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Bidders' Risk Preferences in Discriminative Auctions

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We study multiple unit Discriminative auctions when the bidders share log-concave utility functions and investigate the effects of bidders risk preferences on their bid functions when all bidders share a common utility function and when the bidders exhibit different risk preferences. We extend the existing findings from single unit auctions to multiple unit Discriminative auctions,from concave utility functions to log-concave utility functions, and from identical preference to different preferences.

Key Words: Auction; Bid function; Risk preference.
JEL Classification Numbers: D44, D72, D82.

1. INTRODUCTION

Early work in the auction literature focused on the famous Revenue Equivalence Theorem (RET) for which the principles were established in Vickrey (1961) and then later proven to be more general in Myerson (1981) and Riley and Samuelson (1981).The key assumptions underlying this theorem, as described in Krishna (2002), are independence of bidder values, risk neutrality of bidders, lack of bidder budget constraints and that all bidder values are drawn from the same distribution.In this environment, the reason that so many auctions are revenue equivalent is that they differ only in their payment functions, but have the same expected payment functions when all agents are risk neutral. This equivalency result cannot hold if buyers are risk averse. The lack of equivalency when buyers are risk averse is suggested by the well-known findings that, first-price auctions generate greater expected profit than second-price auctions, found by Harris and Raviv(1981), Holt(1980), Maskin and Riley(1980), Maskin and Riley, (1982), Matthews(1980), Milgrom and Weber(1982) Riley and Samuelson(1981). Vickey (1961) formulated a Nash equilibrium model of bidding by risk neutral economic agents in single unit auctions, and then Vickey (1962)

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generalized the original model to include multiple unit auctions. In both of the Vickrey's papers, individual values for the auctioned object(s) were assumed to be drawn from an uniform distribution. Holt(1980) and Riley and Samuelson(1981), for single unit auctions, and Harris and Raviv(1981), for multiple unit auctions, have extended the Vickrey model to the case in which values are from a general distribution function and all agents have identical concave utility function. Cox, Roberson and Smith(1982) and Cox, Smith and Walker(1982) developed the constant relative risk averse model for single unit auctions. Cox, Smith and Walker(1988) extended the First-price single unit auction theory to the case of heterogeneous bidders characterize by M-parameter log concave model, permits bidder preferences for monetary payoff to be risk averse, risk neutral or risk preferring, although every bidder's utility function for monetary payoff must be less convex than the exponential function. Long(2003a) further studied the constant relative risk averse model for multi-unit auctions.

For the effect of bidders' risk preferences on the bids, the well known result which was found in Riley and Samuelson(1981) suggests that,in the high bid auctions(i.e. the first price auctions) with that all bidders share a common utility function displaying risk aversion, retaining all other assumption but that the bidders are risk neutral, as bidders become more risk avers,they make uniformly higher bids. Harris and Raviv(1981) proved that, in multi-unit concave (risk neutral or averse) utility model, the risk neutral bid function is strictly less than the risk averse bid function. Another classic result is concerned with the heterogeneous risk preferences in the first price auctions. Through a parametric structure, Cox, Smith and Walker(1988) studied the case that the risk preference parameters differ expost among the bidders, but are ex-ante symmetric(i.e. drown from a common probability distribution). They have proved the risk-averse bidders bid higher than the risk-neutral bidders do, and the risk-neutral bidders bid higher than the risk-seeking bidders do.

In this paper, we generalize Cox-Smith-Walker M-parameter log-concave models to include multiple unit auctions in which each of N bidders with log-concave utility functions can bid on one out of a total of Q homogeneous items that are up for auction, where $1 \leq Q < N$, and individual bidders can differ from each other in any way that can he or she be represented by a finite number (M) of parameters, say (v_i, θ_i) of the bidder i , where v_i is his or her (scalar) auctioned object value and θ_i is his or her $M-1$ vector of other individual characteristics that affect bidding behavior in this M-parameter log-concave Q-unit model. Then our model can be looked as a generalization of all above models, which extents exist models from single unit auctions to multiple unit Discriminative auctions,from concave utility functions to log-concave utility functions, and from identical preference to different preferences.

