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Public Debt and the Dynamics of Economic Growth

Alfred Greiner

Department of Business Administration and Economics
Bielefeld University, Bielefeld, Germany
E-mail: agreiner@wiwi.uni-bielefeld.de

We analyze the effects of public debt on economic growth and its dynamics in a basic endogenous growth assuming that the history of debt affects the primary surplus of the government. The economy with a balanced government budget is characterized by a unique balanced growth path and a condition for saddle point stability is derived. With permanent public deficits there is either no balanced growth path, a unique balanced growth path or there exist two balanced growth paths. The balanced growth path is either stable or unstable. Further, the system may undergo a Hopf bifurcation leading to stable limit cycles.

Key Words: Inter-temporal budget constraint; Balanced budget; Endogenous growth; Dynamics.
JEL Classification Numbers: H60, O23.

1. INTRODUCTION

How does public debt affect the allocation of resources in an economy? A first answer to that question is provided by the Ricardian equivalence theorem stating that a rise in public debt today must be accompanied by an equivalent increase of taxes in the future, in present value terms, so that it is irrelevant whether a given stream of public spending is financed by deficits or by taxes. But the relevance of that theorem for real world economies is rather limited since it is based on very restrictive assumptions, such as lump-sum taxes and lump-sum public spending and no GDP growth.

Nevertheless, the Ricardian equivalence theorem contains an important aspect of public finance namely the inter-temporal budget constraint of the government. Hence, in order to guarantee solvency higher public debt today must go along with a corresponding increase of future primary surpluses of the government. The latter can be achieved either through higher taxes, by a reduction of public spending or by higher tax revenues resulting from

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a rise in GDP. As a consequence, the primary surplus becomes a function that positively depends on public debt.

Bohn (1995) has shown that the inter-temporal budget constraint of the government is fulfilled if the primary surplus of the government rises at least linearly with higher public debt for a model formulated in discrete time.[1] Greiner (2011) has analyzed a continuous time setting where the reaction is described by a linear relationship with a time-varying coefficient. In that case, a positive reaction coefficient on average guarantees that the inter-temporal budget constraint is fulfilled.

As regards the empirical relevance, there is strong evidence that governments do raise the primary surplus as public debt rises. Bohn (1998) has demonstrated this for the US using different estimation strategies. Greiner and Fincke (2009) have found empirical support for a positive reaction of the primary surplus to GDP ratio to higher public debt to GDP ratios for countries of the euro area. Hence, the assumption that the primary surplus positively depends on public debt is not only justified on theoretical grounds but there is empirical evidence for that hypothesis, too.

In this paper we want to contribute to that line of research. Our goal is to analyze how public debt affects the allocation of resources in a basic endogenous growth model where growth results from positive externalities of physical capital as in the seminal paper by Romer (1986). Starting point of our analysis is the model by Greiner (2011a) who studies an endogenous growth model with externalities of capital and elastic labor supply where the primary surplus relative to GDP is a positive linear function of the debt to GDP ratio in order to guarantee sustainability of public debt. Public spending is not productive but raises welfare in the economy. In the paper growth and welfare effects of public debt and deficits are analyzed assuming that the primary surplus of a certain period is a positive function of public debt of the same period.

In contrast to the approach in Greiner (2011a) we posit that the primary surplus does not only depend on the public debt of the current period but that the history of government debt is decisive as regards the determination of the primary surplus. Hence, the primary surplus is a function of cumulated past levels of public debt with exponentially declining weights put on debt further back in time. With this assumption we get a more complex outcome. We demonstrate that stability of the economy with the balanced budget depends on the weight given to more recent levels of public debt. The economy with permanent public deficits may either give rise to no balanced growth path, to a unique balanced growth path or to two balanced growth paths. The paths are either stable or unstable and for a

[1]The fact that governments can go bankrupt show the examples of Russia in 1998, Argentine in 2001 and of Greece in 2010.

certain parameter constellation the dynamic system converges to a limit cycle. Finally, as in Greiner (2011a) the economy with a balanced government budget always experiences a higher long-run growth rate than the economy with permanent public deficits.

In the economics literature, there exist some studies that analyze the effects of public debt with respect to the dynamics of market economies. For example, Futagami et al. (2008) present a model with infinitely lived households and productive public spending where the government has to achieve a certain exogenously given debt to GDP ratio, such as the 60 percent debt criterion in the Maastricht treaty for example. These authors demonstrate that there exist two balanced growth paths in their model with one being a saddle point and the other being asymptotically stable. However, they do not study how different debt policies affect the local dynamics and they do not prove that limit cycles may characterize the dynamics of the economy.

For models where the household sector is characterized by an OLG structure, two interesting contributions are presented by Bräuninger (2005) and by Yakita (2008). Bräuninger analyzes an OLG economy and demonstrates that for a fixed deficit ratio there exist two balanced growth paths as long as the deficit ratio is below a certain threshold. As the deficit ratio rises the growth rate declines and once the critical deficit ratio is exceeded sustained growth does not occur any longer. Yakita presents and studies an OLG model with productive public capital and demonstrates that there exists an upper bound for the level of public debt that is compatible with a sustainable debt policy. This critical level is determined by the stock of public capital and once the critical value of public debt relative to public capital is exceeded a sustainable debt policy of the government is excluded. Taking into account that the stock of public capital determines the level of GDP in his model, this result makes sense from an economic point of view.

The rest of the paper is organized as follows. The next section presents the structure of our economy and defines a balanced growth path. Section 3 studies the economy with a balanced government budget and section 4 analyzes the case of permanent public deficits. Section 5, finally, concludes.

