-1}] &= \frac{7}{16}\lambda\sqrt{\mu_{h,t|t-1}} + \frac{11}{16}\alpha\rho\frac{\sqrt{\mu_{h,t-1|t-1}}}{\sqrt{\mu_{h,t|t-1}}} \\ &\quad -\frac{1}{8}\lambda\beta^2\frac{\mu^2_{h,t-1|t-1}}{\sqrt{\mu^3_{h,t|t-1}}}. \end{aligned} \tag{4}$$

This study uses Kalman filter to construct the conditional expectation of volatility, i.e., $\mu_{h,t|t-1}$, and derives the QML function for parameter estimation. Related steps are presented as follows.

Step 1: transform the underlying process denoted in equations (2) and (3) into a linear combination of state-space variables.

Re-arrange equation (2) and take squares and logarithms of both sides,

$$y_t = \ln(h_t) + \ln(x_t+\lambda)^2, \tag{5}$$

where $y_t = \ln\left[\left(R_t - \mu_S + \frac{1}{2}\mu_{h,t|t-1}\right)^2\right]$. The terms, $\ln(h_t)$ and $\sqrt{h_{t-1}}$, in equations (5) and (3), respectively, are still nonlinear. Thus, using Taylor's series expanded at their conditional means $\mu_{h,t|t-1}$ and $\mu_{h,t-1|t-1}$, we obtain equations (6) and (7).

$$\ln(h_t) \cong \left(\frac{1}{\mu_{h,t|t-1}}\right)h_t + \left[\ln(\mu_{h,t|t-1}) - 1\right] \tag{6}$$

$$\begin{aligned} \sqrt{h_{t-1}} &\cong \sqrt{\mu_{h,t-1|t-1}} + \frac{1}{2\sqrt{\mu_{h,t-1|t-1}}}(h_{t-1}-\mu_{h,t-1|t-1}) \\ &\cong \sqrt{\mu_{h,t-1|t-1}}. \end{aligned} \tag{7}$$

Since h_{t-1} is known at time $t-1$, the term, $\frac{1}{2\sqrt{\mu_{h,t-1|t-1}}}(h_{t-1}-\mu_{h,t-1|t-1})$, in Taylor's series expansion will diminish and thus $\sqrt{h_{t-1}} \cong \sqrt{\mu_{h,t-1|t-1}}$.

[10]By applying the Taylor's series to the function $\sqrt{h_t}$ expanded at $h_t = \mu_{h,t|t-1}$, we can obtain the following equation:

$$\begin{aligned} \sqrt{h_t} &\cong \sqrt{\mu_{h,t|t-1}} + \frac{1}{2}\mu^{-1/2}_{h,t|t-1}(h_t-\mu_{h,t|t-1}) - \frac{1}{8}\mu^{-3/2}_{h,t|t-1}(h_t-\mu_{h,t|t-1})^2 \\ &\quad + \frac{1}{16}\mu^{-5/2}_{h,t|t-1}(h_t-\mu_{h,t|t-1})^3 + \cdots. \end{aligned}$$

The approximation of the volatility risk premium, denoted by the conditional expectation of $\sqrt{h_t}(x_t+\lambda)$, becomes $E[\sqrt{h_t}(x_t+\lambda)|I_{t-1}] = \frac{7}{16}\lambda\sqrt{\mu_{h,t|t-1}} + \frac{11}{16}\alpha\rho\frac{\sqrt{\mu_{h,t-1|t-1}}}{\sqrt{\mu_{h,t|t-1}}} - \frac{1}{8}\lambda\beta^2\frac{\mu^2_{h,t-1|t-1}}{\sqrt{\mu^3_{h,t|t-1}}}$.

Re-arrange equations (5)?(7), we can obtain a linear combination of state-space variables as shown in the following equations.

$$y_t = [c_0 + \ln(\mu_{h,t|t-1}) - 1] + \left(\frac{1}{\mu_{h,t|t-1}}\right) h_t + e_t \tag{8}$$

$$h_t \cong \varpi + \beta h_{t-1} + \alpha\sqrt{\mu_{h,t-1|t-1}}\varepsilon_t, \tag{9}$$

where $c_0 = E(v_t)$; $v_t = \ln(x_t + \lambda)^2$; $e_t = v_t - E(v_t) \sim ID(0, \sigma_e^2)$; $\varepsilon_t \sim ID(0,1)$. Note that if the joint probability density function of ε_t and x_t is symmetric, the residuals e_t and ε_t are uncorrelated. Under this circumstance the estimation based on equations (8) and (9) is still feasible, but the correlation, ρ, between ε_t and x_t will be omitted from the squares of observed data. In order to recover the correlation information, or equivalently, to deal with the volatility asymmetry, this study follows Harvey and Shephard's (1996) approach to adjust equations (8) and (9).

Step 2: Recover the correlation information between ε_t and x_t, i.e., ρ.

Harvey and Shepard (1993, 1996) suggest that the missing correlation information can be recovered via the usage of a sign symbol, s_t, corresponding to the residual x_t. Define the value of s_t is $+1(-1)$ if the residual $x_t > 0 (< 0)$. Let $E_+(E_-)$, $var_+(var_-)$ and $cov_+(cov_-)$ denote the expectation, variance and covariance functions, conditional on the value of s_t. Thus, the equation (9) can be rewritten as equation (10),

$$h_t \cong \varpi + \beta h_{t-1} + s_t\mu^* + \eta_t^*, \tag{10}$$

where $\eta_t^* = \alpha\sqrt{\mu_{h,t-1|t-1}}\varepsilon_t - \mu^* s_t$

$$\mu^* s_t = E_+\left(\alpha\sqrt{\mu_{h,t-1|t-1}}\varepsilon_t\right) = \alpha\sqrt{\mu_{h,t-1|t-1}}E(\varepsilon_t|s_t) \tag{11}$$

The residual η_t^* of equation (10) is correlated with the residual e_t of equation (8) in a specific structure shown as in equation (12),

$$\left(\begin{array}{c} e_t \\ \eta_t^* \end{array}\right)\Big| s_t \sim ID\left(\left(\begin{array}{c} 0 \\ 0 \end{array}\right), \left(\begin{array}{cc} \sigma_e^2 & r^* s_t \\ r^* s_t & \alpha^2\mu_{h,t-1|t-1} - \mu^{*2} \end{array}\right)\right), \tag{12}$$

where $r^* s_t = cov_+(\alpha\sqrt{\mu_{h,t-1|t-1}}\varepsilon_t, e_t) = cov_+(\alpha\sqrt{\mu_{h,t-1|t-1}}\varepsilon_t - \mu^* s_t, e_t) = cov_+(\eta_t^*, e_t)$.

The equations of (8), (10) and (12) are our empirical models under stochastic volatility framework. By using Kalman filtration to filter the conditional volatility, we can construct the QML function to estimate parameters, $\Xi = \{\mu_S, \varpi, \beta, \alpha, \sigma_e^2, c_0, \mu^*, r^*\}$. Given these estimates of parameters, $\hat{\Xi}$, and the properties of x_t and ε_t in equation (2), we construct a form for ε_t to recover the information of ρ. Define $\varepsilon_t = \rho(x_t^* - \lambda) +$

$\sqrt{\frac{1-\rho^2}{Var|x_t^*|}}[|x_t^*| - E|x_t^*|]$ where $x_t^* = x_t + \lambda \sim ID(\lambda, 1)$. Thus, we have $E(\varepsilon_t) = 0$, $Var(\varepsilon_t) = 1$ and $corr(x_t, \varepsilon_t) = \rho$. The expectation of ε_t conditional on s_t becomes $s_t\rho\lambda$ derived as follows,

$$\begin{aligned} E(\varepsilon_t|s_t, I_{t-1}; \hat{\Xi}) &= E[\rho(x_t^* - \lambda)|s_t, I_{t-1}; \hat{\Xi}] \\ &\quad + \sqrt{\frac{1-\rho^2}{Var|x_t^*|}}[E(|x_t^*||s_t, I_{t-1}; \hat{\Xi}) - E|x_t^*|] \\ &= s_t\rho E[x_t^*|s_t, I_{t-1}; \hat{\Xi}] = s_t\rho\lambda \\ &= s_t\rho \int_{-\infty}^{\infty} x_t^* f(x_t^*|s_t, I_{t-1}; \hat{\Xi}) dx_t^* - s_t\rho\lambda \\ &= s_t\rho \int_{-\infty}^{\infty} x_t^* \frac{f(x_t^*|I_{t-1}; \hat{\Xi})}{f(s_t|I_{t-1}; \hat{\Xi})} dx_t^* - s_t\rho\lambda \\ &= 2s_t\rho \int_{-\infty}^{\infty} x_t^* f(x_t^*|I_{t-1}; \hat{\Xi}) dx_t^* - s_t\rho\lambda \\ &= 2s_t\rho E(x_t^*|I_{t-1}; \hat{\Xi}) - s_t\rho\lambda \\ &= 2s_t\rho\lambda - s_t\rho\lambda = s_t\rho\lambda \end{aligned} \tag{13}$$

(Supposed that x_t is symmetric around zero, the conditional probability density function of s_t becomes $f(s_t|I_{t-1}; \hat{\Xi}) = f(s_t) = \frac{1}{2}$.)

Combine equations (11) with (13), the correlation ρ between the return and volatility innovations can be recovered as

$$\rho = \frac{\mu^*}{\lambda\alpha\sqrt{\mu_{h,t-1|t-1}}}. \tag{14}$$

Step 3: Kalman filtration of stochastic volatility.

