Optimal Consumption and Portfolio Choice under Ambiguity for a Mean-reverting Risk Premium in Complete Markets

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This paper explicitly solves, in closed form, the optimal consumption and portfolio choice for an ambiguity averse investor in a Merton-type two assets economy where a risk premium follows a mean-reverting process. The investor's preferences are represented by the recursive multiple priors utility model developed by Chen and Epstein (2002). The investor's utility depends on both intermediate consumption and terminal wealth. Under the assumption of complete markets, I use the martingale method to solve the dynamic optimization problem in continuous time. I find that ambiguity can decrease the optimal consumption-to-wealth ratio, the intertemporal hedging demand and the optimal portfolio allocation, but magnifies the importance of hedging demand in the optimal portfolio allocation. In addition, ambiguity also increases riskless savings.

Key Words: Ambiguity; Mean-reverting; Portfolio choice; Recursive multiple priors.

JEL Classification Numbers: G11, D81, C61.

1. INTRODUCTION

The seminal work of Merton (1971) examines dynamic portfolio choice when investment opportunities are time varying. The optimal portfolio allocation, in general, deviates from the mean-variance efficient allocation that only depends on the current risk-return trade-off. For investors with long horizons, the intertemporal hedging demand is induced to hedge against time variation in the future investment opportunities. Along this active line of research, a large body of literature has analyzed how investors engage in intertemporal hedging of stochastic variation in investment op-

portunities. Among them, several papers assume that the risk premium of a risky asset follows a mean-reverting process in the light of the empirical evidence that stock returns are predictable. For example, Campbell and Viceira (1999), Kim and Omberg (1996) and Wachter (2002) find that mean reversion in the risk premium can substantially increase the optimal demand for stocks when the level of risk aversion is greater than that of logarithmic utility. Another important finding is that hedging demand increases in the length of the investment horizon. This is consistent with the popular advice made by financial professionals that younger investors should invest more than older investors do.

These papers usually postulate that investors completely trust the specified law of motion of asset returns, and that their beliefs are represented by a single probability measure. In this paper, however, I consider a multiple priors model in which an investor's beliefs are characterized by a set of priors. I then examine optimal consumption and portfolio choice when the risk premium follows a mean-reverting (Ornstein-Uhlenbeck) process. Rather than relying on a single subjective prior, the investor considers a set of priors that are relevant to his decision making. Multiplicity in beliefs gives rise to ambiguity and ambiguity aversion. I assume that the utility preferences are represented by recursive multiple priors utility (hereafter RMPU) proposed by Chen and Epstein (2002), which is a continuous-time extension of the multiple priors model axiomatized by Gilboa and Schmeidler (1989).[1] In the RMPU model, utility is defined as the minimum of expected utilities over the set of priors, where the minimum captures the investor's concern about model uncertainty. Thus, the multiple priors model provides one way to distinguish ambiguity from risk. Ambiguity refers to the situation where the investor is uncertain about a set of probability distributions that governs investment opportunities, whereas risk refers to the situation where a probability distribution can be precisely known. The distinction has a well-grounded decision theoretic basis in that it is consistent with the Ellsberg-type behavior.

I assume a Merton-type two-asset economy and constant relative risk aversion (CRRA) utility. The investor's utility is defined over both intermediate consumption and terminal wealth. By means of the martingale method developed by Cox and Huang (1989), I solve the optimal portfolio choice and the consumption-to-wealth ratio explicitly in closed-form, which accommodates easy economic interpretation and intuition. In order to explicitly solve the model, I further assume that the risk premium and the stock return are perfectly negatively correlated. This implies that the markets is complete. In this paper, the assumption of complete markets

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[1] Epstein and Schneider (2003) provide the axiomatic foundation for RMPU in discrete time. Epstein and Miao (2003) apply the RMPU model to study asset prices in general equilibrium.

can be justified in the light of the empirical evidence that the estimated correlation is close to -1 for different samples of data.[2]

The analytical and numerical results suggest that in the recursive multiple priors model, ambiguity decreases both myopic demand and hedging demand when the level of risk aversion exceeds that of logarithmic utility. In particular, ambiguity decreases myopic demand by deteriorating the current risk-return trade-off perceived by investors. Ambiguity mitigates the intertemporal hedging demand via the precautionary savings effect. The precautionary savings motive makes the consumption-to-wealth ratio less responsive to the variation in investment opportunities. As a result, the incentive of hedging against low-consumption states in the presence of unfavorable investment opportunities is tempered under ambiguity aversion. Although ambiguity lowers the magnitude of hedging demand, it magnifies the importance of hedging demand in the optimal portfolio allocation. Thus, an ambiguity-averse investor behaves more "conservatively" not only by investing less in stocks but also by steering his portfolio composition more toward the intertemporal hedging demand.

The multiple priors model is sharply different from the smooth ambiguity model recently proposed and axiomatized by Klibanoff et al. (2005). In the multiple priors model, the two concepts of ambiguity and ambiguity aversion are tied together, whereas in the smooth ambiguity model, ambiguity is reflected by multiple probability distributions while ambiguity attitude is captured by the aversion toward any mean-preserving spread of conditional expected utility induced by the probability distribution over the set of different models. Therefore, the concepts of ambiguity and ambiguity aversion are disentangled in the smooth ambiguity model. One major limitation of the multiple priors model is that the size of the set of priors not only reflects the magnitude of ambiguity but also represents the degree of ambiguity aversion, making the comparative statics results difficult to interpret. Klibanoff et al. (2009) extends the static model of Klibanoff et al. (2005) to the recursive formulation. Hayashi and Miao (2011) further generalize the model to allow for the separation between risk aversion and intertemporal substitution and also provides the axiomatic foundation. Ju and Miao (2011) employ the generalized recursive smooth ambiguity model to study asset prices in endowment economies. Jahan-Pavar and Liu (2011) further extend the model of Ju and Miao (2011) to production economies.

This paper is closely related to several others studying portfolio choice and ambiguity. Maenhout (2004) derives an explicit solution to the optimal portfolio rule when investment opportunities are constant and investors are ambiguity averse. Uppal and Wang (2003) consider investing in multiple assets with different degrees of ambiguity attitude toward different assets.

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[2]See Barberis (2000) for details on parameter values.

Maenhout (2006) studies the effects of ambiguity in the presence of a mean-reverting risk premium. Liu (2010) generalizes Maenhout (2006)'s model to recursive preferences. Using the recursive multiple priors utility model, both Miao (2009) and Liu (2011) investigate dynamic portfolio choice under incomplete information and ambiguity aversion. Chen et al. (2011) use the recursive smooth ambiguity model to examine dynamic portfolio decisions when investors engage in learning about ambiguous predictability.

In a comparison of this paper to Maenhout (2006), several differences are noteworthy. First, Maenhout assumes utility over terminal wealth only. The consumption-savings decision, however, is another important aspect of the multiperiod optimization problem and would also be incorporated into the portfolio choice problem. Further, assuming utility over consumption can allow us to relate hedging demand to the consumption-to-wealth ratio in a way that the analysis assuming utility over terminal wealth cannot. Second, Maenhout uses the robust control approach of Anderson et al. (2003). To obtain a closed-form solution, he further assumes that levels of ambiguity must be scaled by some function of the investor's lifetime utility and the relative risk aversion parameter. The explicit solution to the optimal portfolio choice is derived by solving the Hamilton-Jacobi-Bellman (HJB) equation resulting from the optimization problem. Third, Maenhout finds that with regard to the impact on the optimal portfolio choice, an increase in the degree of ambiguity aversion is equivalent to an increase of the same magnitude in effective risk aversion. This result implies a form of observational equivalence. Ambiguity, therefore, has a second order effect on the optimal portfolio choice in Maenhout's model. In this paper, however, the effect of ambiguity is of the first order.

The remainder of this paper is organized as follows: Section 2 presents the investor's optimization problem and solves the model using the martingale method. Section 3 derives the value function, the optimal consumption-to-wealth ratio and the optimal portfolio choice. Section 3 also discusses the effects of ambiguity and provides economic explanation. Section 4 concludes. Appendices include proofs and some properties of the solution.

2. THE MODEL

2.1. Recursive Multiple Priors Utility

The investor's utility is defined on a terminal wealth W_T (a non-negative random variable which is $\mathcal{F}_T-$ measurable, where $\mathcal{F}_T$ is the information filtration at time T) and a consumption process c. Suppose that the consumption process is nonnegative, progressively measurable with respect to the filtration $\{\mathcal{F}_t\}$ and square integrable with $E[\int_0^T c_t^2 dt] < \infty$. Also assume that terminal wealth satisfies $E(|W_T|^2) < \infty$. The RMPU process

is defined on a set of priors $\mathcal{P}$, which is constructed through $\{\mathcal{F}_t\}$-adapted *density generators* $\theta = (\theta_t) \in \Theta$ satisfying $\sup |\theta_t| \leq \kappa$, where $\kappa \geq 0$. According to Chen and Epstein (2002), this specification is referred to as κ-*ignorance*. It can be shown that regular technical conditions (e.g. Novikov condition, rectangularity, etc.) are satisfied under the κ-ignorance specification (see Section 2.4, Chen and Epstein (2002)). The parameter κ can also be interpreted as an ambiguity aversion parameter. Each density generator θ delivers a $(P, \{\mathcal{F}_t\})$-martingale (z_t^θ)

$$z_t^\theta = \exp\left(-\frac{1}{2}\int_0^t |\theta_s|^2\, ds - \int_0^t \theta_s dB_s\right), \quad 0 \leq t \leq T.$$

where (B_t) is a Brownian motion under P, which is the reference probability measure. The set of priors is constructed as

$$\mathcal{P} = \left\{Q^\theta : \theta \in \Theta, \frac{dQ^\theta}{dP} = z_T^\theta\right\}$$

Because z_t^θ is a martingale, it follows that

$$\left.\frac{dQ^\theta}{dP}\right|_{\mathcal{F}_t} = z_t^\theta$$

In particular, the investor is ambiguous whether (B_t) is a Brownian motion with respect to the investor's information filtration. Girsanov's Theorem implies that $B_t^Q \equiv B_t + \int_0^t \theta_s ds$ is a Brownian motion under any alternative probability measure Q. The multiplicity of the set of priors captures the investor's doubt on the true model governing investment opportunities.