2. STRUCTURE OF THE MODEL

In this section, we present the structure of our growth model. The household sector consists of many identical households which are represented by one household that maximizes the discounted stream of utility. As regards the utility function we adopt the function presented by Benhabib and Farmer (1994). Thus, utility arises from per-capita consumption, $C(t)$, and the household has disutility from labour, $L(t)$. The household maximizes its utility over an infinite time horizon subject to its budget constraint, taking factor prices as given. Thus, the maximization problem of the household can be written as[2]

$$\max_{C,L} \int_0^\infty e^{-\rho t} \left(\ln C - L^{1+\gamma}/(1+\gamma)\right) dt, \tag{1}$$

subject to

$$(1-\tau)\left(wL + rK + r_B B + \pi\right) = \dot{W} + C + \delta K. \tag{2}$$

$\rho \in (0,1)$ is the household's rate of time preference, $\gamma \geq 0$ gives the inverse of the Frisch labour supply elasticity. The parameter $\delta \in (0,1)$ is the depreciation rate of capital, w denotes the wage rate and r is the return to capital and r_B is the interest rate on government bonds. The variable $W := B + K$ gives wealth which is equal to public debt, B, and capital, K, and π gives possible profits of the productive sector, the household takes as given in solving its optimization problem. Finally, $\tau \in (0,1)$ is the constant income tax rate. The dot gives the derivative with respect to time.

A no-arbitrage condition requires that the return to capital equals the return to government bonds yielding $r_B = r - \delta/(1-\tau)$. This assures that the household is indifferent whether its savings are used for investment or for financing government expenditures because both types of assets yield the same return. Thus, the budget constraint of the household can be written as

$$\dot{W} = (1-\tau)\left(wL + rW + \pi\right) - \delta W - C. \tag{3}$$

Necessary optimality conditions are given by

$$C = w\,(1-\tau)\,L^{-\gamma} \tag{4}$$

$$\dot{C} = C\,(1-\tau)\,r - C\,(\rho+\delta) \tag{5}$$

If the transversality condition $\lim_{t\to\infty} e^{-\rho t} W/C = 0$ holds, which is the standard household's no-Ponzi game condition, that is fulfilled for a time path on which assets grow at the same rate as consumption, the necessary conditions are also sufficient.

2.1. The productive sector

The productive sector is represented by one firm which behaves competitively and which maximizes static profits. The production function of the firm is given by,

$$Y = K^\alpha \bar{K}^\xi L^\beta, \tag{6}$$

[2]From now on we omit the time argument t if no ambiguity arises.

with $\alpha \in (0,1)$ the capital share, $\beta \in (0,1)$ the labour share and $\alpha+\beta \leq 1$. Y is output and $\bar{K}$ represents the average economy-wide level of capital and we assume constant returns to capital in the economy, i.e. $\alpha+\xi=1$.

Using $\alpha+\xi=1$ and that $K=\bar{K}$ in equilibrium, profit maximization gives

$$r = \alpha L^{\beta} \tag{7}$$
$$w = \beta L^{\beta-1} K \tag{8}$$

2.2. The government

The government in our economy receives tax revenues from income taxation and has revenues from issuing government bonds and it finances public spending, G, that are a pure waste of resources, i.e. it neither enhances welfare nor raises production possibilities on the economy. The reason for that assumption is that we are interested in effects of government debt per se, meaning that we neglect any distortions resulting from variations in government spening going along with changes in public debt The period budget constraint of the government describing the accumulation of public debt in continuous time is given by:

$$\dot{B} = r_B B(1-\tau) - S, \tag{9}$$

where S is government surplus exclusive of net interest payments.

Public debt could be positive or negative with the latter implying that the government would be a lender to the private sector. In this paper, however, we limit our analysis to the case $B \geq 0$, i.e. we do not assume that the government is a net lender. This is definitely the more relevant case for real world economies.

The inter-temporal budget constraint of the government is fulfilled if

$$B(0) = \int_0^{\infty} e^{-\int_0^{\mu}(1-\tau)r_B(\nu)d\nu} S(\mu)d\mu \Leftrightarrow \lim_{t\to\infty} e^{-\int_0^{t}(1-\tau)r_B(\mu)d\mu} B(t) = 0 \tag{10}$$

holds, which is the no-Ponzi game condition.

Now, assume that the government runs into debt today. Then, in order to meet its inter-temporal budget constraint, it has to repay that amount in the future in present value terms. This implies that the primary surplus must rise as public debt increases. But, nevertheless, the government has some discretionary scope in setting the primary surplus so that assuming that public debt is the only determinant of the primary surplus would be too short-sighted. We posit that it is the level of GDP that determines the primary surplus, besides public debt, which seems to be reasonable.

In addition, we assume that the history of government debt is decisive as regards the determination of the primary surplus. We do so because governments will make their budget plans dependent on how the public debt has evolved over time. Thus, a continuous rise in public debt relative to GDP may affect the budget plans of a government differently compared to a time path of the public debt ratio that has also shown a tendency to decline in the past. Thus, Legrenzi and Milas (2011) find empirical evidence that governments take corrective actions only when debt exceeds a certain threshold that depends on the history of government debt. Thus, it is the history of public debt that is decisive as regards the determination of the primary surplus. Therefore, we posit that the primary surplus relative to GDP depends on cumulated past debt with an exponentially declining weight put on investment flows further back in time. Further, in all empirical estimations performed by Greiner and Fincke (2009) it is the debt ratio of the previous period that has a statistically significant effect on the primary surplus to GDP ratio.