Kalman filtration uses the linear projection to construct the conditional mean of variance, $\mu_{h,t+1|t}$, and conditional prediction error, $P_{t+1|t}$, shown as follows,

$$\begin{aligned} \mu_{h,t+1|t} &= \varpi + \beta\mu_{h,t|t-1} + \mu^* s_{t+1} \\ &\quad + \beta\left(\frac{P_{t|t-1}}{\mu_{h,t|t-1}} + r^* s_t\right)\left(\frac{P_{t|t-1}}{\mu_{h,t|t-1}^2} + \sigma_e^2 + \frac{2r^* s_t}{\mu_{h,t|t-1}}\right)^{-1}(y_t - y_{t|t-1}) \end{aligned} \tag{15}$$

$$
\begin{aligned}
P_{t+1|t} &= \beta^2 P_{t|t-1} - {\mu^*}^2 + \alpha^2 \mu_{h,t|t-1} \\
&+ \left(\frac{P_{t|t-1}}{\mu_{h,t|t-1}} + r^* s_t\right)\left(\frac{P_{t|t-1}}{\mu^2_{h,t|t-1}} + \sigma_e^2 + \frac{2r^* s_t}{\mu_{h,t|t-1}}\right)^{-1} \\
&\times \left[\alpha^2 (y_t - y_{t|t-1}) - \beta^2 \left(\frac{P_{t|t-1}}{\mu_{h,t|t-1}} + r^* s_t\right)\right. \\
&\left. + 2\beta^2 \left(\frac{\omega}{1-\beta} - \mu_{h,t|t-1}\right)^2 \mu^{-1}_{h,t|t-1}\right]
\end{aligned} \tag{16}
$$

Equation (15) will also generate a term structure of volatility, $\mu_{h,t+k|t}$, which summarizes all the forecasting properties of second moments of returns of various maturities, all starting at date t. Given the observations of $\{R_t\}_{t=1}^T$ and initial values of $\mu_{h,1|0}$ and $P_{1|0}$, we can use equations (15) and (16) to estimate the conditional mean of variance and prediction error in the next period, $\mu_{h,2|1}$ and $P_{2|1}$. By iteration, $\mu_{h,t|t-1}$ and $P_{t|t-1}$ with $t = 3, 4, \ldots, T$ can be computed. The Kalman filter is initialized using the unconditional mean of variance, $E(h_t) = \varpi/(1-\beta)$, as $\mu_{h,1|0}$. Similarly, the conditional variance of variance $P_{1|0}$ adopts the unconditional variance of variance, i.e. $var(h_t) = \alpha^2\varpi/(1-\beta)(1-\beta^2)$, as its starting value.

The predicted value $y_{t+1|t}$ and the prediction error $E(y_{t+1} - y_{t+1|t})^2$ of y_{t+1} are expressed, respectively, by equations (17) and (18).

$$
y_{t+1|t} = c_0 + \ln(\mu_{h,t+1|t}) \tag{17}
$$

$$
E(y_{t+1} - y_{t+1|t})^2 = \frac{P_{t+1|t}}{\mu^2_{h,t+1|t}} + \sigma_e^2 + \frac{2r^* s_{t+1}}{\mu_{h,t+1|t}}. \tag{18}
$$

Given the assumption that the residuals $\{e_t, \eta_t^*\}_{t=1}^T$ follow a Gaussian distribution, the logarithmic QML function for $\{y_t\}_{t=1}^T$ is expressed in equation (19). By maximizing this logarithmic QML function, we can obtain the parameter estimates, $\hat{\Xi} = \{\hat{\mu_S}, \hat{\varpi}, \hat{\beta}, \hat{\alpha}, \hat{\sigma}_e^2, \hat{c}_0, \hat{\mu}^*, \hat{r}^*\}$.

$$
\begin{aligned}
\max_{\Xi=\{\mu_S,\varpi,\beta,\alpha,\sigma_e^2,c_0,\mu^*,\gamma^*\}} L &= \frac{-T}{2}\ln(2\pi) - \frac{1}{2}\sum_{t=1}^{T} \ln\left[E(y_t - y_{t|t-1})^2\right] \\
&- \frac{\sum_{t=1}^T (y_t - y_{t|t-1})^2}{2E(y_t - y_{t|t-1})^2}.
\end{aligned} \tag{19}
$$

Given the estimates of parameters, the residual $\sqrt{h_t}(x_t+\lambda)$ of equation (2) can be calculated as follows,

$$R_t^{res} = R_t - \hat{\mu}_S + \frac{1}{2}\hat{\mu}_{h,t|t-1} \cong \sqrt{h_t}(x_t+\lambda), \tag{20}$$

where R_t^{res} is the residual return containing the information of volatility risk premiums.

3.2. Constant Volatility

The benchmark model for option pricing is the classic Black-Merton-Scholes (BMS) model published in 1973 which was awarded the Nobel Prize in Economics in 1997.[11] The BMS model assumes volatility is constant, and its discrete-time process can be approximated by,

$$R_t = \mu_S - \frac{1}{2}\sigma^2 + \sigma x_t \tag{21}$$

where σ is standard deviation of index returns; x_t is the residual term following a normal distribution with mean 0 and variance 1. Given a set of observations $\{R_t\}_{t=1}^T$, the logarithmic maximum likelihood function becomes,

$$\max_{\Xi=\{\mu_S,\sigma\}} L = \frac{-T}{2}\ln(2\pi) - T\ln(\sigma) - \frac{\sum_{t=1}^T (R_t - \mu_S + \sigma^2/2)^2}{2\sigma^2}. \tag{22}$$

By maximizing the logarithmic maximum likelihood function, we can obtain the parameter estimates, $\hat{\Xi} = \{\hat{\mu}_S, \hat{\sigma}\}$, and further calculate the residual σx_t according to equation (21).

$$R_t^{res} = R_t - \hat{\mu}_S + \frac{1}{2}\hat{\sigma}^2 \cong \sigma x_t. \tag{23}$$

Since $\{R_t\}_{t=1}^T$ are in accordance with stylized facts of observed leptokurtosis, the calculated R_t^{res} thus contains the information about the market price of volatility risk.

[11] Although the BMS model assumes volatility is constant, its creators knew that volatility is itself volatile — an observation underlined by the Nobel Prize awarded in 2003 to professor Engle for his pioneering work (1982) on modeling volatility dynamics.

4. THE PRICE AND VOLATILITY RISK PREMIUMS

4.1. The Market Price of Volatility Risk under Stochastic Volatility

The volatility risk premium, on one hand, can be expressed as the conditional expectation of the residual term in the return process (equation (4)). On the other hand, the residual return R_t^{res} in equation (20) contains the information about the volatility risk premium. Thus, we can express the relationship between the volatility risk premium and residual returns by,

$$R_t^{res} = \frac{11}{16}\alpha\rho\frac{\sqrt{\mu_{h,t-1|t-1}}}{\sqrt{\mu_{h,t|t-1}}} + \lambda\left(\frac{7}{16}\sqrt{\mu_{h,t|t-1}} - \frac{1}{8}\beta^2\frac{\mu^2_{h,t-1|t-1}}{\sqrt{\mu^3_{h,t|t-1}}}\right) + \xi_t, \quad (24)$$

where λ is the market price of volatility risk, and the residual term is expressed by

$$\xi_t = \sqrt{h_{h,t|t-1}}(x_t+\lambda) - E[\sqrt{h_{h,t|t-1}}(x_t+\lambda)|I_{t-1}] \sim ID(0,1).$$

Given the estimates of parameters $\hat{\Xi} = \{\hat{\mu}_S, \hat{\varpi}, \hat{\beta}, \hat{\alpha}, \hat{\sigma}_e^2, \hat{c}_0, \hat{\mu}^*, \hat{r}^*\}$, we can estimate λ by minimizing the difference between R_t^{res} and its analytical formula for the volatility risk premium,

$$\min_{\lambda}\sum_{t=1}^{T}\left[R_t - \hat{\mu}_S + \frac{1}{2}\hat{\mu}_{h,t|t-1} - \frac{11}{16}\frac{1}{\lambda}\frac{\hat{\mu}^*}{\sqrt{\hat{\mu}_{h,t|t-1}}} - \lambda\left(\frac{7}{16}\sqrt{\hat{\mu}_{h,t|t-1}} - \frac{1}{8}\hat{\beta}^2\frac{\hat{\mu}^2_{h,t-1|t-1}}{\sqrt{\hat{\mu}^3_{h,t|t-1}}}\right)\right]^2 \quad (25)$$

Given $\hat{\lambda}$ and $\hat{\Xi} = \{\hat{\mu}_S, \hat{\varpi}, \hat{\beta}, \hat{\alpha}, \hat{\sigma}_e^2, \hat{c}_0, \hat{\mu}^*, \hat{r}^*\}$, the estimates of price and volatility risk premiums are respectively obtained through $\hat{\mu}_S - \frac{1}{2}\hat{\mu}_{h,t|t-1}$ and $\frac{7}{16}\hat{\lambda}\sqrt{\hat{\mu}_{h,t|t-1}} + \frac{11}{16}\frac{1}{\hat{\lambda}}\frac{\hat{\mu}^*}{\sqrt{\hat{\mu}_{h,t|t-1}}} - \frac{1}{8}\hat{\lambda}\hat{\beta}^2\frac{\hat{\mu}^2_{h,t-1|t-1}}{\sqrt{\hat{\mu}^3_{h,t|t-1}}}$. The information of ρ can be also recovered from $\hat{\rho} = \frac{\hat{\mu}^*}{\hat{\lambda}\hat{\alpha}\sqrt{\hat{\mu}_{h,t-1|t-1}}}$.

4.2. The Market Price of Volatility Risk under Constant Volatility

The constant-volatility model itself as shown in equation (21) cannot capture the volatility risk premium.[12] However, the residual return R_t^{res} in

[12]This is due to the assumption of $E(\sigma x_t|I_{t-1}) = 0$.

equation (23), as a function of realized returns, $\{R_t\}_{t=1}^T$, contains the information of the volatility risk premium. Therefore, we express the market price of volatility risk, λ, under the constant-volatility assumption, by

$$R_t^{res} = \lambda\sigma + \xi_t, \tag{26}$$

where $\xi_t \sim ID(0,1)$.