The utility process V_t^Q under each probability measure Q is defined as

$$V_t^Q = E_Q\left[\int_t^T \alpha e^{-\rho(s-t)}u(c_s)ds + (1-\alpha)e^{-\rho(T-t)}u(W_T)\right], \quad 0 \leq t \leq T.$$

where α determines the relative importance of intermediate consumption versus terminal wealth in the utility process, $u(\cdot)$ is the instantaneous utility function, and $\rho > 0$ is the subjective discount rate.

The RMPU process $V_t(c, W_T)$ is defined as:

$$V_t(c, W_T) = \min_{Q \in \mathcal{P}} E_Q\left[\int_t^T \alpha e^{-\rho(s-t)}u(c_s)ds + (1-\alpha)e^{-\rho(T-t)}u(W_T) \mid \mathcal{F}_t\right]$$
$$(1)$$

where the minimization operator is taken over the set of priors. RMPU is denoted by $V_0(c, W_T)$. The standard expected utility model can be obtained

by setting $\kappa = 0$. It can be shown that the utility process $\{V_t(c, W_T)\}$ is dynamically consistent and satisfies the backward stochastic differential equation (BSDE)[3]:

$$dV_t = [-u(c_t) + \rho V_t + \max_{\theta \in \Theta} \theta_t \times \sigma_t^V]dt + \sigma_t^V dB_t, \quad V_T = (1 - \alpha)u(W_T).$$

where the volatility term σ_t^V is endogenous and is part of the complete solution to the BSDE. Further, the optimal density generator can be characterized by the following equation

$$\max_{\theta \in \Theta} \theta_t \times \sigma_t^V = \theta_t^* \times \sigma_t^V, \text{ where } \theta_t^* = \kappa \times sgn(\sigma_t^V) \qquad (2)$$

where $sgn(x_i) = |x_i|/x_i$ if $x_i \neq 0$ and $= 0$ otherwise. It follows from (2) that the equilibrium value of ambiguity is given by $\theta_t^* = \kappa$ if $\sigma_t^V > 0$. In this paper, I consider constant and positive levels of ambiguity ($\theta_t^* = \kappa$ $\forall t$), which is crucial for me to obtain a closed-form solution. Although it is difficult to verify this condition explicitly in analytical form, the condition is indeed verified to be true in the numerical analysis that follows.

The constant relative risk aversion (CRRA) utility function is:

$$u(c) = \begin{cases} \dfrac{c^{1-\gamma}}{1 - \gamma} & \gamma \neq 1 \\ \log(c) & \gamma = 1 \end{cases}$$

where γ is the coefficient of relative risk aversion. Thus, $V_t(\cdot)$ is continuous, increasing and strictly concave.

2.2. The Investor's Optimization Problem

Suppose two assets, a risky asset and a risk-free asset, are available for investments. The price S of the risky asset follows the process

$$\frac{dS_t}{S_t} = \mu_t dt + \sigma dB_t$$

where B is Brownian under P. The risk premium (market price of risk) X is given by

$$X_t = \frac{\mu_t - r}{\sigma}$$

where r is the risk-free interest rate. Following Wachter (2002), we assume that X follows an Ornstein-Uhlenbeck process

$$dX_t = -\lambda_X(X_t - \bar{X})dt - \sigma_X dB_t.$$

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[3]See Chen and Epstein (2002) and El Karoui and Quenez (2001) for rigorous proofs and detailed discussions.

The parameters σ and σ_X are assumed to be constant and strictly positive, and λ_X is assumed to be greater than or equal to zero. The stock price and the state variable (X_t) are perfectly negatively correlated, implying that that the market is complete.

The investor possesses multiple priors that exist in the neighborhood of the reference probability measure P. Under certain distorted probability measure Q, the state processes become

$$\frac{dS_t}{S_t} = \mu_t dt + \sigma(dB_t^Q - \theta dt)$$

and

$$dX_t = -\lambda_X(X_t - \bar{X})dt - \sigma_X(dB_t^Q - \theta dt)$$

where B^Q is Brownian under Q. Ambiguity thereby imputes two distortions to the price process and the mean-reverting process of the risk premium. Under Q, the risk premium is $X - \theta$.

In the multiple-priors model, the investor's optimization problem is given by

$$\max_{c_t, \psi_t} V_0(c, W_T) \tag{3}$$

$$s.t. \quad dW_t = [(\psi_t(\mu_t - r) + r)W_t - c_t]\, dt + \psi_t \sigma W_t dB_t$$

where V_0 is the utility defined in the RMPU (1). When the set of priors collapses to a singleton, that is when $\kappa = 0$, one can obtain the model of Wachter (2002) with $\alpha = 1$. In that case, ambiguity is ignored, and the investor maximizes utility over intermediate consumption.

Since the market is complete, I use the martingale method of Cox and Huang (1989) to solve the dynamic optimization problem. First, I transform the problem into a static one. As in Cox and Huang (1989) and Wachter (2002), I formulate a linear partial differential equation to characterize the solution by the no-arbitrage argument. Finally, I use the guess-verification method to explicitly solve for the optimal consumption-to-wealth ratio and the optimal portfolio choice.

2.3.   The Martingale Method under Ambiguity Aversion

No-arbitrage and market completeness imply that a state price deflator process (state density process) exists and is unique. When the Novikov's condition holds, that is

$$E\left(\exp\left\{\frac{1}{2}\int_0^T X_t^2 dt\right\}\right) < \infty,$$

the state price deflator process ξ is given by

$$\frac{d\xi_t}{\xi_t} = -rdt - X_t dB_t.$$

In the martingale formulation, the solution to Problem (3) is equivalent to the solution of the following static problem:

$$\max_{c,X_T} V_0(c, W_T) \tag{4}$$

subject to the static budget constraint

$$E\left[\int_0^T \xi_t c_t dt + \xi_T W_T\right] \leq W_0. \tag{5}$$

The first-order conditions for the optimal consumption and terminal wealth can be expressed in terms of the supergradients of utility defined over the optimal c_t and W_T.[4] The first-order conditions are given by

$$\alpha e^{-\rho t} u'(c_t^*) z_t^{\theta^*} = y\xi_t$$

$$(1-\alpha)e^{-\rho T} u'(W_T^*) z_T^{\theta^*} = y\xi_T$$

where $p_t(c) = \alpha e^{-\rho t} u'(c_t^*) z_t^{\theta^*}$ and $p_T(c) = (1-\alpha)e^{-\rho t} u'(c_t^*) z_t^{\theta^*}$ are the utility supergradients. The optimal consumption c_t^* and terminal wealth W_T^* are then given by

$$c_t^* = (y\xi_t/z_t^{\theta^*})^{-\frac{1}{\gamma}} e^{-\frac{1}{\gamma}\rho t} \alpha^{\frac{1}{\gamma}} \tag{6}$$

$$W_T^* = (y\xi_T/z_T^{\theta^*})^{-\frac{1}{\gamma}} e^{-\frac{1}{\gamma}\rho T} (1-\alpha)^{\frac{1}{\gamma}} \tag{7}$$

where the constant y is the Lagrange multiplier associated with the optimization problem (4). The multiplicity in priors is captured by the term z^{θ^*}, which distorts the state density process ξ. The Lagrange multiplier y can be derived by substituting the optimal policies c_t^* and W_T^* into the complementary-slackness condition (5), and can be explicitly expressed as

$$y = \left(E\left[\int_0^T (\xi_t)^{\frac{\gamma-1}{\gamma}} (e^{-\rho t} z_t^{\theta^*})^{\frac{1}{\gamma}} dt + (\xi_T)^{\frac{\gamma-1}{\gamma}} (e^{-\rho T} z_T^{\theta^*})^{\frac{1}{\gamma}}\right]/W_0\right)^{\gamma}.$$

Define a new variable $K_t = (y\xi_t)^{-1} z_t^{\theta^*}$. By Ito's lemma, it follows that

$$\frac{dK_t}{K_t} = (r + X_t(X_t - \theta^*)) dt + (X_t - \theta^*)dB_t.$$

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[4]See Chen and Epstein (2002) for the definition of the supergradient of utility.

In the martingale formulation, the value of wealth at time t depends on the state price density and the optimal consumption stream and terminal wealth and is given by

$$W_t = \xi_t^{-1} E_t \left[\int_t^T \xi_s c_s^* ds + \xi_T W_T^* \right].$$

Thus, W_t gives the discounted present value over time of the process (c, W_T), where the discount factor is the state-price density. In deriving a closed-form solution, I follow the approach proposed in Cox and Huang (1989) and construct a candidate function for W_t explicitly in terms of the state variables X_t and K_t. I verify that the function admits a closed-form solution and indeed delivers the solution to the optimal portfolio and consumption decisions. Define

$$F(K_t, X_t, t) \equiv W_t. \tag{8}$$

By the no-arbitrage condition, one can show that $F(K_t, X_t, t)$ satisfies the following partial differential equation (PDE)

$$\mathcal{L}F + \frac{\partial F}{\partial t} + \alpha^{\frac{1}{\gamma}} K_t^{\frac{1}{\gamma}} e^{-\frac{1}{\gamma}\rho t} = \left(\frac{\partial F}{\partial K} K_t (X_t - \theta^*) - \frac{\partial F}{\partial X} \sigma_X \right) X_t + rF \tag{9}$$

where the operator $\mathcal{L}F$ is defined as

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\begin{aligned}
\mathcal{L}F = & \frac{1}{2} \frac{\partial^2 F}{\partial^2 X} \sigma_X^2 + \frac{1}{2} \frac{\partial^2 F}{\partial^2 K} K^2 (X - \theta)^2 - \frac{\partial^2 F}{\partial K \partial X} K (X - \theta) \sigma_X \\
& + \frac{\partial F}{\partial X} (-\lambda_X (X - \bar{X})) + \frac{\partial F}{\partial K} K (r + X(X - \theta))
\end{aligned}
$$

and the boundary condition is

$$F(K_T, X_T, T) = K_T^{\frac{1}{\gamma}} e^{-\frac{1}{\gamma}\rho T} (1 - \alpha)^{\frac{1}{\gamma}}.$$

The first two terms on the left-hand side of Eq.(9) gives the instantaneous drift of wealth X_t and the last term is the optimal consumption. As a result, the left-hand side is the instantaneous expected return on the investor's wealth. On the right-hand side, the term $\left(\frac{\partial F}{\partial K} K_t (X_t - \theta^*) - \frac{\partial F}{\partial X} \sigma_X \right)$ gives the diffusion term of the wealth process. Therefore, the right-hand side also gives the instantaneous expected return on the investor's wealth. The no-arbitrage condition requires that both sides must be equal, leading to Eq.(9). Once I obtain an explicit solution to the differential Eq.(9), the optimal portfolio can be derived by matching the instantaneous variance of the wealth dynamics to that of the portfolio value.