Therefore, the equation determining the primary surplus is written as,

$$S = \phi Y + \int_{-\infty}^{t} e^{\kappa(\mu - t)} \, \psi \, B(\mu) d\mu, \tag{11}$$

with $\phi \in \mathbb{R}$ the parameter determining whether a rise in GDP goes along with a higher or lower level of the primary surplus. The parameter $\psi \in \mathbb{R}_{++}$, which is the average of the reaction parameter that may vary over time, determines how strong the primary surplus reacts to cumulated past levels of public debt with an exponentially declining weight put on debt further back in time. The parameter κ determines how strong more recent levels of public debt affect the primary surplus where the influence of more recent public debt is the stronger the higher κ. It should be noted that for $\kappa \to \infty$ we get the limit case where only public debt of the current period affects the primary surplus.

In appendix A.1 we show that a positive reaction coefficient ψ guarantees a sustainable debt policy. But, it must be pointed out that a sustainable debt policy is only given if the debt ratio converges to a constant value in the long-run. If the debt to GDP ratio continuously increased, the primary surplus to GDP would also have to rise permanently. That, however, is not possible because the primary surplus must be financed out of GDP so that the theoretical upper bound of the primary surplus to GDP ratio is 1 (for details see Greiner, 2011).[3] Thus, an economy where the debt ratio does

[3]The actual political upper bound will be definitely much smaller. Yakita (2008) shows that the sustainable level of public debt depends on the stock of public capital if public capital is productive.

not converge to a constant in the long-run implies that the government does not fulfill its inter-temporal budget constraint.

In the next subsection we define equilibrium conditions and the balanced growth path.

2.3. Equilibrium conditions and the balanced growth path

Before we analyze our model we give the definition of an equilibrium and of a balanced growth path. An equilibrium allocation for our economy is defined as follows.

Definition 2.1. An equilibrium is a sequence of variables $\{C(t), K(t), B(t)\}_{t=0}^{\infty}$ and a sequence of prices $\{w(t), r(t)\}_{t=0}^{\infty}$ such that
(a) equations (3), (4) and (5) hold,
(b) equations (7) and (8) hold and
(c) equations (9) and (11) hold.

In Definition 2.2 we define a balanced growth path.

Definition 2.2. A balanced growth path (BGP) is a path such that the economy is in equilibrium and such that consumption and capital grow at the same strictly positive constant growth rate, i.e. $\dot{C}/C = \dot{K}/K = g$, $g > 0$, $g = constant$, and either
(i) $\dot{B} = 0$ or
(ii) $\dot{B}/B = \dot{C}/C = \dot{K}/K = g$.

Definition 2 shows that we consider two different budgetary rules. Rule (i) is the balanced-budget rule where the government has at each point in time a balanced budget. Rule (ii) describes a situation which is characterized by public deficits where public debt grows at the same rate as all other economic variables in the long-run. But, since the government sets the primary surplus according to equation (11), it does not play a Ponzi game in this case but fulfills the inter-temporal budget constraint.[4]

To study our model, we introduce the new variable

$$R = \int_{-\infty}^{t} e^{\kappa(\mu - t)}\, \psi\, B(\mu) d\mu$$

giving the reaction of the primary surplus to cumulated past public debt. Further, we note that (4) and (8) imply $L = (C/K)^{1/(\beta-\gamma-1)} \upsilon^{1/\beta}$, with $\upsilon = (\beta(1-\tau))^{\beta/(1-\beta+\gamma)}$. Thus, in equilibrium our economy is completely

[4] Of course, GDP grows at the same rate as capital and consumption on a BGP.

described by the following differential equations,

$$\frac{\dot{C}}{C} = (1-\tau)\alpha\upsilon\,(C/K)^{-\beta/(1-\beta+\gamma)} - (\rho+\delta), \tag{12}$$

$$\frac{\dot{K}}{K} = \upsilon\,(C/K)^{-\beta/(1-\beta+\gamma)}\,(1-\tau+\phi) - (C/K) - \delta + R/K, \tag{13}$$

$$\begin{aligned}\frac{\dot{B}}{B} &= (1-\tau)\alpha\upsilon\,(C/K)^{-\beta/(1-\beta+\gamma)} \\ &- \phi\upsilon\,(C/K)^{-\beta/(1-\beta+\gamma)}\,(K/B) - \delta - R/B,\end{aligned} \tag{14}$$

$$\frac{\dot{R}}{R} = \psi\,(B/R) - \kappa \tag{15}$$

where we used $r_B = r - \delta/(1-\tau)$. The initial conditions with respect to capital, K_0, public debt, B_0, and the reaction to cumulated past debt, R_0, are assumed to be given while consumption can be chosen by the household at time $t = 0$. It should be noted that (13) gives the resource constraint of the economy.

It should be mentioned that equation (12) is obtained from equation (5) where equations (4), (7) and (8) have been used. Equation (13) is obtained by combining the budget constraint of the household, (3), with the government budget constraint, (9) and (11), where again (7) and (8) have been resorted to. Equation (14), finally, is obtained from (9), where equations (4), (7) and (8) as well as the production function (6) have been used. The last equation, (15), finally is obtained by differentiating R with respect to time.