In section 2, we present the general model, and give the differential equation the equilibrium bid functions satisfy. In section 3, we consider the effect of risk preference on equilibrium bids. For the auction with identical bidders' risk preferences, we relax the assumption that the bidders'common utility function is concave in Harris and Raviv(1981)'s multi-unit one-parameter model to permit the utility function to be log-concave, thus permit the bidders to be risk neutral or averse or risk-seeking. On the other hand, Harris and Raviv(1981) compared the bids only between the case of risk neutral and the case of risk averse, we extend this result and show that as the bidders become more risk avers,they make uniformly higher bids. Easy to see, it also means that the result of Riley and Samuelson(1981) for one-unit first price auction with concave utility function of bidders retain to be valid in the multi-unit discriminative auctions with log-concave utility function of bidders. For the auction with heterogeneous risk preferences, we extend the single unit M-parameter log-concave utility model of Cox, Smith and Walker (1988) to the multi-units model described in the section 2. We show that the result of Cox, Smith and Walker (1988) which is limited in the comparison of bids between risk-neutral bidders,risk-averse bidders, and risk-seeking bidders can be extended to make comparison based on the extent of bidder's preference. We conclude that if one bidder avers risk more than another bidder,then he will bid higher than the later.

2. MODEL OF MULTI-UNIT DISCRIMINATIVE AUCTION

Let $Q \geq 1$ unit(s) of a homogeneous good be offered in perfectly inelastic supply to $N > Q$ bidders. Each bidder submits a bid for single unit with understanding that each of the Q highest bidders will be awarded a unit of the good at price equal to his bid, i.e, the institution is a discriminative sealed-bid auction. Let v_i be the monetary value of a unit of the good to bidder i, where $i = 1, 2, \cdots, N$. Bidders are assumed to know their own v_i , but to know only the probability distribution with cdf $H(\cdot)$ on $[0, \bar{v}]$, from which their rivals' values are independently drawn. $H(\cdot)$ is assumed to have a continuous density function $h(\cdot)$ that is positive on $(0, \bar{v})$.

The utility to any bidder i of a winning bid in the amount b_i is the Von Neumann Morgenstern utility $U(v_i - b_i, \theta_i)$, where θ_i is an $M-1$ vector of parameters that is independently drawn from the probability distribution with integrable cdf $\Phi(\cdot)$ on the convex set Θ . Each bidder knows his or her own θ_i , but knows only that his or her rivals' θ's are drawn from the distribution $\Phi(\cdot)$. Thus, a bidder is represented by M-parameter (v_i, θ_i) , where v_i is his or her auctioned object value and θ_i is his or her $M-1$ vector of other individual characteristics that affect bidding behavior in this M-parameter log-concave model.

Assume that $u(y, \theta)$ is twice continuously differentiable and strictly increasing in monetary payoff y and normalized such that $u(0, \theta) = 0$, for all $\theta \in \Theta$. Finally, assume that $u(y, \theta)$ is strictly log-concave in y, for each $\theta \in \Theta$; i.e. $u_1(y, \theta)/u(y, \theta)$ is strictly decreasing in y for each $\theta \in \Theta$; i.e. $u_1(y, \theta)/u(y, \theta)$ is strictly decreasing in y or each $\theta \in \Theta$ (where $u_1(y, \theta)$ is the derivative of $u(y, \theta)$ with respect to y). This means that bidder preferences for risky monetary payoff can be risk averse, risk neutral, or risk preferring, but they must be less convex than e^y .