The following proposition shows that if the level of ambiguity is constant, the solution to Eq.(9) can be explicitly characterized by a system of ordinary differential equations (ODEs), which admits a closed-form solution.

PROPOSITION 1. *If θ^* is constant, the general form of the solution to the PDE* (9) *is given by*

$$F(K_t, X_t, t) = K_t^{\frac{1}{\gamma}} e^{-\frac{1}{\gamma}\rho t} H(X_t, t) \tag{10}$$

$$H(X_t, t) = \left[\alpha^{\frac{1}{\gamma}} \int_t^T \hat{H}(X_t, \tau) d\tau + (1-\alpha)^{\frac{1}{\gamma}} \hat{H}(X_t, t) \right] \tag{11}$$

$$\hat{H}(X_t, \tau) \equiv \exp\left\{ \frac{1}{\gamma} \left(A_1(\tau) \frac{X_t^2}{2} + A_2(\tau) X_t + A_3(\tau) \right) \right\} \tag{12}$$

with the boundary conditions

$$A_1(T) = A_2(T) = A_3(T) = 0$$

for a system of ordinary differential equations (ODEs)

$$\frac{dA_1(t)}{dt} = -b_1 A_1^2(t) - b_2 A_1(t) - b_3 \tag{13}$$

$$\frac{dA_2(t)}{dt} = -b_1 A_2(t) A_1(t) - \frac{1}{2} b_2 A_2(t) - b_4 A_1(t) + b_3 \theta^* \tag{14}$$

$$\frac{dA_3(t)}{dt} = -\frac{1}{2} b_1 A_2^2(t) - b_4 A_2(t) - \frac{1}{2} \sigma_X^2 A_1(t)$$
$$- (1-\gamma) \left(\frac{(\theta^*)^2}{2\gamma} + r \right) + \rho \tag{15}$$

where

$$b_1 = \frac{\sigma_X^2}{\gamma} \tag{16}$$

$$b_2 = 2 \left(\frac{\gamma - 1}{\gamma} \sigma_X - \lambda_X \right) \tag{17}$$

$$b_3 = \frac{1 - \gamma}{\gamma} \tag{18}$$

$$b_4 = \frac{\sigma_X \theta^*}{\gamma} + \lambda_X \bar{X} \tag{19}$$

When $\gamma > 1$, it can be shown that $b_2^2 - 4b_1b_3 > 0$.[5] *The explicit solutions to A_1, A_2 and A_3 are given in the proof.*

Proof. See Appendix A.1. ▮

If κ is equal to zero, the solution above is very similar to those in Kim and Omberg (1996) and Wachter (2002). Maenhout (2006) uses the robust control framework and assumes that the degree of ambiguity is scaled by the value function and risk aversion to keep the desired homogeneity property. This assumption results in observational equivalence, that is, an increase in the degree of ambiguity aversion is equivalent to an increase of the same magnitude in effective risk aversion. This is in contrast to the solution shown in this paper. In the multiple priors model, ambiguity directly affects how the investor perceives the risk premium rather than changing the effective risk aversion. As a result, observational equivalence does not hold in this paper.

The corollary below shows that constant ambiguity can be supported as an optimum under certain condition. It is suggested in Eq.(2) that the sign of the diffusion term of the utility process determines the optimal density generator θ^*. In the proof of the following corollary, I also derive the solution to the diffusion term of the utility process.

COROLLARY 1. *If the following condition holds*[6]

$$X_t - \kappa - \frac{\gamma \sigma_X}{1 - \gamma} \frac{\partial \ln H_t}{\partial X_t} > 0$$

then the level of ambiguity is given by $\theta^* = \kappa$ in the optimum.

Proof. See Appendix A.1. ▮

In the numerical analysis below, I have verified that this condition does hold for the parameter values given in Table 1. I at first simulate a large number of sample paths over the horizon $[0, T]$ given a certain risk premium at time 0, where the horizon T is set to 60 months. I then verify the condition in Corollary 1 to be true for every grid point in the simulated sample paths. The parameter values are taken from Barberis (2000), using the methodology of mapping discrete-time parameter values to continuous-time ones proposed by Wachter (2002). The mean reversion parameter λ_X

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[5]This is also the condition for a "normal" solution to exist. See Kim and Omberg (1996)

[6]In the numerical analysis below, I have verified that this condition does hold for a large number of simulated sample paths given the empirical parameter values.

is scaled to be higher than the empirical estimate considered by Wachter (2002), for the condition in Corollary 1 to be numerically verified to be true. A high λ_X implies strong tendency of mean reversion and low variation of the risk premium.

TABLE 1.

Parameter values of the model

Parameter descriptions	Parameter values
Rate of time preference ρ	0.0052
Riskless interest rate r	0.0014
Return volatility σ	0.0436
Volatility of risk premium σ_X	0.0189
Mean reversion parameter λ_X	0.0452
Unconditional mean of risk premium $\bar{X}$	0.0788

This table presents the parameter values used in the numerical examples of the paper. Parameters are calculated based on Barberis (2000). The details of the calculation are provided in Appendix D of Wachter (2002). The parameter value λ_X is the empirical estimate scaled by 2.

3. FURTHER RESULTS AND DISCUSSION

In this section, I first use the key relations in the martingale formulation to derive the value function, the optimal consumption-to-wealth ratio and the optimal portfolio choice. I then perform numerical analysis to examine the impacts of ambiguity on the consumption-to-wealth ratio and the optimal portfolio choice.

3.1. The Value Function and the Optimal Consumption-Wealth Ratio

The value function (the indirect utility function) for RMPU can be derived from the martingale solution obtained in the previous section. Although the value function is a by-product in the martingale solution, it can provide useful insights to understand the effects of ambiguity on the consumption-to-wealth ratio and therefore the consumption-saving trade-off.

Formally, the value function for RMPU is defined by

$$J^\kappa(W_t, X_t, t) = \max_{c, W_T} \min_{Q \in \mathcal{P}} E_Q \left[\int_t^T \alpha e^{-\rho(s-t)} u(c_s) ds + (1-\alpha) e^{-\rho(T-t)} u(W_T) \mid \mathcal{F}_t \right]$$

where the superscript κ means that the value function is associated with the worst-case prior Q^κ.

As shown by Cox and Huang (1989) and Wachter (2002), the value function J^κ and the function $F(K, X, t)$ are related via the following formula,

$$\frac{\partial J^\kappa}{\partial W} = \frac{1}{K} = \frac{1}{F^{-1}(W, X, t)} \tag{20}$$

where the inverse function F^{-1} is implicitly defined by $F^{-1}\left(F(K, X, t), X, t\right) = K$. From (20), one can obtain the value function

$$J^\kappa(W_t, X_t, t) = e^{-\rho t} \frac{W_t^{1-\gamma}}{1-\gamma} \left(H\left(X_t, t\right)\right)^\gamma$$

with the boundary condition

$$J^\kappa(W_T, X_T, T) = (1-\alpha)e^{-\rho T} \frac{W_T^{1-\gamma}}{1-\gamma}$$

where $H\left(X_t, t\right)$ is given in (11).

The value function for expected utility, denoted by $\bar{J}(W_t, X_t, t)$, is given by

$$\bar{J}(W_t, X_t, t) = \max_{c, W_T} E_P \left[\int_t^T \alpha e^{-\rho(s-t)} u(c_s) ds + (1-\alpha)e^{-\rho(T-t)} u(W_T) \mid \mathcal{F}_t \right]$$

where the expectation is taken under the reference probability measure $P \in \mathcal{P}$. It can be shown that $\bar{J}(W_t, X_t, t)$ is given by

$$\bar{J}(W_t, X_t, t) = e^{-\rho t} \frac{W_t^{1-\gamma}}{1-\gamma} \left(\bar{H}\left(X_t, t\right)\right)^\gamma$$

where the function $\bar{H}\left(X_t, t\right)$ is obtained from $H\left(X_t, t\right)$ by setting κ to zero. In general, as shown below, the value function for RMPU is strictly less the value function for expected utility for non-singleton $\mathcal{P}$.