To analyze our economy around a BGP we define the new variables $c := C/K$, $b := B/K$ and $z := R/K$. Differentiating these variables with respect to time leads to a three dimensional system of differential equations given by,

$$\dot{c} = c\left(c - c^{-\beta/(1-\beta+\gamma)}\upsilon((1-\tau)(1-\alpha)+\phi) - \rho - z\right) \tag{16}$$

$$\begin{aligned}\dot{b} &= \\ &b\left(c - c^{-\beta/(1-\beta+\gamma)}\upsilon((1-\tau)(1-\alpha)+\phi) - \phi\upsilon c^{-\beta/(1-\beta+\gamma)}b^{-1} - z(1+b^{-1})\right)\end{aligned} \tag{17}$$

$$\dot{z} = z\left(\psi(b/z) - \kappa + c + \delta - z - c^{-\beta/(1-\beta+\gamma)}\upsilon(1-\tau+\phi)\right) \tag{18}$$

In the next section we study the balanced budget scenario.

3. BALANCED GOVERNMENT BUDGET

To model the balanced budget scenario we set $\phi = 0$ and ψ and κ are set such that $\dot{B} = 0$ holds. Setting $\dot{B} = 0$ implies that the ratio of public debt to private capital equals zero on the BGP, i.e. $b^\star = 0$ holds.[5] A constant value of public debt implies $R = B(r(1-\tau)-\delta) = const.$ so that $\dot{R} = 0$ and $z^\star = 0$ hold along the BGP. The condition $\dot{R} = 0$ gives $\psi B = \kappa R$ which, together with $R = B(r(1-\tau)-\delta)$, leads to $\psi = \kappa(r(1-\tau)-\delta)$. With $b^\star = z^\star = 0$ the economy is completely described by equation (16) and a rest point of that equation gives a balanced growth path for the economy. Proposition 1 shows that there exists a unique BGP for the balanced budget scenario under a slight additional assumption.

Proposition 1. *Assume that the rate of time preference and the depreciation rate are sufficiently small. Then, there exists a unique balanced growth path if the government runs a balanced budget.*

Proof. A rest point of the equation $\dot{c}/c = c - c^{-\beta/(1-\beta+\gamma)}\upsilon(1-\tau)(1-\alpha) - \rho$ gives a BGP. It is easily seen that $\lim_{c\to 0}\dot{c}/c = -\infty$ and $\lim_{c\to\infty}\dot{c}/c = +\infty$ hold. Further, we have $\partial(\dot{c}/c)/\partial c > 0$ so that there exists a unique $c^\star$ that solves $\dot{c}/c = 0$. ▮

Proposition 1 shows that there exists a unique balanced growth path if the government runs a balanced budget. Hence, multiplicity of long-run growth rates can be excluded in this case. It should be noted that the assumption of a sufficiently small rate of time preference and of the depreciation rate of capital must be made because we can only prove that there exists a unique rest point of the system (16)-(18) but we cannot show that this rest point implies a positive growth rate.

In order to analyze stability of the BGP we first state the following lemma.

Lemma 1. *The eigenvalues of the Jacobian evaluated at the balanced growth path are given by $\lambda_1 = \partial\dot{c}/\partial c > 0$, $\lambda_2 = -g$, $\lambda_3 = \rho - \kappa$.*

Proof. See appendix A.2. ▮

Given this lemma it is easy to derive a condition assuring that the BGP is saddle point stable. This is done in proposition 2.

Proposition 2. *The balanced growth path is saddle point stable if and only if $\kappa > \rho$ holds.*

[5]The $\star$ denotes BGP values.

Proof. Follows immediately from lemma 1. ▮

Proposition 2 states that the government must put a sufficiently high weight on more recent levels of public debt when setting the primary surplus so that saddle point stability is given. We should also like to point out that on the BGP we have $\rho = (r(1-\tau)-\delta)-g$, i.e. the rate of time preference is equal to the net return on wealth minus the balanced growth rate. Thus, we can state that the parameter κ must exceed the difference between the net return on wealth and the growth rate for the model economy to be stable. We also recall that the balanced budget scenario implies $\psi = \kappa(r(1-\tau)-\delta)$ so that a high value for κ implies a high value for ψ, too. Hence, saddle point stability is given if the government puts a sufficiently high weight on stabilization, i.e. if the primary surplus reacts strongly to past public debt, and if the primary surplus reacts soon to higher public debt, i.e. the weight given to more recent levels of public debt in setting the primary surplus must be large.

In the next section, we analyze the economy with permanent public d-eficits.

4. PERMANENT PUBLIC DEFICITS

In the following we limit the analysis to the case $\upsilon((1-\tau)(1-\alpha)+\phi) > 0$. From an economic point of view this states that the government can reduce the primary surplus as GDP grows but that effect must not be too large, i.e. ϕ may become negative but its absolute value must be smaller than $(1-\tau)(1-\alpha)$. For the model with permanent public deficits we then see that the long-run dynamics is more complex than for the balanced budget case. Proposition 3 gives the different possible long-run outcomes.

PROPOSITION 3. *Assume that the rate of time preference and the depreciation rate are sufficiently small. Then, the following holds true:*

(i) For $(\psi/\rho) + \rho + \delta < \kappa$ *and* $\phi < 0$ *there exists no balanced growth.*

(ii) For $(\psi/\rho) + \rho + \delta < \kappa$ *and* $\phi > 0$ *there exists a unique balanced growth path.*

(iii) For $(\psi/\rho) + \rho + \delta > \kappa$ *and* $\phi < 0$ *there exists a unique balanced growth path.*

(iv) For $(\psi/\rho) + \rho + \delta > \kappa$ *and* $\phi > 0$ *there exists either no balanced growth path or*

there exists two balanced growth paths.