Suppose that bidder i expects each of his or her rivals to bid according to the differentiable bid function

$$b_j = b(v_j, \theta_j) \tag{1}$$

let $b(v, \theta)$ be strictly increasing in v and has the property that $b(0, \theta) = 0$ for all $\theta \in \Theta$, denote by $\pi(b, 0)$ the v-inverse of bid function $b(v, \theta)$. The probability that each of rival's bidder i will bid less than or equal to some amount b in the range of (1) is

$$F(b) = \int_{\Theta} H(\pi(b, \theta)) d\Phi(\theta) \tag{2}$$

Hence, the probability that a bid b by i will win is the probability $G(b)$, that at least $N - Q$ of i's rivals will bid less than b , this probability given by the distribution of $(N - Q)$th order statistic for a sample of size $N - 1$ from the distribution F :

$$G(b) = \frac{(N-1)!}{(N-Q-1)!(Q-1)!} \int_0^b F(x)^{N-Q-1} [1 - F(x)]^{Q-1} dF(x) \tag{3}$$

Thus, if bidder i believes that his or her rivals will use bid function $b(v, \theta)$, with v-inverse function $\pi(b, \theta)$, then the expected utility to bidder i of a bid in the amount b_i , is given by (2) , (3) , and

$$U(b_i | v_i, \theta_i) = G(b_i) u(v_i - b_i, \theta_i) \tag{4}$$

The first order condition for $b_i > 0$ to maximize equation (4) is

$$\begin{aligned} U'(b_i | v_i, \theta_i) &= G'(b_i) u(v_i - b_i, \theta_i) - G(b_i) u_1(v_i - b_i, \theta_i) \\ &= 0 \end{aligned} \tag{5}$$

If $\pi(b, \theta)$ is to be the v-inverse of an equilibrium bid function, then it must be a best reply for bidder i, substituting $\pi(b_i, \theta_i)$ for v_i in equation (4) yields

$$G'(b_i) u(\pi(b_i, \theta_i) - b_i, \theta_i) - G(b_i) u_1(\pi(b_i, \theta_i) - b_i, \theta_i) = 0 \tag{6}$$

On the other hand, for $Q = 1$, Cox, Smith and Walker(1988) have proved that $\pi(b,\theta)$ given by equation (6) is certainly the v-inverse of an equilibrium bid function. By a same proof as for one-unit auction in Cox, Smith and Walker(1988) , we can show that the result retains to be true for any $1 \leq Q < N$ (a complete proof can be found in Long (2003b)). This means that we have the following theorem.

THEOREM 1. *A differentiable function $b(v,\theta)$ which is strictly increasing in v , is a equilibrium bid function if and only if its v-inverse $\pi(b,\theta)$ satisfies the following equation:*

$$G'(b)u(\pi(b,\theta) - b, \theta) = G(b)u_1(\pi(b,\theta) - b, \theta) \tag{7}$$

where $G(b)$ is given by (2) and (3) .

3. THE EFFECT OF RISK PREFERENCES ON THE EQUILIBRIUM BIDS

In the case of $N > Q \geq 1$, where all bidders are known to be risk neutral or all bidders are known to have the same strictly concave utility function. Harris and Raviv(1981) have proved that

$$b_n(v) < b_\alpha(v) < v \qquad for\ all\ v \in (0, \bar{v}) \tag{8}$$

Where $b_n(v)$ is the Nash equilibrium risk neutral bid function, and $b_\alpha(v)$ is Nash equilibrium risk averse bid function.

In the case of $Q = 1$, where any bidder i has a strictly log-concave utility function $u(y, \theta_i)$, and each bidder knows his or her own parameter θ_i , but knows only that his or her rivals's θ's are drawn from the distribution $\Phi(\cdot)$, Cox, Smith, and Walker(1988) have proved the following result for the Nash equilibrium bid function :

$$b(v, \theta^A) > b(v, \theta^N) > b(v, \theta^P) \tag{9}$$

Where θ^N , θ^A , θ^P are the characteristic vectors of bidders who are risk neutral, risk averse and risk preferring.

In this section, we not only extend the result of Cox, Smith and Walker to the case of $Q \geq 1$, and the result of Harris and Raviv to the case of log-concave utility function, but also get more intensive results which show that higher equilibrium bids will be submitted by the bidders whoever are more risk averse.

Let $A(\cdot)$ and $B(\cdot)$ be two strictly increasing, (May be concave, or convex, or linear)we call that $A(\cdot)$ is more concave than $B(\cdot)$, if there exist a strictly increasing and strictly concave function $S(\cdot)$ such that

$$A(\cdot) = S(B(\cdot)) \tag{10}$$

If so, we also call that the agent with utility function $A(\cdot)$ is more risk averse than the agent utility function $B(\cdot)$.