PROPOSITION 2. *(i) Suppose $\kappa > 0$, then the following relation holds:*

$$J^\kappa(W_t, X_t, t) < \bar{J}(W_t, X_t, t).$$

(ii) Given two values of κ, κ_1 and κ_2, and the corresponding sets of priors, $\mathcal{P}_1$ and $\mathcal{P}_2$, suppose $\kappa_2 > \kappa_1$, then the following relation holds:

$$J^{\kappa_2}(W_t, X_t, t) < J^{\kappa_1}(W_t, X_t, t).$$

Proof.   See Appendix A.1   ▊

The definition of RMPU implies that given a consumption process, RM-PU delivers the least utility value among all the utility values over the set of priors. Built on this definition, the above proposition demonstrates that when the optimal feedback controls account for ambiguity, the value function for RMPU is still lower than that for expected utility. In fact, the value function for RMPU is the minimum of all the value functions associated with the corresponding priors in $\mathcal{P}$. This implies that ambiguity (or ambiguity aversion) can cause certain amount of welfare loss to the investor. Further, the higher the magnitude of ambiguity, the more welfare loss he is subject to. Intuitively, high levels of ambiguity represent highly unfavorable investment opportunities perceived by the investor. In general, when states deteriorate, it is impossible for the investor to achieve a even higher welfare level. However, a similar result does not generally hold for the multiplier formulation of the robust control problem (see Maenhout (2006) for an example).[7]

Suppose $\alpha > 0$, it follows from (6), (8) and (10) that the optimal wealth-to-consumption ratio under the worst-case prior Q^κ, denoted by $\left.\frac{W_t}{c_t}\right|_{Q^\kappa}$, is given by

$$\left.\frac{W_t}{c_t}\right|_{Q^\kappa} = \alpha^{-\frac{1}{\gamma}} H\left(X_t, t\right) = \left[\int_t^T \hat{H}\left(X_t, \tau\right) d\tau + \left(\frac{1-\alpha}{\alpha}\right)^{\frac{1}{\gamma}} \hat{H}\left(X_t, t\right)\right]$$

The optimal wealth-to-consumption ratio, denoted by, in the expected utility model is given by

$$\left.\frac{W_t}{c_t}\right|_P = \alpha^{-\frac{1}{\gamma}} \bar{H}\left(X_t, t\right)$$

By Proposition 2, it is straightforward to derive a relationship between the optimal consumption-to-wealth ratio for RMPU and that for expected utility, which is summarized in the following corollary.

COROLLARY 2.   *(i) Suppose $\kappa > 0$, then the optimal consumption-to-wealth ratio in the multiple priors model is strictly less than that in the expected utility model:*

$$\left.\frac{c_t}{W_t}\right|_P > \left.\frac{c_t}{W_t}\right|_{Q^\kappa}$$

---

[7] In the multiplier formulation, the relative entropy appears to be an extra term in the value function, which penalizes the distortion of alternative models relative to the reference model. This term may not lead to a counterpart of the result shown in Proposition 2.

(ii) Given two values of the degree of ambiguity, κ_1 and κ_2, and the corresponding sets of priors, $\mathcal{P}_1$ and $\mathcal{P}_2$, suppose $\kappa_2 > \kappa_1$, the optimal consumption-to-wealth ratios for the two multiple priors models satisfy the following inequality:

$$\left.\frac{c_t}{W_t}\right|_{Q^{\kappa_1}} > \left.\frac{c_t}{W_t}\right|_{Q^{\kappa}}$$

Proof.   See Appendix A.1. ▌

The above results on the consumption-to-wealth ratio can shed light on the consumption-saving trade-off, which is another important aspect in the multiperiod problem. As shown by Wachter (2002), the function $\hat{H}(X_t, \tau)$ gives the value of consumption in τ periods after being scaled by the current consumption. Here, it is shown that ambiguity increases the scaled value of future consumption stream. Thus, the current consumption-to-wealth ratio becomes lower in the multiple priors model than in the expected utility model. Moreover, the ratio decreases when the level of ambiguity increases. The explanation is that as the level of ambiguity (or equivalently, the degree of ambiguity aversion) increases, the investor has a more pessimistic view about investment opportunities, which gives rise to two effects. The income effect tends to decrease the current consumption and increase the future consumption, while the substitution effect tends to increase the current consumption because investment opportunities seem to be less attractive. When $\gamma > 1$, the income effect dominates the substitution effect. As a result, the consumption-to-wealth ratio decreases with ambiguity. This is in contrast to previous results on the effect of risk aversion. Campbell and Viceira (1999) and Wachter (2002) find that the consumption-to-wealth ratio is non-monotonic in risk aversion. This relationship results from the fact that for CRRA utility, the coefficient of relative risk aversion and the elasticity of intertemporal substitution are reciprocals of each other. These findings can be confirmed in Figure 1 and Figure 2 where numerical results are computed using the parameter values in Table 1. Figure 1 reveals that under expected utility the consumption-to-wealth ratio is non-monotonic in γ, however, as shown in Figure 2, the consumption-to-wealth ratio is monotonically decreasing in the level of ambiguity.

To see more clearly the effect of ambiguity on the substitution of the current consumption and the investment in the risky asset, I next derive the optimal portfolio choice and hedging demand taking into account ambiguity.

FIG. 1. Consumption-to-wealth ratio: different levels of risk aversion
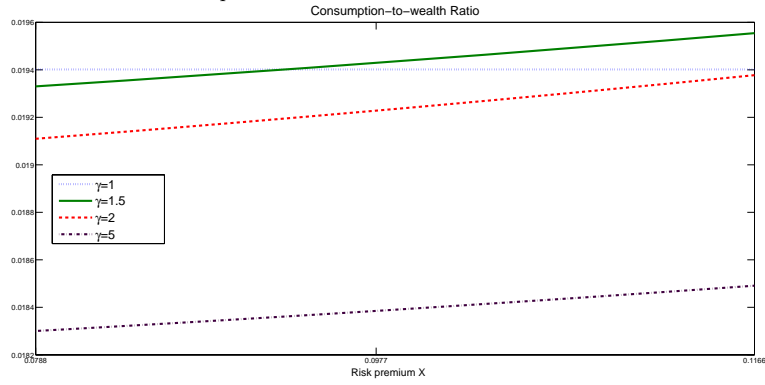
This figure plots the optimal consumption-to-wealth ratio for different levels of risk aversion (γ). The risk premium, X, ranges from $\bar{X}$ to $\bar{X} + 2\sigma_X$. The horizon is 60 months.

FIG. 2. Consumption-to-wealth ratio: different levels of ambiguity
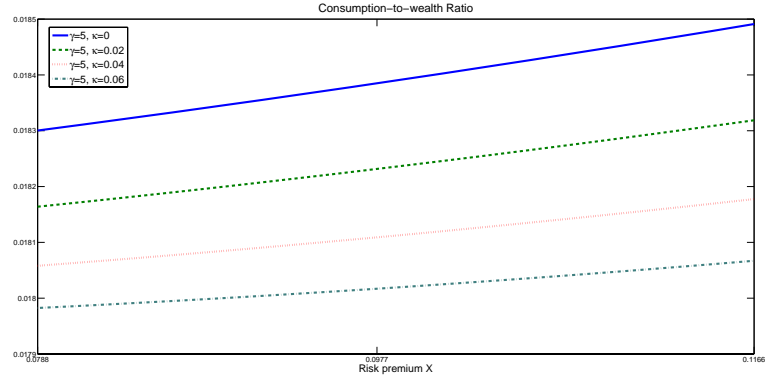
This figure plots the optimal consumption-to-wealth ratio for different levels of ambiguity (κ). The risk premium, X, ranges from $\bar{X}$ to $\bar{X} + 2\sigma_X$. The horizon is 60 months.

3.2.   Optimal Portfolio Choice and Hedging Demand

The optimal portfolio allocation rule should make the instantaneous variance of the portfolio value be equal to that of the optimal wealth, leading to the following equation:

$$\psi_t F \sigma = \frac{\partial F}{\partial K} K_t \left(X_t - \theta^* \right) - \frac{\partial F}{\partial X} \sigma_X$$

where $\theta^* = \kappa$ if the condition in Corollary 1 holds. The equation can be rewritten as

$$\psi_t = \left(\frac{\partial F}{\partial K} \frac{K}{F} \right) \frac{X_t - \kappa}{\sigma} - \left(\frac{\partial F}{\partial X} \frac{1}{F} \right) \frac{\sigma_X}{\sigma}. \tag{21}$$

Substituting the function F given in (10) into (21) delivers a closed-form solution for the optimal portfolio:

$$\psi_t = \frac{\mu_t - r}{\gamma \sigma^2} - \frac{\kappa}{\gamma \sigma} - \frac{\sigma_X}{\gamma \sigma} \times$$

$$\frac{\alpha^{\frac{1}{\gamma}} \int_t^T \hat{H}(X_t, \tau) \left(A_1(\tau) X_t + A_2(\tau) \right) d\tau + (1 - \alpha)^{\frac{1}{\gamma}} \hat{H}(X_t, t) \left(A_1(t) X_t + A_2(t) \right)}{\alpha^{\frac{1}{\gamma}} \int_t^T \hat{H}(X_t, \tau) d\tau + (1 - \alpha)^{\frac{1}{\gamma}} \hat{H}(X_t, t)}$$

$$\tag{22}$$

$$= \frac{X_t - \kappa}{\gamma \sigma}$$

$$- \frac{\sigma_X}{\gamma \sigma} \frac{\alpha^{\frac{1}{\gamma}} \int_t^T \hat{H}(X_t, \tau) \left(A_1(\tau) X_t + A_2(\tau) \right) d\tau + (1 - \alpha)^{\frac{1}{\gamma}} \hat{H}(X_t, t) \left(A_1(t) X_t + A_2(t) \right)}{\alpha^{\frac{1}{\gamma}} \int_t^T \hat{H}(X_t, \tau) d\tau + (1 - \alpha)^{\frac{1}{\gamma}} \hat{H}(X_t, t)}$$

The first term in (22) is myopic demand, which is instantaneously mean-variance efficient and would be optimal if the investor ignored future variation in investment opportunities and disregarded ambiguity as well. The second term captures the effect of ambiguity about the current risk premium. This term reduces myopic demand by decreasing the risk premium perceived by the investor. The first two terms together can be called "ambiguity-adjusted" myopic demand. If σ_X is equal to zero, the investment opportunity set is constant and the optimal portfolio only contains the first two terms. The third term is the optimal fraction of wealth allocated to hedge against risk-premium uncertainty, which is also be affected by ambiguity. However, it is difficult to separate a hedge component that is solely attributed to ambiguity because κ appears implicitly in $\hat{H}(X_t, t)$, $A_1(t)$ and $A_2(t)$. For logarithmic utility ($\gamma = 1$), the intertemporal hedge component vanishes but the investor still hedges for ambiguity about the current risk premium.