Proof. See appendix A.3. ▮

Proposition 3 demonstrates that the reaction of the government to higher public debt is crucial as regards existence of a BGP, given the structural parameters ρ and δ. If the parameter ψ is relatively small given a certain value for κ, such that $(\psi/\rho) + \rho + \delta < \kappa$ (case (i) and (ii)), the primary surplus must increase as GDP rises so that a BGP exists, i.e. $\phi > 0$ must hold. Otherwise, i.e. for $\phi < 0$, no BGP exists since fiscal policy is too loose in the sense that the primary surplus declines with a rising GDP and the reaction of the primary surplus to cumulated past debt is small, too. Thus, with a governmental policy that does not pay sufficient attention to stabilizing public debt sustained growth is not possible.

If ψ is relatively large given a certain value for κ, so that $(\psi/\rho)+\rho+\delta > \kappa$ holds (case (iii) in proposition 3), there exists a unique BGP for $\phi < 0$. This holds because the reaction of the government to higher cumulated public debt, ψ, is relatively large so that a BGP can exist even if the primary surplus declines as GDP rises. If the reaction of the government to cumulated public debt is relatively large and the primary surplus rises with GDP, $\phi > 0$, case (iv), there exists either no BGP or two BGPs. If a BGP does not exist, the government puts too high a weight on stabilizing public debt in the sense that the primary surplus rises as GDP increases and it strongly reacts to cumulated past levels of public debt. Hence, a situation may exist where the government puts too high a weight on stabilizing debt implying that the growth rate of public debt falls short of the growth rates of capital and consumption. In this case, reducing the reaction to cumulated past debt, i.e. lowering ψ, or reducing ϕ, giving the increase of the primary surplus as GDP rises, can lead to endogenous growth. However, in this situation there then exist two BGPs, the stability of which remains to be determined, meaning that the economy is globally indeterminate.

Before we study stability of the BGP we first derive a lemma that gives the relation between the balanced growth rate and the debt ratio. That is the contents of lemma 2.

LEMMA 2. *Assume that there exists at least one BGP. Then, on the BGP the following relation holds:*

$$\frac{dg}{db} < 0$$

Proof. See appendix A.4. ∎

Lemma 2 shows that the balanced growth rate is the smaller the higher the debt ratio. The economic mechanism behind that result is that a higher debt ratio implies that more resources in the economy must be used for the debt service. As a consequence the shadow price of wealth is smaller which gives a lower incentive to save and invest. Therefore, the balanced growth rate and the debt ratio are negatively correlated. Proposition 4 gives an immediate consequence of that result.

PROPOSITION 4. *The long-run growth rate in the economy with permanent public deficits is smaller than in the economy with a balanced government budget.*

Proof. Follows immediately from lemma 2. ∎

That proposition which is an immediate consequence of lemma 2 states that economies with a balanced government budget will always experience a higher growth rate than economies with permanent deficits that are such public debt grows at the same rate as capital and GDP. The reason is that the shadow price of wealth is smaller if the government runs deficits because it implies that less of the household's savings is used for the formation of productive private capital. That leads to a lower return to capital, r, and also to a lower labour supply, L, so that there will be less saving and less investment when the government requires a certain part of the savings in the economy for its debt service.

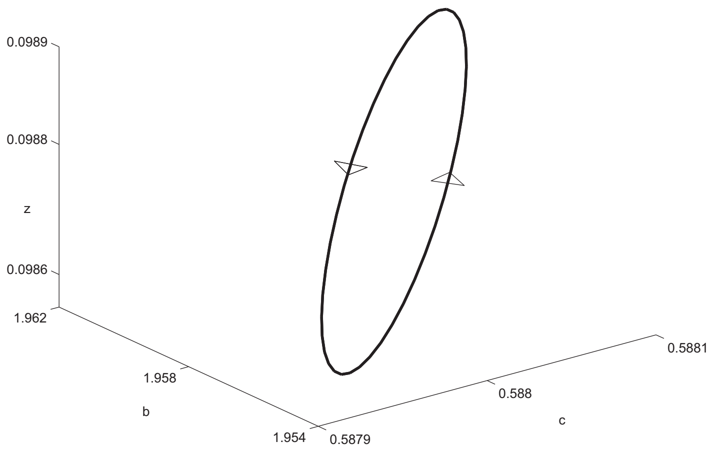
In order to study stability of the BGP for the situation with permanent public deficits we resort to numerical examples. We do so because the analytical model turns out to be too complex to gain insight into its stability properties. We should also like to point out that we do not intend to perform a calibration exercise that replicates real economies. The goal of our simulations is to get additional insight into the qualitative behaviour of our model economy.

As regards the parameter values we set the capital share to 30 percent, $\alpha = 0.3$, and the labour share is set to 70 percent, $\beta = 0.7$. The depreciation rate of capital is 7.5 percent, $\delta = 0.075$, and the income tax rate is 25 percent, $\tau = 0.25$. The rate of time preference is 5 percent, $\rho = 0.05$, and the inverse of the labour supply elasticity is set to $\gamma = 0.15$ which is in the range of the values considered by Benhabib and Farmer (1994) for example. For a further discussion of plausible values for the labour supply elasticity we refer to Benhabib and Farmer (1994), p. 32-33.

First, we consider the case (ii) in proposition 3 where we set $\kappa = 0.25$, $\phi = 0.001$ and $\psi = 0.003$. With these parameter values the system is unstable with two positive real eigenvalues and one negative. When we increase the parameter ψ implying that the reaction to cumulated public debt becomes larger, the qualitative outcome does not change. The largest value of ψ we considered was $\psi = 0.006$ because for values of ψ larger than 0.006 gives case (iv) of proposition 3.