Similarly, by minimizing the difference between the residual return R_t^{res} and its theoretical volatility risk premium, the parameter estimate $\hat{\lambda}$ can be obtained via the following equation.

$$\min_{\hat{\lambda}} \sum_{t=1}^{T} \left[R_t - \hat{\mu}_S + \frac{1}{2}\hat{\sigma}^2 - \lambda\hat{\sigma} \right]^2. \tag{27}$$

Given the values of $\hat{\lambda}$ and $\hat{\Xi} = \{\hat{\mu}_S, \hat{\sigma}\}$, the estimates of price and volatility risk premiums are, respectively, calculated by $\hat{\mu}_S - \frac{1}{2}\hat{\sigma}^2$ and $\hat{\lambda}\hat{\sigma}$ under the constant-volatility assumption.

5. ESTIMATION RESULTS

This section presents empirical results from the QML estimation procedures outlined in the previous sections. The estimation results for individual models are presented first and then characteristics of residual returns in terms of the information of the market prices of price and volatility risk are diagnosed later.

5.1. Estimation Results

QML estimates of parameters with asymptotic t statistics are reported in Table 2 for various models for data covering alternative frequencies. The long-term means of the volatility process ($\sqrt{\hat{\varpi}/(1-\hat{\beta})}$) appear time-varying and seem to revert their medians of various frequencies (0.64, 0.69, 0.66 and 0.71). The QML estimates of the autoregressive parameter $\hat{\beta}$ with significant asymptotic t-statistics imply persistence of the volatility. Given the dynamics of the volatility process in equation (3), the half-life is determined by finding the date t_s, for which

$$E(h_{t_s}|h_t) = \frac{1}{2}\left(h_t + \frac{\varpi}{1-\beta}\right), \quad t_s > t. \tag{28}$$

Thus, it is necessary to know the process driving $h_{t+\Delta t}$ in the future in order to obtain the value of t_s. The estimate for the expected future spot

TABLE 2.

Parameter Estimates of Constant- and Stochastic-Volatility Models

	Constant Volatility				Stochastic Volatility			
Parameters	1 day	30 days	100 days	300 days	1 day	30 days	100 days	300 days
μ	0.0706	0.0658	0.0871	0.1155	0.0383	0.0472^{**}	0.0404^{**}	0.0434^{**}
	(0.47)	(0.46)	(0.55)	(0.66)	(1.09)	(2.94)	(14.83)	(22.23)
σ	0.3987	0.3950	0.3882	0.3772				
	(0.99)	(0.94)	(0.85)	(0.45)				
ϖ					0.0728^{*}	0.0764^{**}	0.0781^{**}	0.0804^{**}
					(2.23)	(9.59)	(30.34)	(311.25)
β					0.8214^{**}	0.8388^{**}	0.8239^{**}	0.8388^{**}
					(50.12)	(2983.73)	(167.11)	(427.72)
α					0.0397^{**}	0.0014	0.0014	0.0014^{**}
					(21.54)	(0.56)	(1.37)	(28.69)
σ_e^2					0.1474	0.1468^{**}	0.1468^{**}	0.1468^{**}
					(1.18)	(15.16)	(14.10)	(3.16)
c_0					-2.8628^{**}	-3.0723^{**}	-3.9200^{**}	-3.0702^{**}
					(−7.65)	(−81.93)	(−3.44)	(−6.53)
μ^*					−0.0056	−0.0005	−0.0006	-0.0006^{**}
					(−0.30)	(−0.08)	(−0.93)	(−15.24)
r^*					−0.0000	−0.0001	-0.0000^{**}	-0.0000^{**}
					(−0.00)	(−0.33)	(−4.83)	(−3.96)
Q_{BL}	2.6749	16765.1266^{**}	92795.4964^{**}	119310.1977^{**}	140.8119^{**}	16460.8644^{**}	92639.1171^{**}	119245.2475^{**}
$corr(R_t^{res}, \mu_{h,t\|t-1})$					0.3375	−0.8011	−0.8767	−0.9061
$corr(R_t^{res}, R_t)$	1.0000	1.0000	1.0000	1.0000	0.9435	0.9999	1.0000	1.0000

QML estimates with asymptotic t-statistics in the parentheses are reported. The symbol of * (**) indicates that given a 5% (1%) significance level, the t test statistic rejects the null hypothesis of a zero parameter. Q_{BL} presents the Box-Ljung statistics for residual returns, based on 20 lags, and is distributed χ^2 under the null hypothesis of identical and independent observations. The symbol of * (**) along with Q_{BL} indicates that given a 5% (1%) significance level, the Q_{BL} statistic show an indication of residual serial correlation. The residual return R_t^{res} is the realized return R_t after subtracting its expectation, i.e., the premium for price risk. By contruction, the residual return under constant volatility is $R_t^{res} = \sigma x_t \cong R_t - (\hat{\mu}_S - \hat{\sigma}^2/2)$, whereas the one under stochastic volatility is expressed by $R_t^{res} = \sqrt{h_t} x_t \cong R_t - (\hat{\mu}_S - \hat{\mu}_{h,t|t-1}/2)$ with $\hat{\mu}_{h,t|t-1}$ being the Kalman-filtered volatility. $corr(\cdot,\cdot)$ represents the correlation coefficient between variables. All figures presented in this table are the averages within the 180 one-month subperiods starting from January 1969 and ending in January 1990—December 2004.

volatility is given by

$$E(h_{t_s}|h_t) = h_t e^{-(1-\beta)(t_s-t)} + \frac{\varpi}{(1-\beta)}[1 - e^{-(1-\beta)(t_s-t)}]. \qquad (29)$$

According to equations (28) and (29),

$$t_s - t = \frac{\ln 2}{1-\beta} \times \tau_i, \qquad (30)$$

where $\tau_i = 1, 30, 100, 300$ days and $(t_s - t)$ is the number of working days. The persistence parameter of $\hat{\beta}$ gives about 4, 129, 395 or 1290 working days for an arbitrary volatility to revert halfway to its 1-, 30-, 100- or 300-day long-term mean. The volatility of variance, as measured by $\hat{\alpha}$, is possibly responsible for the high kurtosis in short-term raw returns.[13] Given the estimated parameter values, together with the corresponding $\hat{\sigma}$ or $\hat{\mu}_{h,t|t-1}$, we fit our models to the residuals R_t^{res} as shown in equations (20) and (23). The Box-Ljung statistics of the residual returns for 20 lags indicate that there is some linear structure left in the models, except for the 1-day constant-volatility model. This could be removed by fitting a more complex dynamic model to the mean of the process, and experiments along these lines indicate a substantial degree of correlation between R_t^{res} and Kalman filtered $\hat{\mu}_{h,t|t-1}$, which is increased in magnitude across investment horizons by 0.34, -0.80, -0.88 and -0.92 for the 1, 30, 100 and 300 days. Except for the 1-day frequency, the residual returns also have a substantial degree of correlation with raw returns. This partially explains why this study exploits the residual returns to recover the information of the market price of volatility risk. Nevertheless parameter estimates can be obtained by treating $\ln x_t^2$ as though it were $NID(-1.27, \pi^2/2)$ and maximizing the resulting QML function (Ruiz, 1994). A further extraction of applying QML to stochastic-volatility models proposed here is that the assumption of normality for x_t can be relaxed, in which case its mean c_0 and variance σ_e^2 are estimated unrestrictedly. The resultant statistical significance of $\hat{c}_0$ and $\hat{\sigma}_e^2$ implies that the distribution of x_t is nonnormal for the S&P 500 returns. This is important because the kurtosis in many financial series is greater than the kurtosis that results from incorporating stochastic volatility into a Gaussian process (Harvey et al., 1994). The introduction of μ^* and r^* is aimed to recover the lost information of the correlations between the return shock x_t and the volatility shock ε_t due to the usage of squared returns. The signs of the cross-correlations are determined by the corresponding pair of estimated values $(\hat{\mu}^*, \hat{\lambda})$ or $(\hat{r}^*, \hat{\lambda})$ and noting that the sign of each of the pairs $(\hat{\mu}^*, \hat{r}^*)$ is negative.

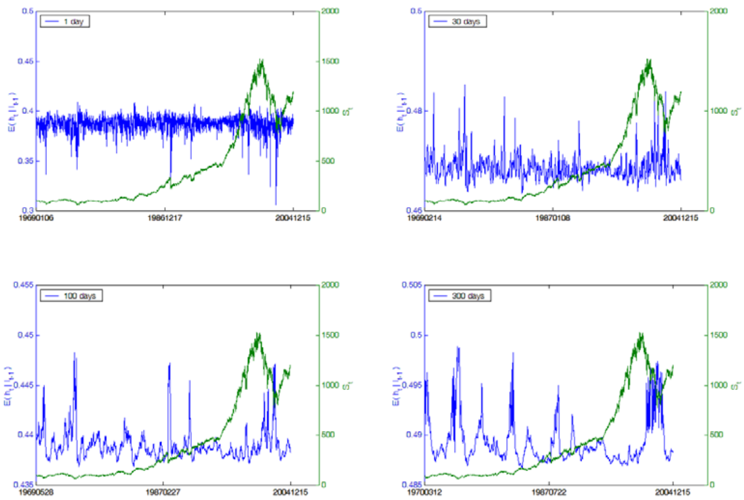
5.2. Filtered Volatilities

During periods of financial turmoil, which are often accompanied by steep market declines, filtered volatilities tend to rise. As the char in Figure 4 illustrates, the filtered volatility mirrored the peaks and troughs as the market suffered through steep declines during the Long Term Capital Management and Russian Debt Crises in 1998. A high level of the filtered volatility seems to have a power in predicting the market's subsequent di-

[13] The annualized volatility of variance of various frequencies (1, 30, 100 and 300 days) is presented by $\hat{\alpha} \times 252/\tau_i = 10.0139$, 0.0117, 0.0035 and 0.0012, respectively.