To further explore how ambiguity affects the hedge component for risk-premium uncertainty, I first assume that the investor maximizes utility over terminal wealth and then consider the more general case of intermediate consumption. When $\alpha = 0$, the optimal portfolio can be expressed as

$$\psi_t = \frac{\mu_t - r}{\gamma \sigma^2} - \frac{\kappa}{\gamma \sigma} - \frac{\sigma_X}{\gamma \sigma} \left[A_1(t) X_t + A_2(t) \right] \tag{23}$$

In the absence of ambiguity, both A_1 and A_2 have negative sign for $\gamma > 1$.[8] This results in positive hedging demand, which increases the optimal

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[8]See Appendix B in Wachter (2002) for a proof.

demand for the risky asset over myopic demand. The intuition has been well illustrated in, for example, Kim and Omberg (1996) and Campbell and Viceira (1999). Because the risky asset returns and the risk premium are perfectly negatively correlated, shocks to the risk premium are always associated with better payoff of the risky asset and tend to increase the investor's wealth. To hedge against poor investment opportunities that are associated with high marginal utility, the investor wants to hold a portfolio that can produce more wealth when the risk premium is low. The risky asset provides such hedging opportunities. Thus, the optimal demand for the risky asset exceeds myopic demand.

Ambiguity also has an impact on intertemporal hedging of risk-premium uncertainty. It can be shown that ambiguity affects hedging demand only through the term $A_2(t)$.[9] In the expression of $A_2(t)$ (Eq. A.2 in Appendices 5.1), ambiguity affects the first term in the numerator and also gives rise to the second term. Suppose $\sigma_X < \lambda_X$, it immediately follows that $A_2(t)$ is decreasing in absolute value in the level of ambiguity provided that $A_2(t)$ does not switch its sign. It might be helpful to decompose the optimal portfolio into four components:

$$\psi = \frac{\mu_t - r}{\gamma \sigma^2} - \frac{\kappa}{\gamma \sigma} + \text{hedge}^{\text{EU}} + \text{hedge}^{\text{ambiguity}}$$

where hedge^{EU} is the hedge component under expected utility, and $\text{hedge}^{\text{ambiguity}}$ quantifies the impact of ambiguity on hedging demand. It is worth noting that the term $-\kappa/\gamma\sigma$ accounts for the ambiguity adjustment to the current risk premium. On the other hand, due to the negative correlation between risk premium and asset returns, the hedge component $\text{hedge}^{\text{ambiguity}}$ has a negative sign as the density generator also distorts the risk premium process.

When $0 < \alpha \leq 1$, the hedge component for risk-premium uncertainty in (22) takes the form of a weighted average of hedge components in (23) for different horizons, where the weights depend on the value of the function $\hat{H}$ at different horizons. In the more general case of intermediate consumption, the effect of ambiguity on hedging demand seems less obvious from simply inspecting the analytical form of the optimal portfolio, since both the averaged functions and the weights depend on the level of ambiguity. In addition, the strength of the dependence may vary across horizons.

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[9]See Appendices 5.1.

Assuming $\alpha = 1$, the optimal portfolio can be rewritten as

$$
\begin{aligned}
\psi_t &= \frac{\mu_t - r}{\gamma\sigma^2} - \frac{\kappa}{\gamma\sigma} - \frac{\sigma_X}{\gamma\sigma}\int_t^T \frac{\hat{H}(X_t, \tau)\left(A_1(\tau)X_t + A_2(\tau)\right)}{\int_t^T \hat{H}(X_t, \tau)d\tau}d\tau \\
&= \frac{\mu_t - r}{\gamma\sigma^2} - \frac{\kappa}{\gamma\sigma} - \frac{\sigma_X}{\gamma\sigma}\int_t^T \omega(X_t, \tau)\left(A_1(\tau)X_t + A_2(\tau)\right)d\tau.
\end{aligned}
$$

As suggested by Wachter (2002), the optimal portfolio choice problem with utility over consumption can be viewed as a multiperiod problem in which the investor applies the terminal wealth analysis to each future consumption event. The overall portfolio allocation then takes the form of a weighted average in which the averaged terms are the hedging terms for different horizons in the terminal wealth analysis and the weights depend on the values of $\hat{H}$ for different horizons. Unlike the terminal wealth analysis, the impact of ambiguity on hedging demand when utility is defined over consumption is not obvious by only inspecting the analytical form because the nonlinear weights in the hedge term bring in complication. Thus, I provide numerical examples to illustrate how ambiguity affects hedging demand.

TABLE 2.

Results: optimal portfolio allocation

|	$\kappa = 0$			$\kappa = 0.02$			$\kappa = 0.04$			$\kappa = 0.06$		
γ	ψ_{myopic}	$\psi_{optimal}$	$\frac{\psi_{hedge}}{\psi_{optimal}}$	ψ_{myopic}	$\psi_{optimal}$	$\frac{\psi_{hedge}}{\psi_{optimal}}$	ψ_{myopic}	$\psi_{optimal}$	$\frac{\psi_{hedge}}{\psi_{optimal}}$	ψ_{myopic}	$\psi_{optimal}$	$\frac{\psi_{hedge}}{\psi_{optimal}}$
Panel A: $X = \bar{X}$												
1	1.8073	1.8073	0.0000	1.3486	1.3486	0.0000	0.8899	0.8899	0.0000	0.4312	0.4312	0.0000
2	0.9037	1.0496	0.1390	0.6743	0.7895	0.1459	0.4450	0.5293	0.1593	0.2156	0.2689	0.1982
5	0.3615	0.4705	0.2319	0.2697	0.3560	0.2424	0.1780	0.2414	0.2626	0.0862	0.1266	0.3191
10	0.1807	0.2460	0.2650	0.1349	0.1865	0.2767	0.0890	0.1269	0.2987	0.0431	0.0674	0.3591
Panel B: $X = \bar{X} + \sigma_X$												
1	2.0241	2.0241	0.0000	1.5654	1.5654	0.0000	1.1067	1.1067	0.0000	0.8647	0.8647	0.0000
2	1.1204	1.3006	0.1385	0.8911	1.0408	0.1439	0.6617	0.7809	0.1526	0.4323	0.5207	0.1696
5	0.4482	0.5829	0.2313	0.3564	0.4686	0.2394	0.2647	0.3541	0.2525	0.1729	0.2395	0.2781
10	0.2241	0.3048	0.2648	0.1782	0.2454	0.2734	0.1323	0.1859	0.2878	0.0865	0.1264	0.3157

This table shows the optimal portfolio allocation to the risky asset, ambiguity-adjusted myopic demand and the ratio of hedging demand to the optimal demand for the risky asset for different levels of risk aversion and ambiguity aversion. Myopic demand (ambiguity-adjusted) is defined by $\psi_{myopic} = \frac{\mu - r}{\gamma\sigma^2} - \frac{\kappa}{\gamma\sigma}$. Hedging demand is defined by $\psi_{hedging} = -\frac{\sigma_X}{\gamma\sigma}\frac{\partial \ln H}{\partial X}$. The optimal demand for the risky asset is defined by $\psi_{optimal} = \psi_{myopic} + \psi_{hedging}$. Panel A and B present the results for $X = \bar{X}$ and $X = \bar{X} + \sigma_X$ respectively. The horizon is 60 months. The investor is assumed to maximize utility over consumption ($\alpha = 1$).
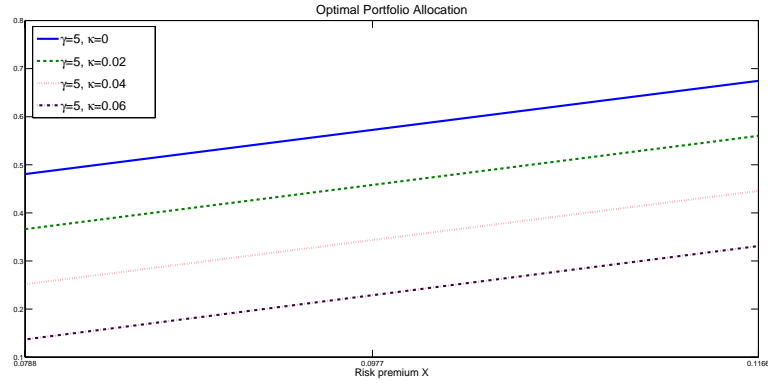
Table 2 presents the optimal portfolio allocation, myopic demand and the fraction of hedging demand in the optimal portfolio allocation for different levels of ambiguity. The risk premium is assumed to equal its long-run mean $(\bar{X})$ and $\bar{X} + \sigma_X$ in, respectively, Panel A and Panel B. The table shows that ambiguity decreases myopic demand and the optimal demand for the risky asset but magnifies the importance of hedging demand in the optimal demand. This result suggests that the effect of ambiguity on hedging demand is proportionally smaller than the effect on myopic demand. In addition, Table 2 reveals that a small level of ambiguity can have a similar impact on the optimal portfolio as a significant increase in the degree of risk aversion. For example, Panel A shows that under expected utility, the optimal portfolio allocation is $\psi = 0.47$ when $\gamma = 5$ and $\psi = 0.24$ when $\gamma = 10$. On the other hand, when the level of ambiguity is $\kappa = 0.04$, the optimal portfolio allocation is $\psi = 0.24$ even when $\gamma = 5$.