For case (iii) in proposition 3 we set $\kappa = 0.01$, $\phi = -0.001$ and $\psi = 0.0039$. With these parameter values the economy is stable with one pair of complex conjugate eigenvalues and one positive real eigenvalue. If we continuously decrease the parameter ψ implying that the government puts less weight on stabilizing public debt, we realize that for $\psi = \psi_{crit} = 3.712708 \cdot 10^{-3}$ two eigenvalues are purely imaginary and a Hopf bifurcation occurs.[6] The Hopf bifurcation gives rise to stable limit cycles since the first Lyapunov coefficient L_1 is negative,[7] $L_1 = -5.36271 \cdot 10^{-2}$. The limit cycles exist for an interval of values of ψ which are smaller than ψ_{crit}. From an economic point of view the emergence of limit cycles means that the economy is not characterized by a constant growth rate at which all variables grow but the growth rates are cyclically fluctuating over time. If ψ is further decreased the economy becomes unstable. Figure 1 shows the limit cycle in the $(b - z - c)$ phase space where the orientation is as indicated by the arrows.[8]

FIG. 1. Limit cycle in the $(b - z - c)$ phase space with $\psi = 3.7127 \cdot 10^{-3}$.



In case (iv) there exist two BGPs where the one with the higher growth rate is unstable (two positive and one negative real eigenvalue) while the BGP yielding the lower balanced growth rate is stable (one positive real eigenvalue and one complex conjugate with negative real part) for $\kappa = 0.01$, $\phi = 0.001$ and $\psi = 0.0039$. As in case (iii) two eigenvalues become purely

[6] A formal statement of the Hopf bifurcation theorem is given in appendix A.5.

[7] For those computations we used the software LOCBIF, see Khibnik et al. (1993).

[8] To detect the limit cycle we used the software MATCONT, see Dhooge et al. (2003).

imaginary as ψ is continuously reduced and for $\psi = \psi_{crit} = 3.638542 \cdot 10^{-3}$ a Hopf bifurcation occurs giving rise to limit cycles. Again, the limit cycles are stable because the first Lyapunov coefficient is negative, $L_1 = -5.85141 \cdot 10^{-2}$.

5. CONCLUSION

In this paper we have analyzed how public debt affects the allocation of resources in a simple inter-temporal model of a market economy. Given the high indebtedness of most industrialized countries this question is not only of academic interest but has also relevance for the real world. The decisive aspect in analyzing effects of public debt is that a rise in public debt must be accompanied by future increases in primary surpluses so that the government can fulfill its inter-temporal budget constraint. In this paper we have posited that the primary surplus of the government depends on the history of past public debt with exponentially declining weight put on debt further back in time.

We have seen that the higher the debt ratio the lower the balanced growth rate. This implies that a balanced government budget yields a higher long-run growth rate than a debt policy where public debt grows at the same rate as all other economic variables. That result is rather robust and also holds when the primary surplus is a linear function of the current debt level alone as in Greiner (2011a).

However, the assumption that the primary surplus reacts to cumulated past debt gives a more complex outcome as regards the dynamics of the economy. Thus, it turned out that the balanced budget rule yields a stable balanced growth path only if the reaction of the government to higher public debt is sufficiently high and if the weight given to more recent levels of public debt in the function determining the reaction to higher debt is large.

In case of permanent public deficits such that public debt grows at the same rate as all other economic variables, the outcome is more complex. In that case, existence of a BGP cannot be guaranteed and, in case it exists, its stability properties may crucially depend on how the government reacts to higher debt ratios. In particular, we have seen that higher values of the coefficient determining the reaction of the primary surplus to higher debt tend to stabilize the economy. When that coefficient is reduced the economy may loose stability and for certain critical values of that coefficient endogenous growth cycles can arise, implying that the economy is characterized by cyclical growth rates.

APPENDIX A

A.1. THE PRIMARY SURPLUS AND SUSTAINABILITY OF PUBLIC DEBT

The inter-temporal budget constraint budget constraint of the government is given by,

$$\lim_{t\to\infty} B(t)\, e^{-\int_0^t r_B(\nu)(1-\tau)d\nu} = 0.$$

Assuming that the primary surplus in t is a function of public debt in t alone and allowing for a time-varying coefficient $\psi(t)$, the primary surplus rule (11) gives the primary surplus as,

$$S = \phi Y(t) + \psi(t)B(t).$$

The evolution of public debt, then, is described by:

$$\dot{B} = (r_B(t)(1-\tau) - \psi(t))B(t) - \phi Y(t) \quad (A.1)$$

Solving $(A.1)$ and multiplying both sides by $e^{-\int_0^\nu r_B(\mu)(1-\tau)d\mu}$ to get present values yields

$$e^{-C_3(t)}B(t) = e^{-C_1(t)}B(0) - \phi Y(0)e^{-C_1(t)} \int_0^t e^{C_1(\nu)}e^{C_2(\nu)}e^{-C_3(\nu)}d\nu \quad (A.2)$$

with

$$\int_0^\nu \psi(\mu)d\mu := C_1(\nu), \int_0^\nu g_y(\mu)d\mu := C_2(\nu), \int_0^\nu r_B(\mu)(1-\tau)d\mu := C_3(\nu),$$

where g_y gives the growth rate of GDP.

Equation $(A.2)$ demonstrates that $\lim_{t\to\infty} C_1(t) = \lim_{t\to\infty} \int_0^t \psi(\nu)d\nu = \infty$ must hold so that the first term in that equation converges to zero. The second term on the right hand side in $(A.2)$ can be written as

$$\frac{\int_0^t e^{C_1(\nu)}e^{C_2(\nu)}e^{-C_3(\nu)}d\nu}{e^{C_1(t)}} := C_4(t),$$

where we have set $\phi\, Y(0) = 1$.