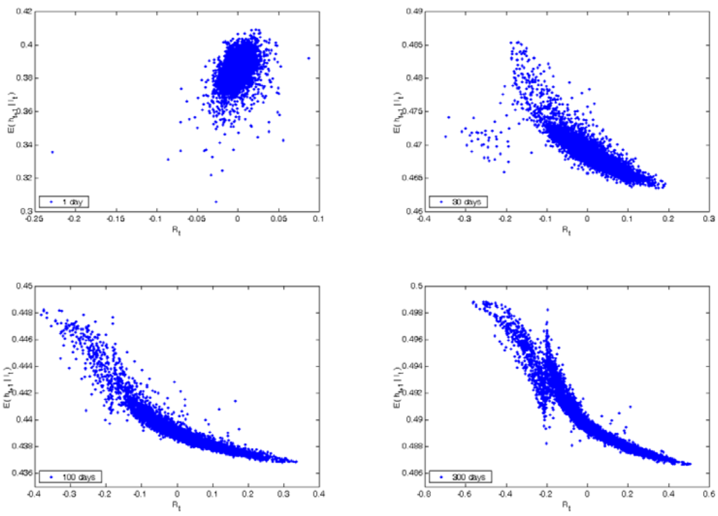
rection. A reversion of the filtered volatility occurs both after upside and downside extremes. Another interesting aspect of filtered volatility is that it tends to move opposite its underlying index, i.e. asymmetric volatility shocks. This tendency is illustrated in Figure 5 comparing future filtered volatilities with current S&P 500 returns. Note that the negatively sloping trend line, except for 1-day frequency, confirms the negative correlation between filtered volatility and market movement. Thus, the filtered volatility paves the way for both listed and over-the-counter volatility contracts at a time of increased market demand for such products. Thus, the filtered volatility could capture the pulse of the market. Extreme filtered volatility readings and reversals often signal quick reversals in the stock market, making it an effective tool for trading strategies. The graph also shows that filtered volatilities are statistically significant leading indicators of market returns.

FIG. 4. Expected conditional variance of S&P 500 index returns of various frequencies.



$E(h_t|I_{t-1})$ in the left Y-axis denotes the estimated expected conditional variance, $\hat{\mu}_{h,t|t-1}$, and S_t in the right Y-axis presents the index level. The filtered volatilities are graphed over the period January 1969—December 2004 with monthly updated parameters. The averages of 1-, 30-, 100- and 300-day filtered variances since 1969 are roughly 38.76%, 46.87%, 44.17% and 49.08%, respectively.

FIG. 5. Future filtered variances and current S&P 500 returns.



$E(h_{t+1}|I_t)$ in the Y-axis denotes the estimated future expected conditional variance, $\hat{\mu}_{h,t+1|t}$. The correlations between current return R_t and future filtered variance $E(h_{t+1}|I_t)$ are given by 0.4188, -0.8418, -0.8871 and -0.9095 for 1-, 30-, 100- and 300-day returns, respectively.

5.3. The Relative Contributions of Price and Volatility Risk Premiums to the Residual Returns

The magnitude of volatility risk premiums depends on the market price, λ, the amplitude of volatility risk, $var(R_t|I_{t-1})$, and the correlation ρ between volatility shocks and return shocks. The estimates of the market price of volatility risk, $\hat{\lambda}$, in Table 3 are in general significantly positive at a given 90% confidence interval, indicating the existence of volatility risk premiums in the residual returns. In particular, the stochastic volatility provides a greater and more significant $\hat{\lambda}$ than the constant volatility, indicating a superior capability to capture volatility risk premiums. As shown in the left axis of Figure 6, a drop in stock price occurs immediately due to the higher expected return required to compensate for the added risk, as an indicative of the existence of time-varying volatility premiums. Since the volatility of the index increases as well, return and volatility are negatively correlated by -0.8089, -0.8785 and -0.9071 for 30, 100 and 300 days to horizon.[14] The observed negative correlation between return innovations and volatility shocks could be often referred to as the volatility feedback effect displayed in Figure 7. The presence of asymmetric volatility

[14]The 1-day return is slightly positively correlated with filtered variance by 0.0066.

TABLE 3.

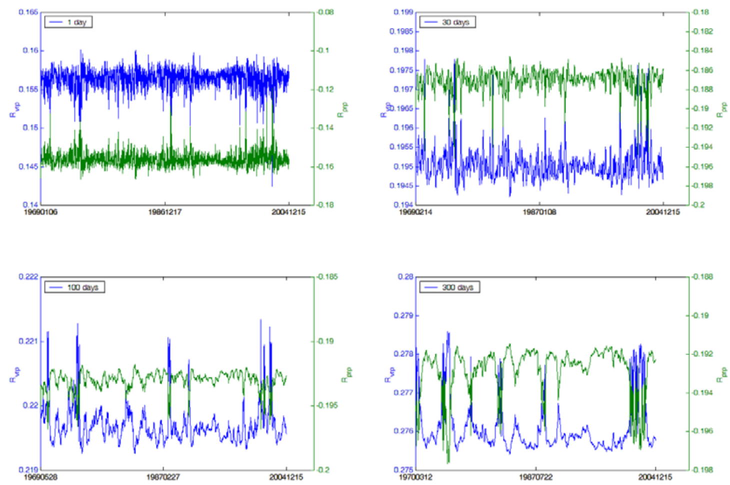
The Market Prices of Price and Volatility Risks

Model	Constant Volatility				Stochastic Volatility			
Horizon	1 day	30 days	100 days	300 days	1 day	30 days	100 days	300 days
$\hat{\lambda}$	0.0230	0.0514**	0.0397*	0.1072**	0.7480**	0.8177**	0.8859**	1.1717**
	(1.09)	(2.42)	(1.85)	(4.76)	(20.61)	(23.47)	(24.76)	(33.70)
$v\hat{a}r(R_t\|I_{t-1})$	0.1590	0.1561	0.1507	0.1422	0.3876	0.4687	0.4417	0.4908
$\hat{\rho}$					−0.3002	−0.7032	−0.7025	−0.4586
R_{vrp}	0.0092	0.0203	0.0154	0.0404	0.1557	0.1951	0.2065	0.2866
R_{prp}	−0.0089	−0.0122	0.0118	0.0444	−0.1554	−0.1871	−0.1795	−0.2021

$\hat{\lambda}$ is the estimated market price of volatility risk along with asymptotic t-statistics in the parentheses. The symbol of * (**) denotes that at the significance level of 5% (1%) the t-statistic rejects the null hypothesis of $\lambda = 0$. $v\hat{a}r(R_t|I_{t-1})$ is the filtered variance of R_t, equal to $\hat{\mu}_{h,t|t-1}$ for the stochastic volatility and $\hat{\sigma}^2$ for the constant volatility. $\hat{\rho} = \hat{\mu}^*/\hat{\lambda}\hat{\alpha}^*\sqrt{\hat{\mu}_{h,t-1|t-1}}$ illustrates the correlation between return shocks and volatility shocks. R_{vrp} means the "volatility risk premium" computed by $7\hat{\lambda}\sqrt{\hat{\mu}_{h,t|t-1}}/16 + 11\hat{\mu}^*/16\hat{\lambda}\sqrt{\hat{\mu}_{h,t|t-1}} - \hat{\lambda}\hat{\beta}^2\hat{\mu}^2_{h,t-1|t-1}/8\sqrt{\hat{\mu}^3_{h,t|t-1}}$ (stochastic volatility) or $\hat{\lambda}\hat{\sigma}$ (constant volatility). R_{prp} denotes the "price risk premium" calculated as $\hat{\mu}_S - \hat{\mu}_{h,t|t-1}/2$ (stochastic volatility) or $\hat{\mu}_S - \hat{\sigma}^2/2$ (constant volatility). The figures in the table are the averages of corresponding estimates in the sample period starting in January 1969 and ending in January 1990—December 2004.

is most apparent during market crashes when a large decline in stock price is associated with a significant increase in market volatility. The economic significance of a negative value of ρ slows down the increase in asset price when the asset price is climbing up, whereas speeds up the decrease in asset price when the asset is falling. Thus, investors in general are not in favor of the consequence brought by a negative ρ. Given a negative value of ρ, the greater the volatility risk, the more premiums demanded by investors for compensation of bearing the risk and thus the market price of volatility risk is positive. In contrast, given a positive value of ρ, the greater the volatility risk, the less premiums required by investors due to the favor of the risk and thus the market price of volatility risk is negative. Our empirical results show a negative estimate of $\hat{\rho}$ and thus indicate a positive volatility risk premium, which is consistent with the result of a positive estimate of the market price of volatility risk $\hat{\lambda}$. As a result, $\hat{\rho}$ plays an important role of passing the volatility information into the underlying market. In other words, a great volatility shock cannot attract investors' attention to demand required premiums for bearing volatility risk if $\hat{\rho}$ is close to zero that blocks away the linkage between the volatility and underlying markets. On the other hand, a drop in the value of the stock (negative return) increases financial leverage, which makes the stock riskier and increases its volatility. Thus, the price risk premium presented in Table 3 and in the right axis of Figure 6 captures the leverage effect, which generates the volatility asymmetry independent of the volatility feedback.[15] Hence, the causality is different: the leverage effect contends that returns shocks lead to changes in conditional volatility, whereas the time-varying risk premium claims that return shocks are caused by changes in conditional volatility. The volatility feedback effect is enhanced by strong asymmetries in conditional covariances. The estimated structural models show that the volatility feedback effect is stronger than the leverage effect, in particular for the stochastic-volatility model. In addition, the volatility risk premium grows along the time to maturity, whereas the price risk premium shrinks across maturities. This is justified by the fact that the volatility feedback effect also relies on the persistence in volatility (Bekaert and Wu, 2000). Thus, the price risk premium is more contributable to short-term returns whereas the volatility risk premium plays an important role in long-dated returns.