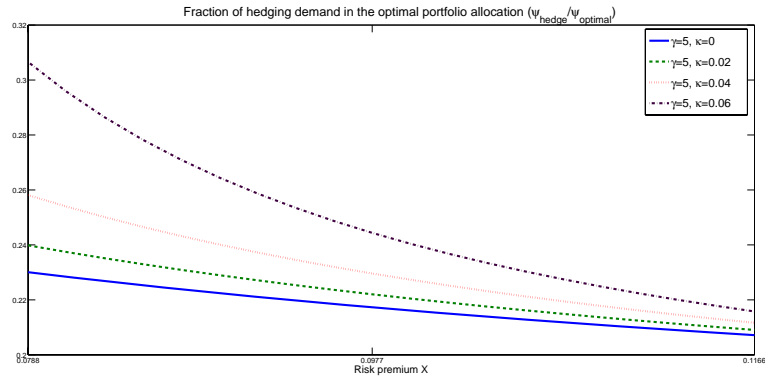**FIG. 3.** Hedging demand: different levels of ambiguity
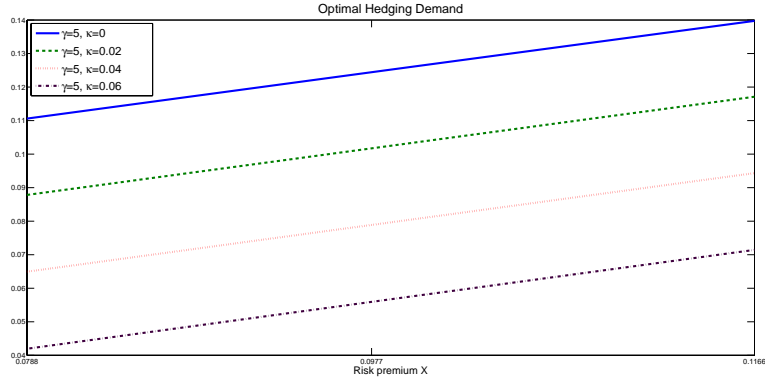
This figure plots the optimal hedging demand for different levels of ambiguity. The risk premium, X, ranges from $\bar{X}$ to $\bar{X} + 2\sigma_X$. The horizon is 60 months.

Figure 3 and 4 plot, respectively, hedging demand and the optimal demand for the risky asset against the risk premium, which ranges from $\bar{X}$ to $\bar{X} + 2\sigma_X$, for different levels of ambiguity. Figure 5 plots the fraction of hedging demand in the optimal demand. Not surprisingly, ambiguity decreases the optimal allocation to the risky asset in all states of the economy. Since ambiguity also reduces the consumption-to-wealth ratio, the amount of wealth invested in the riskless asset increases greatly. In other words, ambiguity generates the substitution of current consumption and investment in the risky asset with riskless savings. Although ambiguity has a significant impact on the optimal portfolio choice, it has almost no effect on the sensitivity of the optimal portfolio allocation and hedging demand to the risk premium. Figure 3 and 4 show that the sensitivity of

FIG. 4. Optimal portfolio demand: different levels of ambiguity

This figure plots the optimal portfolio allocation for different levels of ambiguity. The risk premium, X, ranges from $\bar{X}$ to $\bar{X} + 2\sigma_X$. The horizon is 60 months.

FIG. 5. Fraction of hedging demand in the optimal portfolio demand: different levels of ambiguity

This figure plots the ratio of hedging demand in the optimal portfolio allocation for different levels of ambiguity. The risk premium, X, ranges from $\bar{X}$ to $\bar{X} + 2\sigma_X$. The horizon is 60 months.

the optimal portfolio allocation and hedging demand to the risk premium remains almost the same regardless of the level of ambiguity.

In the expected utility model, Wachter (2002), among others, points out that the consumption-savings decision provides a mechanism to examine hedging demand when investment opportunities are time varying. An increase in the risk premium affects the optimal current consumption relative to wealth in two directions, giving rise to income effect and substitution effect. The income effect allows for more consumption in the light of better investment opportunities, which causes the consumption-to-wealth

ratio to rise. The substitution effect tends to decrease the optimal current consumption relative to wealth because investing seems more attractive. When $\gamma > 1$, the income effect outweighs the substitution effect, causing the consumption-to-wealth ratio to vary positively with investment opportunities. To smooth consumption across different states of the investment opportunity set, the investor wants to hold a portfolio that can generate more wealth when investment opportunities are unfavorable. Moreover, due to the perfect negative correlation between the risk premium and asset returns, the optimal demand for the risky asset exceeds myopic demand.

Turning to the effect of ambiguity on hedging demand when utility is defined over intermediate consumption, I first investigate the optimal consumption-to-wealth ratio as a function of the risk premium. Figure 2 shows that the consumption-to-wealth ratio varies less significantly for an ambiguity-averse investor in response to changes in the risk premium than for an expected utility investor. For instance, the investor with $\gamma = 5$ and $\kappa = 0.06$ has smoother consumption-to-wealth ratios for different values of the risk premium than the investor with $\gamma = 5$ and $\kappa = 0$. The intuition lies in the precautionary savings motive. As the risk premium increases, which represents an improvement of investment opportunities, the ambiguity-averse investor has less willingness to consume but wants to save more to hedge against the future adverse effect of ambiguity on investment opportunities. Since the optimal consumption becomes more stable under ambiguity as investment opportunities change, the incentive of hedging against states with rather low consumption and high marginal utility has been dampened. Thus, ambiguity leads to lower hedging demand.

When investment opportunities are time varying, the optimal portfolio allocation depends on the investment horizon. As the horizon rises, hedging demand increases. For investors with utility over terminal wealth, the horizon effect primarily relies on the derivatives of the functions $A_1(t)$ and $A_2(t)$ with respect to the length of the horizon, as is obvious from (23). Large absolute values of the derivatives of $A_1(t)$ and $A_2(t)$ imply strong horizon effect, all else being equal. With regard to hedging demand, the impact of ambiguity is manifested through the term $A_2(t)$. In Appendices 5.2, I show that ambiguity tempers the horizon effect by decreasing the magnitude of the derivative of $A_2(t)$ with respect to the length of the horizon. This analysis is further confirmed in the numerical example shown in Figure 6. The leftmost panel of Figure 6 presents the case of utility over terminal wealth. For the case of utility over intermediate consumption, it is much more difficult to analytically derive the horizon effect on hedging demand, again since both the averaged functions and the weights in the optimal portfolio formula (22) depend on the level of ambiguity. Nevertheless, the rightmost panel of Figure 6 reveals that the horizon effect is still mitigated under ambiguity when utility is defined over intermediate

FIG. 6.
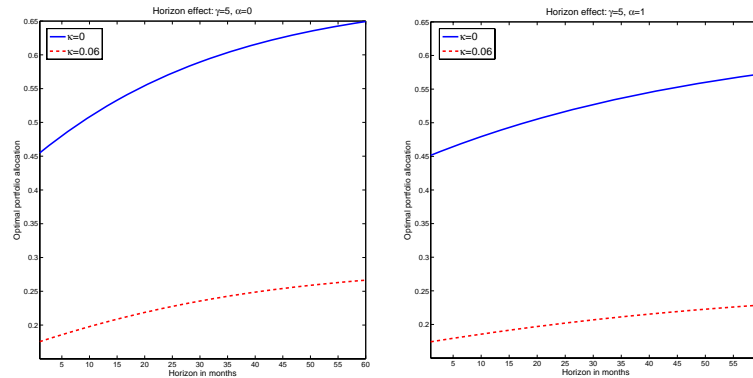
This figure plots the optimal portfolio allocation for different horizons ranging from 1 month to 60 months respectively, under expected utility ($\kappa = 0$) and RMPU ($\kappa = 0.06$). The leftmost panel shows the case of utility defined over terminal wealth only. The rightmost panel shows the case of utility defined over intermediate consumption.

consumption. Thus, an important implication derived from the analysis is that younger investors, if they are ambiguity averse, should reduce aggressiveness in their investments not only by decreasing the total demand for stocks but also by reducing the extent of the horizon dependence.

3.3. Calibrating ambiguity aversion parameter

The last issue to be dealt with is how to calibrate the level of ambiguity given a sample of data. For constant ambiguity aversion, it is straightforward to employ the technique of detection-error probabilities developed by Anderson et al. (2003). Maenhout (2004) applies the same technique to calibrate the preference for robustness for i.i.d. returns.[10] According to Anderson et al. (2003), given a finite sample of data, a reasonable level of ambiguity should render a set of candidate models statistically difficult to distinguish from one another, and thus make the model selection problem obscure to the decision-maker.

Two models P and Q are difficult to distinguish if the probability of rejecting one model mistakenly in favor of the other is high. Specifically, suppose the log of the Radon-Nikodym derivative of the distorted proba-

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[10]Maenhout (2006) further develops a scheme for computing the detection error probabilities for a mean-reverting risk premium in the robust control framework. There the determination of the worst-case model relies on functions of the state variable and the preference for robustness. Maenhout (2006) finds that the detection of model misspecification becomes easier for mean-reverting returns than for i.i.d. returns, which implies less scope for model uncertainty.

bility measure Q with respect to the probability measure P is

$$\eta_{1,t} \equiv \log(\frac{dQ}{dP} \mid \mathcal{F}_t^S) = -\int_0^t \theta_s^* d\hat{B}_s - \frac{1}{2}\int_0^t (\theta_s^*)^2 ds$$

The log of the Radon-Nikodym derivative of the probability measure P with respect to the probability measure Q is

$$\eta_{2,t} \equiv \log(\frac{dP}{dQ} \mid \mathcal{F}_t^S) = \int_0^t \theta_s^* d\hat{B}_s + \frac{1}{2}\int_0^t (\theta_s^*)^2 ds$$

Given that model P is true, the decision maker will reject it mistakenly in favor of model Q based on a finite sample with size N when $\eta_{1,N} > 0$. Conversely, if model Q is correct, it will be rejected erroneously when $\eta_{2,N} > 0$. Assuming an equal prior on each model, the detection error probability $\varepsilon_N(\theta)$ based on a sample size N is defined as

$$\varepsilon_N(\theta) = 0.5\Pr(\eta_{1,N} > 0 \mid P) + 0.5\Pr(\eta_{2,N} > 0 \mid Q)$$

The detection error probability depends on κ in that as κ increases, models P and Q are easier to distinguished statistically from each other and the detection error probability shrinks. For constant ambiguity, it is easy to show that $\varepsilon_N(\kappa)$ is given by

$$\varepsilon_N(\kappa) = \Pr\left(Z < -\frac{\kappa}{2}\sqrt{N}\right)$$

where Z is from standard normal distribution. Table 3 tabulates the detection error probabilities for $\kappa = 0.01, 0.02, \cdots, 0.06$ for the sample period Jan 1952–Dec 1995.