If $\int_0^\infty e^{C_1(\nu)}e^{C_2(\nu)}e^{-C_3(\nu)}d\nu$ remains bounded $\lim_{t\to\infty} C_1(t) = \infty$ guarantees that C_4 converges to zero. Boundedness of $\int_0^\infty e^{C_1(\nu)}e^{C_2(\nu)}e^{-C_3(\nu)}d\nu$ is given for $\lim_{t\to\infty}(C_1(t) + C_2(t) - C_3(t)) = -\infty$. If

$$\lim_{t\to\infty} \int_0^t e^{C_1(\nu)}e^{C_2(\nu)}e^{-C_3(\nu)}d\nu = \infty,$$

applying l'Hôpital gives the limit of C_4 as

$$\lim_{t\to\infty} C_4(t) = \lim_{t\to\infty} \frac{e^{C_2(t)}e^{-C_3(t)}}{\psi(t)} .$$

In a dynamically efficient economy the net interest rate exceeds the growth rate of GDP so that $r_B(1-\tau) > g_y$ which always holds in our economy along the BGP. This shows that C_4 converges to zero in the limit if $\lim_{t\to\infty} C_1(t) = \infty$ holds. It should be noted that $\lim_{t\to\infty} C_1(t) = \infty$ excludes the possibility that $\psi(t)$ converges to zero exponentially. Thus, a positive reaction coefficient on average, that implies $\lim_{t\to\infty} \int_0^t \psi(\nu)d\nu = \infty$, guarantees that the present value of public debt converges to zero.

If the primary surplus depends on cumulated past debt with exponentially declining weights on debt further back in time, the reaction of the government to public debt is larger than in the case where public debt depends on the current level of public debt only, since $\psi B(t) < \psi B(t) + \int_{-\infty}^{t-\epsilon} e^{\kappa(\mu-t)}\psi B(\mu)d\mu$. Thus, C_1 is larger and a positive average reaction coefficient guarantees sustainability if the reaction of the government to public debt is as in equation (11).

A.2. PROOF OF LEMMA 1

The Jacobian is given by:

$$J = \begin{bmatrix} \partial\dot{c}/\partial c & 0 & -c \\ 0 & \rho & -1 \\ 0 & \psi & -g-\kappa \end{bmatrix}$$

with c evaluated at the BGP and where we used $\rho = c - c^{-\beta/(1-\beta+\gamma)}\upsilon((1-\tau)(1-\alpha)+\phi)$ and $-\dot{K}/K - \kappa = -g - \kappa = c + \delta - \kappa - c^{-\beta/(1-\beta+\gamma)}\upsilon(1-\tau)$. Note that $\partial\dot{c}/\partial c > 0$ and $\psi = \kappa(r(1-\tau)-\delta) = \kappa(g+\rho)$.

The eigenvalues of the Jacobian can be computed as

$$\lambda_1 = \partial\dot{c}/\partial c,\ \lambda_{2,3} = 0.5((\rho - g - \kappa) \pm \sqrt{(\rho+g+\kappa)^2 - 4\kappa(g+\rho)})$$

λ_1 is strictly positive and $\lambda_{2,3}$ can be written as:

$$\lambda_{2,3} = 0.5((\rho - g - \kappa) \pm \sqrt{(\rho+g-\kappa)^2})$$

It is immediately seen that we get $\lambda_2 = -g < 0$ and $\lambda_3 = \rho - \kappa$.

A.3. PROOF OF PROPOSITION 3

From $\dot{c}/c = 0$ we get

$$z = c - c^{-\beta/(1-\beta+\gamma)}\upsilon((1-\tau)(1-\alpha)+\phi) - \rho$$

Inserting that in $\dot{b}/b$ and setting $\dot{b}/b = 0$ gives

$$\rho b = c - c^{-\beta/(1-\beta+\gamma)}\upsilon((1-\tau)(1-\alpha)+\phi) - \rho + \phi\upsilon c^{-\beta/(1-\beta+\gamma)}$$

Using those two expressions to substitute for z and b in $\dot{z}/z$ leads to

$$q = \frac{\psi}{\rho}\left(1 + \frac{\phi\upsilon}{h(c,\cdot) - \upsilon((1-\tau)(1-\alpha)+\phi)}\right) - c^{-\beta/(1-\beta+\gamma)}\upsilon(1-\tau)\alpha + \rho + \delta - \kappa,$$

with $h(c,\cdot) = c^{(1+\gamma)/(1-\beta+\gamma)} - \rho c^{\beta/(1-\beta+\gamma)}$. A solution c such that $q(\cdot) = 0$ gives a BGP.

The expression $h(c,\cdot) - \upsilon((1-\tau)(1-\alpha)+\phi)$ is equivalent to $zc^{\beta/(1-\beta+\gamma)} > 0$. Note that we limit the analysis to the case $\upsilon((1-\tau)(1-\alpha)+\phi) > 0$ which implies $h > 0$. The function $q(\cdot)$ has a discontinuity (a pole) at c^{pol} with c^{pol} such that $h(c,\cdot) - \upsilon((1-\tau)(1-\alpha)+\phi) = 0$ for $c = c^{pol}$. Since we only analyze the economy with $z \geq 0$ the BGP value of c, $c^{\star}$, must lie to the right of $\upsilon((1-\tau)(1-\alpha)+\phi)$.