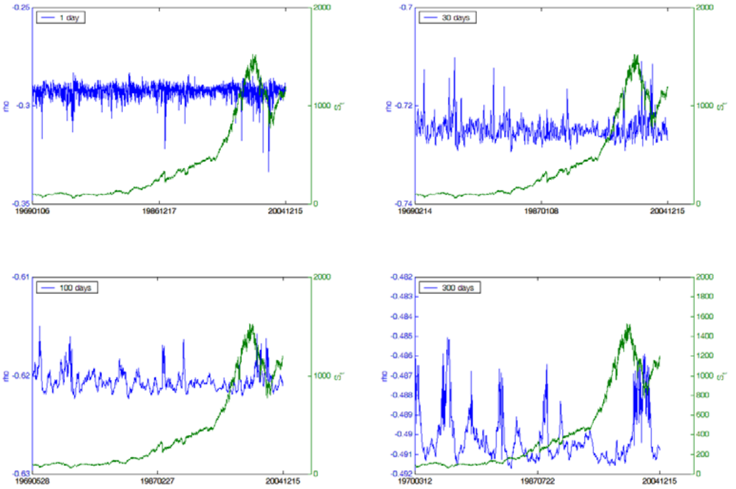
FIG. 6. The price risk premium (in the right axis) and the volatility risk premium (in the left axis) under stochastic volatility.



Under the stochastic-volatility framework, the price risk premium is calculated by $R_{prp} = \hat{\mu}_S - \hat{\mu}_{h,t|t-1}/2$, whereas the volatility risk premium is computed as $R_{vrp} = 7\hat{\lambda}\sqrt{\hat{\mu}_{h,t|t-1}}/16 + 11\hat{\mu}^*/16\hat{\lambda}\sqrt{\hat{\mu}_{h,t|t-1}} - \hat{\lambda}\hat{\beta}^2\hat{\mu}^2_{h,t-1|t-1}/8\sqrt{\hat{\mu}^3_{h,t|t-1}}$.

[15]Due to the limitation of the constant-volatility model, the estimated parameter $\hat{\sigma}$ using observed returns R_t cannot capture the randomness in volatility and the parameter estimate $\hat{\mu}_S$ thus contains the information of price and volatility risk premiums as well as the market prices of other risk factors.

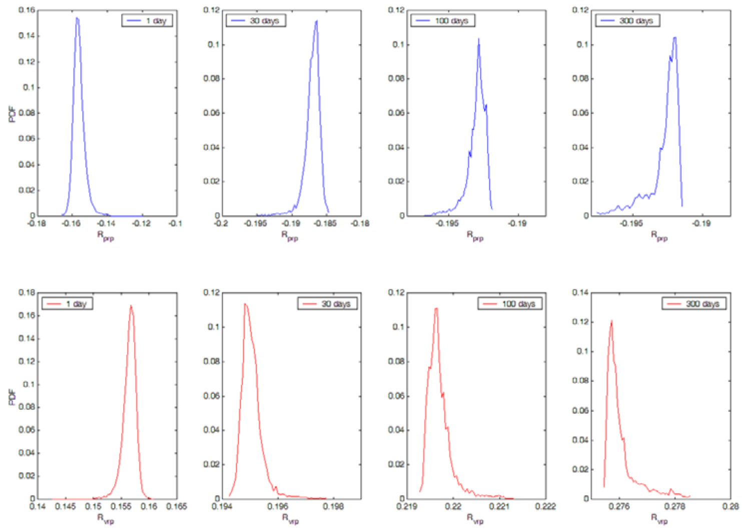
FIG. 7. The correlation between return shocks and volatility shocks in the S&P 500 index market.



rho (ρ) in the left axis denotes the correlation coefficient between return shocks and volatility shocks under stochastic volatility, which can be estimated through the formula, $\hat{\rho} = \hat{\mu}^*/\hat{\lambda}\hat{\alpha}^*\sqrt{\hat{\mu}_{h,t-1|t-1}}$, in which $\hat{\mu}^*$ and $\hat{\alpha}$ are parameter estimates, and $\hat{\mu}_{h,t-1|t-1}$ is the filtered variance at time $t-1$ given by the information set I_{t-1}. The average values (standard deviations) of $\hat{\rho}$ over 180 subperiods are -0.3002 (0.0033), -0.7032 (0.0021), -0.7025 (0.0014) and -0.4586 (0.0011) for 1, 30, 100 and 300 days to expiration. The negative value of $\hat{\rho}$ represents the asymmetric impact of volatility shocks on returns. In addition, the averaged correlations between $\hat{\rho}$ and $\hat{\mu}_{h,t|t-1}$ over 180 subperiods are 0.9988, 0.9994, 0.9946 and 0.9622 for 1, 30, 100 and 300 days, respectively. The highly positive correlation represents the situation that the more volatile the index, the more pronounced the volatility asymmetry.

Figure 8 reports the distributions of price and volatility risk premiums under stochastic volatility. With one exception for the 1-day return, the volatility (price) risk premiums have positive (negative) skewness. The existence of volatility risk premiums in the S&P 500 return may help explain the pricing puzzle of CAPM that often, in practice, underestimates (overestimates) the expected rate of return with a low (high) beta coefficient. As motivated by the significant correlation between aggregate consumption growth and S&P 500 index returns (EBRI, 1993), the S&P 500 index could be used as a proxy of the market portfolio. CAPM defines the beta coefficient of an asset i as $\beta_i = \sigma_{i,m}/\sigma_m^2$ where $\sigma_{i,m}$ is the covariance between the asset return, R_i, and the market return, R_m, and σ_m^2 is the variance of the market portfolio. If volatility is priced, the relevant measure of risk

FIG. 8. Distributions of the price risk premium (R_{prp}) and the volatility risk premium (R_{vrp}) under stochastic volatility.



With one exception for the 1-day price (volatility) risk premium with a positive (negative) skewness coefficient of 1.3382 (−1.4013), the price (volatility) risk premiums for 30, 100 and 300 days to expiration have negative (positive) skewness coefficients of −2.0468 (2.0407), −2.6609 (3.1589) and −1.9704 (2.2480), respectively.

is the covariance with the market portfolio that should response positively to increases in market volatility. The beta coefficient of an asset i could be decomposed into two components,

$$\beta_i = \frac{cov(R_i, R_m)}{\sigma_m^2} = \frac{cov(R_i, R_m^{prp})}{\sigma_m^2} + \frac{cov(R_i, R_m^{vrp})}{\sigma_m^2} = \beta_i^{prp} + \beta_i^{vrp},$$

where R_m^{prp} and R_m^{vrp}, respectively, present the price and volatility risk premiums in the market return. The CAPM uses standard statistical techniques (simple linear regression with constant volatility) to compute the intercept and the slope (beta) of a line that analyzes the relationship between the periodic returns of the asset i and those of the market (e.g. the S&P 500). While considering an asset with a high (low) β_i, much of the contribution to the expected return comes from the market price of volatility (price) risk (see Table 3). This is justified by our empirical findings that the increased volatility then raises expected returns and lowers current stock prices, dampening volatility in the case of good news and increasing volatility in the case of bad news. The estimated $\hat{\beta}_i$ is thus greater (lower) than its theoretical value of β_i^{CAPM} (under the constant-volatility assumption).

6. OUT-OF-SAMPLE FORECASTS IN MEAN RETURNS

6.1. Out-of-Sample Forecasts Using Price and Volatility Risk Premiums

In order to measure whether calculated price and volatility risk premiums contain information contents about returns, an out-of-sample forecast in returns is performed. The out-of-sample period is one-month ahead and lasts for 15 years, from February 1990 to December 2004,[16] in total 179 one-month periods. For the stochastic-volatility model, we use the estimates of model parameters, $\hat{\Xi} = \{\hat{\mu}_S, \hat{\varpi}, \hat{\beta}, \hat{\alpha}, \hat{\sigma}_e^2, \hat{c}_0, \hat{\mu}^*, \hat{r}^*\}$, and the market price of volatility risk, $\hat{\lambda}$, obtained from the estimation periods to calculate the volatility risk premium, $7\hat{\lambda}\sqrt{\hat{\mu}_{h,t|t-1}}/16 + 11\hat{\mu}^*/16\hat{\lambda}\sqrt{\hat{\mu}_{h,t|t-1}} - \hat{\lambda}\hat{\beta}^2\hat{\mu}^2_{h,t-1|t-1}/8\sqrt{\hat{\mu}^3_{h,t|t-1}}$, and the price risk premium, $\hat{\mu}_S - \hat{\mu}_{h,t|t-1}/2$, for corresponding out-of-sample data periods. Note that the conditional mean and variance of returns, i.e., $\hat{\mu}_S - \hat{\mu}_{h,t|t-1}/2$ and $\hat{\mu}_{h,t|t-1}$ need to be updated daily via Kalman filter. In the constant-volatility model the estimates of parameters, $\hat{\Xi} = \{\hat{\mu}_S, \hat{\sigma}\}$, and the market price of volatility risk, $\hat{\lambda}$, are used to calculate the price risk premium of $\hat{\mu}_S - \hat{\sigma}^2/2$ and the volatility risk premium of $\hat{\lambda}\hat{\sigma}$ for the forecast horizons. The summations of calculated price and volatility risk premiums are then compared to corresponding actual rates of return and the forecast errors are computed. Three error matrices are used to evaluate the forecast performances, including ME (mean error), MAE (mean absolute error) and RMSE (root mean squared error).

$$\begin{aligned}
ME &= \frac{1}{N}\sum_{i=1}^{N} \varepsilon_i \\
NAE &= \frac{1}{N}\sum_{i=1}^{N} |\varepsilon_i| \\
RMSE &= \sqrt{\frac{1}{N}\sum_{i=1}^{N} \varepsilon_i^2}
\end{aligned}$$

[16] The choice of one-month forecast horizon is justified by the consideration that the estimated model parameters only apply to the returns in the following one-month and thus reduces the inappropriateness of parameter estimates. The ending date of each data subperiod is the third Friday of the current month which corresponds to the expiration date of S&P 500 index options.

where ε_i is the out-of-sample forecast error that measures the difference between the observed return and the predicted return computed as the summation of calculated price and volatility risk premiums. $N = 3,848$ is the number of forecasts for the out-of-sample period, 18 January 1990—15 December 2004. As shown in Table 4, the shorter the investment horizon, the smaller the MAE and RMSE, indicating a superior forecast performance for short-term returns relative to long-term returns under either constant or stochastic volatility. As evidenced by significantly high estimates of parameter β, the relatively poor forecast for long-dated returns may result from the existence of long memory in realized volatility that is not explicitly taken into account in our models (Harvey, 1993). With an exception of the 1-day return, the stochastic-volatility model outperforms the constant-volatility model for 30-, 100- and 300-day returns. According to ME values, both models overprice the realized returns for 30, 100 and 300 days to expiration, but underprice the 1-day return.