If $A(\cdot)$ and $B(\cdot)$ both are twice differentiable, the condition (10) can be equivalently expressed in terms of the Arrow- Pratt measure of risk aversion

$$-\frac{A''(\cdot)}{A'(\cdot)} > -\frac{B''(\cdot)}{B'(\cdot)} \tag{11}$$

THEOREM 2. *Consider the model given in section 2 , let* θ_1 *and* θ_2 *be the risk preference parameters of the bidder* 1 *and bidder* 2 *, assume that the bidder* 1 *is more risk averse than the bidder* 2 *, it means that* $u(\cdot, \theta_1)$ *is more concave than* $u(\cdot, \theta_2)$ *. Then,* $b(v, \theta_1) > b(v, \theta_2)$ *for all positive* v *in the domain of* $b(\cdot)$ *.*

Proof. Suppose that $b(\tilde{v}, \theta_1) \leq b(\tilde{v}, \theta_2)$ for some $\tilde{v} > 0$, since $b(v, \theta)$ is increasing in v and $b(0, \theta) = 0$, for all $\theta \in \Theta$, there exist $\hat{v} \leq \tilde{v}$ such that $b(\hat{v}, \theta_2) = b(\tilde{v}, \theta_1) = b$. Let $S(\cdot)$ be the strictly increasing and strictly concave function such that

$$u(\cdot, \theta_1) = S(u(\cdot, \theta_2)) \tag{12}$$

Then equation (7) and strict concavity of $S(\cdot)$ imply

$$\begin{aligned}
\frac{u(\hat{v} - b, \theta_2)}{u_1(\hat{v} - b, \theta_2)} &= \frac{G(b)}{G'(b)} \\
&= \frac{u(\tilde{v} - b, \theta_1)}{u_1(\tilde{v} - b, \theta_1)} \\
&= \frac{S(u(\tilde{v} - b, \theta_2))}{S'(u(\tilde{v} - b, \theta_2))u_1(\tilde{v} - b, \theta_2)} \\
&> \frac{u(\tilde{v} - b, \theta_2)}{u_1(\tilde{v} - b, \theta_2)}
\end{aligned} \tag{13}$$

Sine $u_1(\cdot, \theta)/u(\cdot, \theta)$ is strictly decreasing, then equation (12) implies $\hat{v} > \tilde{v}$, a contradiction. Therefore we cannot maintain the supposition that there exist $\hat{v} > 0$, such that $b(\tilde{v}, \theta_1) \leq b(\tilde{v}, \theta_2)$. ∎

Next, we consider the case that all bidders have the risk preference, i.e.assume that $\theta_i = \theta$, $i = 1, 2, \cdots, N$.

Denote by $u(y)$ the homogeneous utility function of the bidders. In this special case, a bidder is represented by one-parameter v_i which is his or her auctioned object value. Denote by $b(v)$ the equilibrium bid function with the inverse $\pi(b)$.

$F(b)$ defined in equation 2 can be rewritten as following :

$$F(b) = H(\pi(b)) \tag{14}$$

Hence, we can rewrite equation (3) to be

$$\begin{aligned} G(b) &= G_H(\pi(b)) \\ &= \frac{(N-1)!}{(N-Q-1)!(Q-1)!} \int_0^{\pi(b)} H(v)^{N-Q-1}[1-H(v)]^{Q-1} dH(v) \end{aligned} \tag{15}$$

Where $G_H(v)$ is the probability that at least $N-Q$ of i's rivals' auctioned object value less than v , this probability is given by the distribution of $(N-Q)$th order statistic for a sample of size $N-1$ from the distribution H .