TABLE 3.

Detection error probabilities

κ	0.01	0.02	0.03	0.04	0.05	0.06
$\varepsilon_N(\kappa)$	0.4543	0.4090	0.3652	0.3228	0.2828	0.2451

This table tabulates the detection error probabilities corresponding to different values of the ambiguity aversion parameter κ ranging from 0.01 to 0.06. The sample is drawn from [?] (monthly returns from 1952 to 1995) The detection error probability $\varepsilon_N(\kappa)$ is computed as $\varepsilon_N(\kappa) = \Pr\left(Z < -\frac{\kappa}{2}\sqrt{N}\right)$, where Z is from standard normal distribution and N is the sample size.

4. CONCLUSION AND FUTURE RESEARCH

In this paper, I have explicitly derived closed-form solutions to the optimal portfolio choice and the consumption-to-wealth ratio in a continuous-time setting where an investor has recursive multiple priors utility, and the risk premium follows a mean-reverting process. Markets are assumed to be complete. The analytical and numerical results show that ambiguity generates strong precautionary savings motive and lowers the consumption-to-wealth ratio. With regard to the optimal portfolio choice, ambiguity decreases the optimal demand for the risky asset and hedging demand, while magnifies the relative importance of hedging demand in the optimal portfolio allocation.

This paper can be extended in several directions. For example, one would consider incomplete information and learning. The investor's belief dynamics could enrich the implication of learning and ambiguity on dynamic portfolio choice. Xia (2001) examined the effect of learning about uncertain return predictability in the expected utility framework. Recently, Chen et al. (2011) investigates the impact of ambiguity aversion, allowing for model uncertainty and learning about predictability. This way of extension seems promising. However, explicit solutions are generally difficult to obtain for these models. In addition, how to calibrate ambiguity and quantify its impact would become an interesting but challenging topic.

APPENDIX A

A.1. PROOFS OF PROPOSITIONS, LEMMA AND COROLLARIES

Proof of Proposition 1: To solve the PDE (9), define

$$\hat{F}\left(K_t, X_t, t\right) = K_t^{\frac{1}{\gamma}} e^{-\frac{1}{\gamma}\rho t} \hat{H}(X_t, t)$$

where $\hat{H}$ is given in (12). I show below that $\hat{F}$ satisfies another PDE, the solution of which can be characterized in terms of a system of ODEs. Then I show that the system of ODEs admits a closed-form solution for $\gamma > 1$. Finally, it is shown that the same system of ODEs also characterizes the solution to the PDE (9), and thus the proof is complete.

First, I observe that $\hat{F}$ satisfies the following equation:

$$\mathcal{L}\hat{F} + \frac{\partial \hat{F}}{\partial t} - r\hat{F} = \left(\frac{\partial \hat{F}}{\partial K} K_t(X_t - \theta^*) - \frac{\partial \hat{F}}{\partial X}\sigma_X\right) X_t \qquad \text{(A.1)}$$

with the boundary condition

$$\hat{F}\left(K_T, X_T, T\right) = K_T^{\frac{1}{\gamma}} e^{-\frac{1}{\gamma}\rho T} \ \left(\text{or } \hat{H}(X_T, T) = 1\right).$$

To show this, by plugging

$$\hat{F}\left(K_t, X_t, t\right) = K_t^{\frac{1}{\gamma}} e^{-\frac{1}{\gamma}\rho t} \hat{H}(X_t, t)$$

into (A.1) and matching the coefficients on the constant term, X and X^2, I obtain the ODEs for $A_1(t)$, $A_2(t)$ and $A_3(t)$ given in (13)—(15). Suppose $\gamma > 1$, one can show that $b_2^2 - 4b_1 b_3 > 0$ (see Appendix A, Wachter (2002)). Define

$$\bar{b} = \sqrt{b_2^2 - 4b_1 b_3}.$$

The explicit solution for $A_1(t)$ is standard and has been given in Kim and Omberg (1996). In order to solve for $A_2(t)$, I conjecture that the solution has the form

$$A_2(t) = \frac{1-\gamma}{\gamma} \frac{a_0 + a_1 e^{-\bar{b}(T-t)/2} + a_2 e^{-\bar{b}(T-t)}}{\bar{b}\left[2\bar{b} - \left(b_2 + \bar{b}\right)\left(1 - e^{-\bar{b}(T-t)}\right)\right]}.$$

Substituting $A_2(t)$ into the ODE (14) and matching the coefficients, I obtain

$$
\begin{aligned}
a_0 &= 4b_4 + 2\theta^*(b_2 - \bar{b}) \\
a_1 &= -4\left(2b_4 + \theta^* b_2\right) \\
a_2 &= 4b_4 + 2\theta^*(b_2 + \bar{b}).
\end{aligned}
$$

Rearranging terms gives the solution to $A_2(t)$. Then $A_3(t)$ can be obtained by integration. Thus, the explicit solution to the system of ODEs is given by[1]

$$A_1(t) = \frac{2b_3\left(1 - e^{-\bar{b}(T-t)}\right)}{2\bar{b} - \left(b_2 + \bar{b}\right)\left(1 - e^{-\bar{b}(T-t)}\right)}$$

$$A_2(t) = \frac{1-\gamma}{\gamma} \frac{4\left(\lambda_X \bar{X} + \theta^*\left(\sigma_X - \lambda_X\right)\right)\left(1 - e^{-\bar{b}(T-t)/2}\right)^2 - 2\theta^*\bar{b}\left(1 - e^{-\bar{b}(T-t)}\right)}{\bar{b}\left[2\bar{b} - \left(b_2 + \bar{b}\right)\left(1 - e^{-\bar{b}(T-t)}\right)\right]}$$

(A.2)

---

[1]The explicit solution to A_3 has a very complicated form and thus is not shown here. The solution is available from the author upon request.

$$A_3(t) = \int_t^T \left[-\frac{1}{2} b_1 A_2^2(\tau) - b_4 A_2(\tau) - \frac{1}{2} \sigma_X^2 A_1(\tau) - (1 - \gamma) \left(\frac{(\theta^*)^2}{2\gamma} + r \right) + \rho \right] d\tau.$$

Next, I prove that the solution to Eq. (9) can also be characterized by the ODEs in (13)—(15).

By homogeneity, one can easily show that solving Eq. (9) is equivalent to solving the following differential equation:

$$\frac{\partial H}{\partial t} + \mathcal{G} H + \alpha^{\frac{1}{\gamma}} = 0$$

where the operator $\mathcal{G}$ is defined by

$$\begin{aligned}
\mathcal{G} H &= \frac{1}{2} \frac{\partial^2 H}{\partial^2 X} \sigma_X^2 + \left[-\lambda_X (X - \bar{X}) - \frac{1}{\gamma} (X - \theta) \sigma_X + \sigma_X X \right] \frac{\partial H}{\partial X} \\
&\quad + \left[\frac{1}{2\gamma} \left(\frac{1}{\gamma} - 1 \right) (X - \theta)^2 + \frac{1 - \gamma}{\gamma} r \right] H
\end{aligned}$$

and H is given in (11).

From (A.1) and the definition of $\hat{F}$, I can obtain the following equality

$$\frac{\partial \hat{H}}{\partial t} + \mathcal{G} \hat{H} = 0.$$

Then it follows

$$\begin{aligned}
\frac{\partial H}{\partial t} + \mathcal{G} H &= -\alpha^{\frac{1}{\gamma}} \hat{H}(X_t, t) + \alpha^{\frac{1}{\gamma}} \int_t^T \mathcal{G} \hat{H}(X_t, t) d\tau \\
&= -\alpha^{\frac{1}{\gamma}} \hat{H}(X_t, t) - \alpha^{\frac{1}{\gamma}} \int_t^T \frac{\partial \hat{H}}{\partial t} d\tau \\
&= -\alpha^{\frac{1}{\gamma}} \hat{H}(X_t, t) - \alpha^{\frac{1}{\gamma}} \left[\hat{H}(X_T, T) - \hat{H}(X_t, t) \right] = -\alpha^{\frac{1}{\gamma}}
\end{aligned}$$

with the boundary condition $H(X_T, T) = (1 - \alpha)^{\frac{1}{\gamma}}$.