TABLE 4.

Out-of-Sample Forecast Errors

	Constant Volatility				Stochastic Volatility			
	1 day	30 days	100 days	300 days	1 day	30 days	100 days	300 days
ME	0.0000	−0.0150	−0.0577	−0.0078	0.0027	−0.0142	−0.0570	−0.0072
MAE	0.0078	0.0453	0.0766	0.1154	0.0083	0.0448	0.0760	0.1152
RMSE	0.0098	0.0500	0.0807	0.1174	0.0104	0.0494	0.0801	0.1172

This table shows the performance of out-of-sample forecasts of the constant- and stochastic-volatility models during the forecast periods. The forecast errors are measured by the difference between realized and predicted returns. The predicted returns are computed as the summation of predicted price and volatility risk premiums. Three error matrices are used to evaluate the forecast performances, including ME (mean error), MAE (mean absolute error) and RMSE (root mean squared error). The figures in the table are the averages of forecast errors for 179 forecast subperiods, starting from February 1990—December 2004.

There are two drawbacks while using the RMSE criterion (Fair and Shiller, 1990). First, RMSE cannot really tell the difference between two models when their values are close to each other. Second, the model with higher RMSE may contain useful information for forecasting that is absent from the model with lower RMSE. Consequently, the next section uses the information of price and volatility risk premiums embedded in both constant- and stochastic-volatility models to predict the mean returns.

6.2. Encompassing Regression

As motivated by APT theory that the expected rate of return is composed of premiums of related risk factors, this section considers an encompassing regression to examine the relative contributions of price and volatility risk premiums to the mean returns. This study uses the averages of period returns calculated from the earliest available date in 1969 to the out-of-sample forecast date as the proxy of realized mean returns $\tilde{R}_{t,T}$ where T corresponds to 1-, 30-, 100- and 300-day investment horizons and t is the forecast date, starting from 18 January 1990 to 15 December 2004. Regression $\tilde{R}_{t,T}$ on the risk premiums predicted from both constant and stochastic volatility as shown in the following,

$$\tilde{R}_{t,T} = \beta_0 + \beta_1 \vartheta_t^{const} + \beta_2 \vartheta_t^{sv} + u_t, \tag{31}$$

where ϑ_t^{const} is the summation of the predicted price premium, $\zeta_t^{const} = \hat{\mu}_S - \hat{\sigma}^2/2$, and the predicted volatility risk premium, $\nu_t^{const} = \hat{\lambda}\hat{\sigma}$, under constant volatility where $\{\hat{\mu}_S, \hat{\sigma}\}$ are obtained from the estimation period. Similarly, ϑ_t^{sv} represents the summation of the predicted volatility risk premium,

$$\nu_t^{sv} = 7\hat{\lambda}\sqrt{\hat{\mu}_{h,t|t-1}}/16 + 11\hat{\mu}^*/16\hat{\lambda}\sqrt{\hat{\mu}_{h,t|t-1}} - \hat{\lambda}\hat{\beta}^2\hat{\mu}^2_{h,t-1|t-1}/8\sqrt{\hat{\mu}^3_{h,t|t-1}},$$

and the predicted price premium, $\zeta_t^{sv} = \hat{\mu}_S - \hat{\mu}_{h,t|t-1}/2$, under stochastic volatility where $\hat{\mu}_{h,t-1|t-1}$ and $\hat{\mu}_{h,t|t-1}$ are updated daily via Kalman filter and structural parameters are obtained from the estimation period.

Three hypothesis tests are performed with respective to the encompassing regression of equation (31). First, given running a simple univariate regression, a significant slope coefficient denotes the predictability of ϑ_t^{const} or ϑ_t^{sv} towards $\tilde{R}_{t,T}$. Next, an unbiased forecast will have the result with an intercept of 0 and the slope coefficient of 1. Finally, according to the orthogonality condition of market efficiency, if $\vartheta_t^{sv}(\vartheta_t^{const})$ contains enough efficient information about $\tilde{R}_{t,T}$, the estimated coefficient of $\vartheta_t^{sv}(\vartheta_t^{const})$ turns out to be zero. Besides, if there is no measurement error in $\vartheta_t^{const}, \vartheta_t^{sv}$ and $\tilde{R}_{t,T}$, OLS can provide consistent parameter estimates. Further, this study also runs a multiple regression, shown as follows, to address the relative contribution of individual risk premiums to $\tilde{R}_{t,T}$.

$$\tilde{R}_{t,T} = \beta_0 + \beta_1 \zeta_t^{const} + \beta_2 \nu_t^{const} + \beta_3 \zeta_t^{sv} + \beta_4 \nu_t^{sv} + u_t. \tag{32}$$

Significant slope coefficients indicate the predictability of corresponding ζ_t^{const}, ν_t^{const}, ζ_t^{sv} or towards $\tilde{R}_{t,T}$. The estimation results are shown in Table 5. The first two rows in Table 5 represent average parameter estimates along with t-statistics of a univariate regression of $\tilde{R}_{t,T}$ on ϑ_t^{const} and ϑ_t^{sv}, respectively, for alternate forecast horizons. Although the slope parameter of ϑ_t^{const} is close to 1, it is not statistically significant and thus lacks the evidence of the predictability of ϑ_t^{const} for $\tilde{R}_{t,T}$. In contrast, although the slope parameter of ϑ_t^{sv} is far from 1, it is statistically significant and thus supports for the predictability of $\vartheta_t^{sv}(= \zeta_t^{sv} + \nu_t^{sv})$ for $\tilde{R}_{t,T}$.

However, the intercept is significantly different from zero that rejects the unbiasedness of ϑ_t^{sv} towards $\tilde{R}_{t,T}$. The result of a multiple regression of $\tilde{R}_{t,T}$ on ϑ_t^{const} and ϑ_t^{sv} shows the slope estimate of ϑ_t^{sv} is statistically different from zero, indicating that ϑ_t^{sv} has incremental information relative to ϑ_t^{const}. Finally, the parameter estimates of a multiple regression of $\tilde{R}_{t,T}$ on ζ_t^{const}, ν_t^{const}, ζ_t^{sv} and ν_t^{sv} are statistically insignificant. However, the slope estimates with ζ_t^{sv} and ν_t^{sv} are greater and closer to 1 than those with ζ_t^{const} and ν_t^{const}, indicating that ζ_t^{sv} and ν_t^{sv} are $\tilde{R}_{t,T}$-informative.

7. CONCLUSION

The volatility feedback effect, along with the well-documented persistent volatility dynamics, implies an observationally equivalent negative correlation between current returns and future volatility, as a shock to the volatility will require an immediate return adjustment to compensate for the increased future risk. Empirical evidence also confirms that aggregate market volatility responds asymmetrically to negative and positive returns, and the economic magnitude is statistically significant and time-varying. Importantly, the magnitude also depends on the volatility proxy employed in the estimation, with stochastic volatilities generally exhibiting much more pronounced asymmetry. Since the S&P 500 index is generally used as a proxy of the market portfolio, the volatility shocks negatively correlated to S&P 500 returns may also be negatively correlated to aggregate consumption growth, and thus results in a negative volatility risk premium. However, the negative impact of volatility shocks on the total expected returns is offset and dominated by the volatility asymmetry. The net effect of volatility shocks on the total return rate will turn out to be positive, indicating that investors will demand a positive volatility risk premium to counter with volatility shocks. The contribution of the price risk premium to short-term returns is relatively significant whereas the volatility risk premium is more contributable to long-term returns. Finally, the market price of volatility risk may help to solve the pricing puzzle of CAPM that uses a simple linear regression with constant volatility. To investigate the information contents embedded in the price and volatility risk premiums, this study performs out-of-sample forecasts in returns. The stochastic-volatility model in general outperforms the constant-volatility model for 30-, 100- and 300-day returns. In addition, encompassing regression is used to investigate the information contents of price risk premiums $(\zeta_t^{const}, \zeta_t^{sv})$ and volatility risk premiums $(\nu_t^{const}, \nu_t^{sv})$ towards the mean return $\tilde{R}_{t,T}$ during the out-of-sample data period. The predicted mean return is computed as either $\vartheta_t^{const} (= \zeta_t^{const} + \nu_t^{const})$ under constant volatility or either $\vartheta_t^{sv} (= \zeta_t^{sv} + \nu_t^{sv})$ under stochastic volatility. The slope parameter of ϑ_t^{const} is found close to 1 but not statistically significant and thus cannot

TABLE 5.