From theorem 1 , the equilibrium bid function $b(v)$ is the solution of the equation:

$$G'_H(\pi(b))\pi'(b)u(\pi(b)-b) = G_H(\pi(b))u'(\pi(b)-b) \tag{16}$$

or

$$G'_H(v)u(v-b(v)) = G_H(v)u'(v-b(v))b'(v) \tag{17}$$

for $v \in (0, \bar{v})$, because of $G_H(v) > 0$, we can also rewrite (17) as

$$\frac{G'_H(v)}{G_H(v)} = \frac{u'(v-b(v))}{u(v-b(v))} b'(v) \tag{18}$$

THEOREM 3. *Consider the case that in which all bidders have the identical strictly log-concave utility $A(\cdot)$ or $B(\cdot)$, where $A(\cdot)$ is more concave than $B(\cdot)$. Let $b_A(v)$ and $b_B(v)$ are the equilibrium bid functions for $A(\cdot)$ and $B(\cdot)$, respectably. Then, $b_A(v) > b_B(v)$ for all v in the domain of $b(\cdot)$.*

Proof. Suppose that $b_A(\tilde{v}) \le b_B(\tilde{v})$ for some $\tilde{v} > 0$. Let $\hat{v} = sup\{0 < v < \tilde{v} | b_A(v) > b_B(v)\}$, thus $0 \le \hat{v} \le \tilde{v}$, and $b_A(\hat{v}) = b_B(\hat{v})$. If $\hat{v} < \tilde{v}$, then $b_A(v) \le b_B(v)$ for all $\hat{v} \le v \le \tilde{v}$, and there exists $\hat{v} < \bar{\bar{v}} < \tilde{v}$ such that $b'_A(\bar{\bar{v}}) \le b'_B(\bar{\bar{v}})$. If $\hat{v} = \tilde{v}$, it's easy to see that $b'_A(\tilde{v}) \le b'_B(\tilde{v})$. So, we can always find some $\bar{\bar{v}}$, $0 < \bar{\bar{v}} \le \tilde{v}$, such that $b_A(\bar{\bar{v}}) \le b_B(\bar{\bar{v}})$ and $b'_A(\bar{\bar{v}}) \le b'_B(\bar{\bar{v}})$.

Let $S(\cdot)$ be strictly increasing and strictly concave function such that

$$A(\cdot) = S(B(\cdot)) \tag{19}$$

then equation (18) and strictly concavity of $S(\cdot)$ imply

$$\begin{aligned} b'_B(\bar{\bar{v}})\frac{B'(\bar{\bar{v}} - b_B(\bar{\bar{v}}))}{B(\bar{\bar{v}} - b_B(\bar{\bar{v}}))} &= \frac{G'_H(\bar{\bar{v}})}{G_H(\bar{\bar{v}})} \\ &= b'_A(\bar{\bar{v}})\frac{A'(\bar{\bar{v}} - b_A(\bar{\bar{v}}))}{A(\bar{\bar{v}} - b_A(\bar{\bar{v}}))} \\ &= b'_A(\bar{\bar{v}})\frac{S'(B(\bar{\bar{v}} - b_A(\bar{\bar{v}})))B'(\bar{\bar{v}} - b_A(\bar{\bar{v}}))}{S(B(\bar{\bar{v}} - b_A(\bar{\bar{v}})))} \\ &< b'_A(\bar{\bar{v}})\frac{B'(\bar{\bar{v}} - b_A(\bar{\bar{v}}))}{B(\bar{\bar{v}} - b_A(\bar{\bar{v}}))} \end{aligned} \tag{20}$$

Hence, by $b'_A(\bar{\bar{v}}) \leq b'_B(\bar{\bar{v}})$ we have

$$\frac{B'(\bar{\bar{v}} - b_B(\bar{\bar{v}}))}{B(\bar{\bar{v}} - b_B(\bar{\bar{v}}))} < \frac{B'(\bar{\bar{v}} - b_A(\bar{\bar{v}}))}{B(\bar{\bar{v}} - b_A(\bar{\bar{v}}))} \tag{21}$$

Since $B'(\cdot)/B(\cdot)$ is strictly decreasing, then equation (21) implies $b_B(\bar{\bar{v}}) < b_A(\bar{\bar{v}})$, a contradiction. Therefore we cannot maintain the supposition that there exists $\tilde{v} > 0$ such that $b_A(\tilde{v}) \leq b_B(\tilde{v})$. ▌

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