Thus, we have shown that the function $F(K_t, X_t, t)$ satisfies the PDE (9) with the solution being characterized by (10)—(15). ∎

Proof of Corollary 1: First, I assume that $\theta^* = \kappa$ and then derive a condition for this equality to hold. The value function (the indirect utility function) J^κ has the form:

$$J^\kappa (W, X, t) = e^{-\rho t} \frac{W^{1-\gamma}}{1 - \gamma} (H(X, t))^\gamma.$$

From (6) and (10), it immediately follows that the utility process J_t^κ (J_t^κ abbreviates for $J^\kappa(W, X, t)$) can be written as

$$J_t^\kappa = \frac{c_t^{1-\gamma}}{1-\gamma} G_t$$

where $G_t = e^{-\rho t} H(X_t, t)$. By Ito's lemma and (6), it follows that the consumption process c^* satisfies the SDE

$$\frac{dc_t^*}{c_t^*} = \mu_t^c dt + \sigma_t^c dB_t$$

where μ_t^c and σ_t^c are given by

$$\mu_t^c = \frac{1}{\gamma}(r - \rho) + \frac{1}{2}(1+\gamma)(\sigma_t^c)^2 + \sigma_t^c \theta^*$$

$$\sigma_t^c = \frac{1}{\gamma}(X_t - \theta^*).$$

By Ito's lemma, the process G_t satisfies the SDE

$$\frac{dG_t}{G_t} = \mu_t^G dt + \sigma_t^G dB_t$$

where μ_t^G and σ_t^G are given by

$$\mu_t^G = \frac{1}{G_t}\left(\frac{\partial G_t}{\partial t} - \frac{\partial G_t}{\partial X_t}\lambda_X(X_t - \bar{X}) + \frac{1}{2}\frac{\partial^2 G_t}{\partial X_t^2}\sigma_X^2\right)$$

$$\sigma_t^G = -\frac{1}{G_t}\frac{\partial G_t}{\partial X_t}\sigma_X.$$

Then one can show that the utility process J_t^κ satisfies the following BSDE:

$$dJ_t^\kappa = \mu_t^{J^\kappa} dt + \sigma_t^{J^\kappa} dB_t \quad J_T^\kappa = (1-\alpha)e^{-\beta T}\frac{W_T^{1-\gamma}}{1-\gamma}$$

where $\mu_t^{J^\kappa}$ and $\sigma_t^{J^\kappa}$ are given by

$$\mu_t^{J^\kappa} = J_t^\kappa(1-\gamma)\left(\mu_t^c + \frac{\mu_t^G}{1-\gamma} - \frac{1}{2}\gamma(\mu_t^c)^2 + \mu_t^c \mu_t^G\right)$$

$$\sigma_t^{J^\kappa} = J_t^\kappa(1-\gamma)\left(\sigma_t^c + \frac{\sigma_t^G}{1-\gamma}\right).$$

For the κ-ignorance specification, θ^* is equal to κ when $\sigma_t^{J^\kappa} > 0$. Since $J_t^\kappa(1-\gamma)$ is in positive sign for $\gamma > 1$, $\sigma_t^J > 0$ if and only if $\sigma_t^c + \dfrac{\sigma_t^G}{1-\gamma} > 0$, which is equivalent to the condition given in the corollary. ∎

Proof of Proposition 2: To prove (i), i introduce an auxiliary optimization problem. I consider a candidate probability measure $Q^\theta \in \mathcal{P}$ for a given random variable $\theta \in \Theta$. In this case, the value function is defined by

$$J^\theta(W_t, X_t, t) = \max_{c, W_T} E_{Q^\theta}\left[\int_t^T \alpha e^{-\rho(s-t)}u(c_s)ds + (1-\alpha)e^{-\rho(T-t)}u(W_T) \mid \mathcal{F}_t\right].$$

When Q^θ coincides with P, $J^\theta(W_t, X_t, t)$ is equivalent to $\bar{J}(W_t, X_t, t)$ and gives the value function for expected utility. The value function for RMPU is given by

$$J^{\theta^*}(W_t, X_t, t) = \max_{c, W_T} \min_{Q^\theta \in \mathcal{P}} E_{Q^\theta}\left[\int_t^T \alpha e^{-\rho(s-t)}u(c_s)ds + (1-\alpha)e^{-\rho(T-t)}u(W_T) \mid \mathcal{F}_t\right]$$

with $\theta^* = \kappa$. By the Minimax theorem, one can reverse the order of the minimization operator and the maximization operator and obtain

$$J^{\theta^*}(W_t, X_t, t) = \min_{Q^\theta \in \mathcal{P}} \max_{c, W_T} E_{Q^\theta}\left[\int_t^T \alpha e^{-\rho(s-t)}u(c_s)ds + (1-\alpha)e^{-\rho(T-t)}u(W_T) \mid \mathcal{F}_t\right]$$

which can be rewritten as

$$J^{\theta^*}(W_t, X_t, t) = \min_{Q^\theta \in \mathcal{P}} J^\theta(W_t, X_t, t).$$

Since $\theta^* = \kappa \neq 0$, it follows that the inequality $J^{\theta^*}(W_t, X_t, t) < \bar{J}(W_t, X_t, t)$ must hold.

To prove (ii), notice that $\kappa_2 > \kappa_1$ implies $\mathcal{P}_1 \subset \mathcal{P}_2$. The value functions associated with these two sets of priors are given by

$$J^{\kappa_1}(W_t, X_t, t) = \min_{Q^\theta \in \mathcal{P}_1} J^\theta(W_t, X_t, t) \quad \text{and} \quad J^{\kappa_2}(W_t, X_t, t) = \min_{Q^\theta \in \mathcal{P}_2} J^\theta(W_t, X_t, t)$$

Since $\mathcal{P}_1 \subset \mathcal{P}_2$, it follows that $J^{\kappa_2}(W_t, X_t, t) \leq J^{\kappa_1}(W_t, X_t, t)$. By Corollary 1, we have

$$Q^{\kappa_1} = \arg\min_{Q^\theta \in \mathcal{P}_1} J^\theta(W_t, X_t, t) \quad \text{and} \quad Q^{\kappa_2} = \arg\min_{Q^\theta \in \mathcal{P}_2} J^\theta(W_t, X_t, t).$$

Thus, the inequality strictly holds, that is, $J^{\kappa_2}(W_t, X_t, t) < J^{\kappa_1}(W_t, X_t, t)$. ∎

Proof of Corollary 2: From Proposition 2, it follows $J^\kappa(W_t, X_t, t) < \bar{J}(W_t, X_t, t)$ if $\kappa > 0$. The value function for RMPU and that for expected utility are given by

$$J^\kappa(W_t, X_t, t) = e^{-\rho t} \frac{W_t^{1-\gamma}}{1-\gamma} \left(H(X_t, t)\right)^\gamma \quad \text{and} \quad \bar{J}(W_t, X_t, t) = e^{-\rho t} \frac{W_t^{1-\gamma}}{1-\gamma} \left(\bar{H}(X_t, t)\right)^\gamma.$$

Then we have $H(X_t, t) > \bar{H}(X_t, t)$ for $\gamma > 1$. Thus, one can obtain $(c_t/W_t)_{Q^\kappa} < (c_t/W_t)_P$ for $0 < \alpha \le 1$. In a similar way, it can be shown that $(c_t/W_t)_{Q^{\kappa_1}} > (c_t/W_t)_{Q^{\kappa_2}}$ holds for $\kappa_2 > \kappa_1$. ∎

A.2. PROPERTIES OF DERIVATIVES OF THE FUNCTION A_2

The partial derivative of $A_2(t)$ with respect to θ^* ($\theta^* = \kappa$) is given by

$$\frac{\partial A_2}{\partial \theta^*} = \frac{1-\gamma}{\gamma} \frac{4(\sigma_X - \lambda_X)\left(1 - e^{-\bar{b}(T-t)/2}\right)^2 - 2\bar{b}\left(1 - e^{-\bar{b}(T-t)}\right)}{\bar{b}\left[2\bar{b} - (b_2 + \bar{b})\left(1 - e^{-\bar{b}(T-t)}\right)\right]}.$$

Suppose $\sigma_X < \lambda_X$ and $\gamma > 1$, then $\frac{\partial A_2}{\partial \theta^*} > 0$. To see the effect on $\frac{\partial A_2}{\partial \theta^*}$ as t varies, we compute the derivative of $\frac{\partial A_2}{\partial \theta^*}$ with respect to t:

$$\frac{d\left(\frac{\partial A_2}{\partial \theta^*}\right)}{dt} = \frac{1-\gamma}{\gamma} \frac{-4(\sigma_X - \lambda_X)\bar{b}^2 e^{-\bar{b}(T-t)/2}\left(\bar{b} - b_2 + 2b_2 e^{-\bar{b}(T-t)/2} - (b_2 + \bar{b})e^{-\bar{b}(T-t)}\right)}{\left(\bar{b}\left[2\bar{b} - (b_2 + \bar{b})\left(1 - e^{-\bar{b}(T-t)}\right)\right]\right)^2}.$$

Since $\sigma_X < \lambda_X$ and $\gamma > 1$, b_2 is negative. From (16)—(18), it can be shown $\bar{b} + b_2 > 0$. Then it follows that

$$\bar{b} - b_2 + 2b_2 e^{-\bar{b}(T-t)/2} - (b_2 + \bar{b})e^{-\bar{b}(T-t)} \ \ge\ \bar{b} - b_2 + 2b_2 e^{-\bar{b}(T-t)/2} - (b_2 + \bar{b})$$
$$= -2b_2\left(1 - e^{-\bar{b}(T-t)/2}\right) > 0$$

which implies $\frac{d\left(\frac{\partial A_2}{\partial \theta^*}\right)}{dt} < 0$. This result shows that the effect of ambiguity on the function A_2 is a decreasing function of time t.

To see how ambiguity affects the horizon dependence of the function A_2, I compute the derivative of $A_2(s)$ with respect to s, where s is defined by $s = T - t$. Define

$$C(s) = 2\bar{b} - (b_2 + \bar{b})\left(1 - e^{-\bar{b}s}\right)$$

$$\frac{dA_2(s)}{ds} = \frac{1-\gamma}{\gamma} \left[\frac{4\lambda_X \bar{X} e^{-\bar{b}s/2} \left(1 - e^{-\bar{b}s/2}\right) \left[2\bar{b} - \left(b_2 + \bar{b}\right) \left(1 - e^{-\bar{b}s/2}\right)\right]}{C(s)^2} + \frac{\theta^* B(s)}{C(s)^2} \right]$$

(A.3)

$$B(s) = 2\left(\sigma_X - \lambda_X\right) \left[\bar{b} \left(e^{-\bar{b}s/2} - e^{-3\bar{b}s/2}\right) - b_2 \left(e^{-\bar{b}s/2} + e^{-3\bar{b}s/2}\right)\right]$$
$$+ \left[\frac{8\sigma_X}{\gamma} b_2 + 4(b_2^2 - \bar{b}^2)\right] e^{-\bar{b}s}$$

The first term in the parenthesis of (A.3) is positive because

$$\left[2\bar{b} - \left(b_2 + \bar{b}\right) \left(1 - e^{-\bar{b}s/2}\right)\right] > 0,$$

while the second term is negative since $B(s) < 0$ assuming $\sigma_X - \lambda_X < 0$.

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