Encompassing Regression

	β_0	ϑ_t^{const}	ϑ_t^{sv}	ζ_t^{const}	ν_t^{const}	ζ_t^{sv}	ν_t^{sv}
	0.0000	1.0000					
	(0.00)	(0.00)					
	0.0002**		0.00001**				
1 day	(1182949.90)		(1581.17)				
	0.0001	0.7092	0.0001**				
	(0.00)	(0.00)	(508.57)				
	−0.1213			−0.2180	0.2179	0.5403	1.2903
	(−0.06)			(−0.18)	(0.02)	(0.09)	(0.09)
	0.0000	1.0000					
	(0.00)	(0.05)					
	0.0068**		−0.0364**				
30 days	(766534.77)		(−19335.74)				
	0.0127	−0.8917	−0.0364**				
	(0.02)	(−0.01)	(−14707.80)				
	−0.1439			−0.0911	0.0895	0.4884	1.2317
	(−0.02)			(−0.52)	(0.41)	(0.23)	(0.31)
	−0.0001	0.9999					
	(−0.00)	(0.09)					
	0.0262**	−0.1682**					
100 days	(54436.99)		(−5441.05)				
	0.0412	−0.6681	−0.1682**				
	(0.01)	(−0.00)	(−7976.21)				
	−0.1461			0.1715	0.1704	0.4982	1.2673
	(−0.42)			(0.37)	(0.05)	(0.09)	(0.13)
	0.0000	1.0000					
	(0.00)	(0.11)					
	0.0744**		−0.0237**				
300 days	(62856.18)		(60328.52)				
	0.0813**	−0.0952	−0.0237**				
	(4.31)	(−0.05)	(21699.38)				
	0.1182			0.0394	0.0392	0.6420	1.1451
	(?0.12)			(0.14)	(0.04)	(0.29)	(0.25)

The first two rows in the table represent the results of a univariate regression of $\tilde{R}_{t,T}$ on ϑ_t^{const} and ϑ_t^{sv}, respectively, where $\tilde{R}_{t,T}$ is the realized mean return. The third row shows the estimation results of $\tilde{R}_{t,T} = \beta_0 + \beta_1 \vartheta_t^{const} + \beta_2 \vartheta_t^{sv} + u_t$, whereas the fourth row displays the parameter estimates of $\tilde{R}_{t,T} = \beta_0 + \beta_1 \zeta_t^{const} + \beta_2 \nu_t^{const} + \beta_3 \zeta_t^{sv} + \beta_4 \nu_t^{sv} + u_t$. The figures in the parentheses are the t-statistics of parameter estimates. The symbol of ** indicates that the t-statistic rejects the null hypothesis of a zero parameter at the significance level of 1%.

support for the predictability of ϑ_t^{const} on $\tilde{R}_{t,T}$. The intercept and slope parameters of ϑ_t^{sv} are relatively small but statistically significant, indicating that ϑ_t^{sv} is an informative but biased estimator of $\tilde{R}_{t,T}$. The result of a multiple regression of $\tilde{R}_{t,T}$ on ϑ_t^{const} and ϑ_t^{sv} indicates that ϑ_t^{sv} has incremental information relative to ϑ_t^{const}. Finally, all parameter estimates of a multiple regression of $\tilde{R}_{t,T}$ on ζ_t^{const}, ν_t^{cobst}, ζ_t^{sv} and ν_t^{sv} are not statistically significant. However, slope parameter estimates of ζ_t^{sv} and ν_t^{sv} are greater and closer to 1 than the corresponding values of ζ_t^{const} and ν_t^{const}, indicating a superior information content of ζ_t^{sv} and ν_t^{sv} towards $\tilde{R}_{t,T}$ relative to ζ_t^{const} and ν_t^{const}. In other words, these empirical results point out the importance of volatility risk premiums in mean returns, especially under the stochastic volatility. Thus, the framework of stochastic volatility plays an important role in correctly measuring or capturing the market price of volatility risk, which is crucial to asset pricing.

REFERENCES

Abel, Andrew B., 1988, Stock prices under time-varying dividend risk: An exact solution in an infinite-horizon general equilibrium model. *Journal of Monetary Economics* **22**, 375-393.

Ait-Sahalia, Yacine, 2002, Maximum likelihood estimation of discretely sampled diffusions: A closed-form approach. *Econometrica* **70**, 223-262.

Andersen, Torben G., Luca Benzoni, and Jesper Lund, 2002, An empirical investigation of continuous-time equity return models. *Journal of Finance* **57**, 1239-1284.

Andersen, Torben G., Tim Bollerslev, Francis Diebold, and Heiko Ebens, 2001, The distribution of realized stock return volatility. *Journal of Financial Economics* **61**, 43-76.

Backus, David K., and Allen W. Gregory, 1993, Theoretical relations between risk premiums and conditional variance, *Journal of Business and Economic Statistics* **11**, 177-185.

Bates, David S., 2000, Post-'87 crash fears in the S&P 500 futures option market. *Journal of Econometrics* **94**, 181-238.

Bekaert, Geert, and Guojun Wu, 2000, Asymmetric volatility and risk in equity markets. *Review of Financial Studies* **13**, 1-42.

Black, Fisher, 1976, Studies in stock price volatility changes, Proceedings of the 1976 Business Meeting of the Business and Economic Statistics Section. *American Statistics Association*, 177C181.

Black, Fisher, and Myron Scholes, 1973, The pricing of options and corporate liabilities. *Journal of Political Economy* **81**, 637-659.

Bollerslev, Tim, 1986, Generalized autoregressive conditional heteroscedasticity. *Journal of Econometrics* **51**, 307-327.

Bollerslev, Tim, Robert F. Engle, and Jeffrey M. Wooldridge, 1988, A capital asset pricing model with time varying covariances. *Journal of Political Economy* **95**, 116-131.

Bollerslev, Tim, and Hao Zhou, 2002, Estimating stochastic volatility diffusion using conditional moments of integrated volatility. *Journal of Econometrics* **109**, 33-65.

Brandt, Michael W., and Qiang Kang, 2004, On the relationship between the conditional mean and volatility of stock returns: A latent VAR approach. *Journal Financial Economics* **72**, 217-257.

Breen, William, Lawrence R. Glosten, and Ravi Jagannathan, 1989, Economic significance of predictable variations in stock index returns. *Journal of Finance* **44**, 1177-1189.

Campbell, John Y., 1987, Stock returns and the term structure. *Journal of Financial Economics* **18**, 373-399.

Campbell, John Y., 1993, Intertemporal asset pricing without consumption data. *American Economic Review* **83**, 487-512.

Campbell, John Y., and Ludger Hentschel, 1992, No news is good news: An asymmetric model of changing volatility in stock returns. *Journal of Financial Economics* **31**, 281-318.

Campbell, John Y., and Albert S. Kyle, 1993, Smart money, noise trading and stock price behaviour. *Review of Economic Studies* **60**, 1-34.

Carnero, M. Angeles, Daniel Pena, and Esther Ruiz, 2004, Persistence and kurtosis in GARCH and stochastic volatility models, *Journal of Financial Econometrics* **60**, 319-342.

Chacko, George C., and Luis M. Viceira, 2003, Spectral GMM estimation of continuous-time processes. *Journal of Econometrics* **116**, 259-292.

Chernov, Mikhail, Ronald Gallant, Eric Ghysels, and George Tauchen, 2003, Alternative models for stock price dynamics. *Journal of Econometrics* **116**, 225-257.

Chou, Ray Y., 1988, Volatility persistence and stock valuations: Some empirical evidence using GARCH. *Journal of Econometrics* **3**, 279-294.

Chou, Ray Y., Robert F. Engle, and Alex Kane, 1992, Measuring risk aversion from excess returns on a stock index. *Journal of Econometrics* **52**, 201-224.

Christie, Andrew A., 1982, The stochastic behavior of common stock variances?Value, leverage and interest rate effects. *Journal of Financial Economics* **10**, 407-432.

Conley, Tim, Lars Peter Hansen, Erzo G. J. Luttmer, Jose A. Scheinkman, 1997, Short term interest rates as subordinated diffusions. *Review of Financial Studies* **10**, 525-578.

Corradi, Valentina, 2000, Reconsidering the continuous time limit of the GARCH(1,1) process. *Journal of Econometrics* **96**, 145-153.

Cox, John C., Jonathan E. Ingersoll, and Stephen A. Ross, 1985, A theory of term structure of interest rate. *Econometrica* **53**, 385-407.

Derman, Emanuel, and Iraj Kani , 1994, Riding on a smile. *Risk* **7**, 32-39.

Diebold, Francis X., Til Schuermann, and John D. Stroughair, 2000, Pitfalls and opportunities in the use of extreme value theory in risk management. *Journal of Risk Finance* **1**, 30-35.

Duffee, Gregory R., 1995, Stock returns and volatility: A firm-level analysis. *Journal of Financial Economics* **37**, 399-420.

Duffie, Darrel, and Kenneth Singleton, 1993, Simulated moments estimation of Markov models of asset prices. *Econometrica* **61**, 929-952.

Dumas, Bernard, Jeff Fleming, and Robert Whaley, 1998, Implied volatility smiles: Empirical tests. *Journal of Finance* **53**, 2059-2106.

Dupire, Bruno, 1994, Pricing with a smile. *Risk* **7**, 18-20.

EBRI Issue Brief, 1993, Pension fund portfolio turnover and performance evaluation, the U.S. patent and Trademark Office.

Engle, Robert F., 1982, Autoregressive conditional heteroskedasticity with estimates of the variance of United Kingdom inflation. *Econometrica* **50**, 987-1007.

Engle, Robert F., and Isao Ishida, 2002, Forecasting variance of variance: The square-root, the affine and the CEV GARCH models, Department of Finance Working Papers, New York University.

Engle, Robert F., David M. Lilien, and Russell P. Robins, 1987, Estimating time varying risk premia in the term structure: The ARCH-M model. *Econometrica* **55**, 391-407.

Engle, Robert F., and Victor K. Ng, 1993, Measuring and testing the impact of news on volatility. *Journal of Finance* **48**, 1749-1801.

Engle, Robert F., and Andrew J. Patton, 2001, What good is a volatility model? Working paper, Department of Finance, New York University.

Engle, Robert F., and Joshua V. Rosenberg, 2002, Empirical pricing kernels. *Journal of Financial Economics* **64**, 341-372.

Eraker, Bjorn, 2001, MCMC analysis of diffusion models with application to finance. *Journal of Business and Economic Statistics* **19**, 177-191.

Eraker, Bjorn, 2004, Do stock prices and volatility jump? Reconciling evidence from spot and option prices. *Journal of Finance* **59**, 1367-1404.

Eraker, Bjorn, Michael Johannes, and Nicholas Polson, 2003, The impact of jumps in equity index volatility and returns. *Journal of Finance* **58**, 1269-1300.

Fair, Ray C., and Robert J. Shiller, 1990, Comparing information in forecasts from econometric models. *American Economic Review* **80**, 375-389.

Fama, Eugene F. 1965, The behavior of stock prices. *Journal of Business* **38**, 34-105.

French, Kenneth R., G. William Schwert, and Robert F. Stambaugh, 1987, Expected stock returns and volatility. *Journal of Financial Economics* **19**, 3-30.

Gallant, A. Ronald, and Jonathan R. Long, 1997, Estimating stochastic differential equations efficiently by minimum chi-square. *Biometrika* **84**, 125-141.

Gallant, A. Ronald, Peter E. Rossi, and George Tauchen, 1992, Stock prices and volume. *Review of Financial Studies* **5**, 199-242.

Gallant, A. Ronald, and George Tauchen, 1996, Which Moment to Match? *Econometric Theory* **12**, 657-681.

Ghysels, Eric, Pedro Santa-Clara, and Rossen Valkanov, 2002, There is a risk-return tradeoff after all! Working Paper, UNC and UCLA.

Ghysels, Eric, Andrew C. Harvey, and Eric Renault, 1996, Stochastic volatility. In *Statistical Methods in Finance*. Edited by G.S. Maddala and C. R. Rao. North Holland, Amsterdam.

Glosten, Lawrence R., Ravi Jagannathan, and David Runkle, 1993, On the relation between the expected value and the volatility of the nominal excess return on stocks. *Journal of Finance* **48**, 1779-1801.

Gourieroux, Christian, Alan Monfort, and Eric Renault, 1993, Indirect inference. *Journal of Applied Econometrics* **8**, 85-118.

Guedhami, Omrane, and Oumar Sy, 2004, Investigating the puzzling conditional market risk-return relationship, Working paper, Faculty of Administrative Science, Laval University, Canada.

Guo, Dajiang, 1996, The predictive power of implied stochastic variance from currency options. *Journal of Futures Markets* **16**, 915-942.

Hansen, Lars P., 1982, Large sample properties of generalized method of moment estimator. *Econometrica* **50**, 1029-1054.

Hansen, Lars P., and Jose A. Scheinkman, 1995, Back to the future: Generalized moment implications for continuous time Markov process. *Econometrica* **63**, 767-804.

Harrison, Paul, and Harold H. Zhang, 1999, An investigation of the risk and return relation at long horizons. *Review of Economics and Statistics* **81**, 399-408.

Harvey, Andrew C., 1993, Long memory in stochastic volatility, Discussion Paper, London School of Economics.

Harvey, Andrew C., and Neil Shephard, 1993, Estimation and testing of stochastic variance models, STICERD Econometrics, Discussion Paper, EM93/268, London School of Economics.

Harvey, Andrew C., and Neil Shephard, 1996, Estimation of an asymmetric stochastic volatility model for asset returns. *Journal of Business and Statistics* **14**, 429-434.

Harvey, Andrew C., Esther Ruiz, and Neil Shephard, 1994, Multivariate stochastic variance model. *Review of Economic Studies* **61**, 247-264.

Harvey, Campbell R., 2001, The specification of conditional expectations. *Journal of Empirical Finance* **8**, 573-637.

Heston, Steven L., 1993, A close form solution for options with stochastic volatility. *Review of Financial Studies* **6**, 327-343.

Heston, Steven L., and Saikat Nandi, 2000, A closed-form GARCH option valuation model. *Review of Financial Studies* **13**, 585-625.

Hull, John C., and Allan White, 1987, The pricing of options on assets with stochastic volatility. *Journal of Finance* **42**, 281-300.

Jacquier, Eric, Nicholas G. Polson, and Peter E. Rossi, 1994, Bayesian analysis of stochastic volatility models. *Journal of Business and Economic Statistics* **12**, 371-417.

Jiang, George J., and John L. Knight, 2002, Estimation of continuous time stochastic processes via the empirical characteristic function. *Journal of Business and Economic Statistics* **20**, 198-212.

Kim, Sangjoon, Neil Shephard, and Siddhartha Chib, 1998, Stochastic volatility: Likelihood inference and comparison with ARCH models. *Review of Economic Studies* **65**, 361-394.

Kitagawa, Genshiro, 1987, Non-gaussian state space modeling of nonstationary time series (with discussion). *Journal of the American Statistics Association* **82**, 1032-1041.

Koopman, Siem Jan, and Eugenie Hol Uspensky, 2002, The stochastic volatility in mean model: Empirical evidence from international stock markets, *Journal of Applied Econometrics* **17**,667-689.

Lamoureux, Christopher G., and William D. Lastrapes, 1993, Forecasting stock return variance: Toward an understanding of stochastic implied volatilities. *Review of Financial Studies* **6**, 293-326.

Liesenfeld, Roman, and Robert C., Jung, 2000, Stochastic volatility models: Conditional normality versus heavy-tailed distributions. *Journal of Applied Econometrics* **15**, 137-160.

Lintner, John, 1965, The valuation of risk assets and the selection of risky investments in stock portfolios and capital budgets. *Review of Economics and Statistics* **47**, 13-37.

Lo, Andrew W., 1988, Maximum likelihood estimation of generalized It? process with discretely sampled data. *Econometric Theory* **4**, 231-247.

Loudon, Geoffrey F., Wing H. Watt, and Pradeep K. Yadav, 2000, An empirical analysis of alternative parametric ARCH models. *Journal of Applied Econometrics* **15**, 117-136.

Mandelbrot, Benoit, 1963, The variation of certain speculative prices. *Journal of Business* **36**, 394-419.

Melino, Angelo, and Stuart M. Turnbull, 1990, Pricing foreign currency options with stochastic volatility. *Journal of Econometrics* **45**, 7-39.

Merton, Robert C., 1973, A rational theory of option pricing. *Bell Journal of Economics and Management Science* **4**, 141-183.

Merton, Robert C., 1980, On estimating the expected return on the market. *Journal of Financial Economics* **8**, 323-361.

Nandi, Saikat, 1998, How important is the correlation between returns and volatility in a stochastic volatility model? Empirical evidence from pricing and hedging in the S&P 500 index options market. *Journal of Banking and Finance* **22**, 589-610.

Nelson, Daniel B., 1988, Time series behavior of stock market volatility and returns, Ph.D. dissertation, MIT.

Nelson, Daniel B., 1990, ARCH models as diffusion approximation. *Journal of Econometrics* **45**, 7-39.

Nelson, Daniel B., 1991, Conditional heteroskedasticity in asset returns: A new approach. *Econometrica* **59**, 347-370.

Nelson, Daniel B., 1992, Filtering and forecasting with misspecified ARCH models I: getting the right variance with the wrong model. *Journal of Econometrics* **25**, 61-90.

Nelson, Daniel B., and Douglas Foster, 1994, Asymptotic filtering theory for univariate ARCH models. *Econometrica* **62**, 1-41.

Pan, Jun, 2002, The jump-risk premia implicit in options: Evidence from an integrated time-series study. *Journal of Financial Economics* **63**, 3-50.

Pindyck, Robert, 1984, Risk, inflation and the stock market. *American Economic Review* **74**, 335-351.

Rubinstein, Mark, 1994, Implied binomial trees. *Journal of Finance* **49**, 771-818.

Ruiz, Esther, 1994, Quasi-maximum likelihood estimation of stochastic volatility models. *Journal of Econometrics* **63**, 289-306.

Schwert, G. William, 1989, Business cycles, financial crises, and stock volatility. *Carnegie-Rochester Conference Series on Public Policy* **39**, 83-126.

Schwert, G. William, 1990, Stock volatility and the crash of 1987. *Review of Financial Studies* **3**, 77-102.

Scruggs, John T., 1998, Resolving the puzzling intertemporal relation between the market risk premium and conditional market variance: A two-factor approach. *Journal of Finance* **52**, 575-603.

Sharpe, William F., 1964, Capital asset prices: A theory of market equilibrium under conditions of risk. *Journal of Finance* **19**, 425-442.

Shephard, Neil, 1995, Statistical aspect of ARCH and stochastic volatility, Discussion Paper 1994, Nuffield College, Oxford University.

Singleton, Kenneth J., 2001, Estimation of affine asset pricing models using the empirical characteristic function. *Journal of Econometrics* **102**, 111-141.

Tauchen, George, and Robert Hussey, 1991, Quadrature-based methods for obtaining approximate solutions to nonlinear asset pricing models. *Econometrica* **59**, 371-396.

Tauchen, George, Harold Zhang, and Ming Liu, 1996, Volume, volatility, and leverage: A dynamic analysis. *Journal of Econometrics* **74**, 177-208.

Taylor, Stephen J., 1986, *Modeling Financial Time Series.* John Wiley: Chichester.

Taylor, Stephen J., and Xinzhong Xu, 1993, The magnitude of implied volatility smiles: Theory and empirical evidence for exchange rates. *Review of Futures Markets* **13**, 355C380.

Turner, Christopher M., Richard Startz, and Charles R. Nelson, 1989, A markov model of heteroskedasticity, risk, and learning in the stock market. *Journal of Financial Economics* **25**, 3-22.

Watanabe, Toshiaki, 1993, The time series properties of returns, volatility and trading volume in financial markets, Ph.D. Thesis, Department of Economics, Yale University.

Wu, Guojun, and Zhijie Xiao, 2002, A generalized partially linear model of asymmetric volatility. *Journal of Empirical Finance* **9**, 287-319.