The function $h(c,\cdot)$ has the following properties:

$$\begin{aligned}
h(c,\cdot) &= c^{(1+\gamma)/(1-\beta+\gamma)} - \rho c^{\beta/(1-\beta+\gamma)} > 0, \ \lim_{c\to\infty} h(c,\cdot) = \infty, \\
\frac{dh}{dc} &= \left(\frac{1+\gamma}{1-\beta+\gamma}\right) c^{-1+(1+\gamma)/(1-\beta+\gamma)} \\
&- \rho\left(\frac{\beta}{1-\beta+\gamma}\right) c^{-1+(\beta/(1-\beta+\gamma))} > 0, \text{ for } h > 0
\end{aligned}$$

The function $q(\cdot)$ has the following properties:

$$\lim_{c\to\infty} q = (\psi/\rho) + \rho + \delta - \kappa, \ \lim_{c\searrow c^{pol}} q = \infty\,(-\infty), \text{ for } \phi > (<)\, 0,$$

where $c \searrow c^{pol}$ means that c approaches c^{pol} from above. The derivative of $q(c,\cdot)$ with respect to c is obtained as:

$$\begin{aligned}
\frac{dq}{dc} &= \left(\frac{\beta}{1-\beta+\gamma}\right) c^{-1-(\beta/(1-\beta+\gamma))} \\
&- \frac{\psi}{\rho}\left(\frac{1}{h(c,\cdot) - \upsilon((1-\tau)(1-\alpha)+\phi)}\right)^2 \phi\,\upsilon\left(\frac{dh}{dc}\right)
\end{aligned}$$

Thus, the function $q(c,\cdot)$ does not intersect the horizontal axis for $(\psi/\rho) + \rho + \delta - \kappa < 0$ with $\phi < 0$ and, therefore, no BGP exists. For $\phi > 0$ and

$(\psi/\rho)+\rho+\delta-\kappa<0$ there exists a unique intersection of $q(c,\cdot)$ with the horizontal axis and, therefore, a unique BGP.

For $(\psi/\rho)+\rho+\delta-\kappa>0$ and $\phi<0$ there exists a unique intersection of $q(c,\cdot)$ with the horizontal axis and, therefore, a unique BGP.

For $(\psi/\rho)+\rho+\delta-\kappa>0$ and $\phi>0$ the function $q(c,\cdot)$ starts from $+\infty$ for $c=c^{pol}$ and converges to $(\psi/\rho)+\rho+\delta-\kappa>0$ for $c\to\infty$. Either, $q(c,\cdot)$ does not intersect the horizontal axis (no BGP) or the function is such that there are two points of intersection (2 BGPs) for this case. To show that are maximally 2 BGPs we note that $q(c,\cdot)=0$ implies $q_1(c,\cdot)=(h(c,\cdot)-\upsilon((1-\tau)(1-\alpha)+\phi))\cdot q(c,\cdot)=0$. Recall that a positive level of outstanding public debt, to which we limit our analysis, implies $h(c,\cdot)-\upsilon((1-\tau)(1-\alpha)+\phi)>0$. The function $q_1(c,\cdot)$ is given by:

$$\begin{aligned} q_1 &= (h(c,\cdot)-\upsilon((1-\tau)(1-\alpha)+\phi))((\psi/\rho)+\rho+\delta-\kappa)+\phi\upsilon\psi/\rho+ \\ &\quad \rho\upsilon\alpha(1-\tau)+c^{-\beta/(1-\beta+\gamma)}\upsilon^2((1-\tau)(1-\alpha)+\phi)\alpha(1-\tau) \end{aligned}$$

with the properties $\lim_{c\to 0} q_1=+\infty$, $\lim_{c\to\infty} q_1=+\infty$. The second derivative of $q_1(c,\cdot)$ is given by

$$\frac{d^2q_1}{dc^2}=\left(\frac{\beta}{1-\beta+\gamma}+1\right)c^{-\beta/(1-\beta+\gamma)-2}\cdot C_5+((\psi/\rho)+\rho+\delta-\kappa)\left(\frac{d^2h}{dc^2}\right),$$

with $C_5=\upsilon^2((1-\tau)(1-\alpha)+\phi)\alpha(1-\tau)\beta/(1-\beta+\gamma)>0$. The second derivative of $h(c,\cdot)$ is given by

$$\frac{d^2h}{dc^2}=\frac{\beta}{1-\beta+\gamma}\left(\left(\frac{1+\gamma}{1-\beta+\gamma}\right)c^{((1+\gamma)/(1-\beta+\gamma))-2}+\rho\left(\frac{1+\gamma-2\beta}{1-\beta+\gamma}\right)c^{(\beta/(1-\beta+\gamma))-2}\right)$$

It is positive if and only if

$$\begin{aligned} &\left(\frac{1+\gamma}{1-\beta+\gamma}\right)c^{(1+\gamma)/(1-\beta+\gamma)}-\rho\left(\frac{\beta}{1-\beta+\gamma}\right)c^{\beta/(1-\beta+\gamma)} \\ +\;& \rho\left(\frac{1-\beta+\gamma}{1-\beta+\gamma}\right)c^{\beta/(1-\beta+\gamma)}\geq 0 \end{aligned}$$

which always holds true since $h(c,\cdot)=c^{(1+\gamma)/(1-\beta+\gamma)}-\rho c^{\beta/(1-\beta+\gamma)}>0$. Thus, the second derivative of $h(c,\cdot)$ is positive and, therefore, the second derivative of $q_1(c,\cdot)$, too. Consequently, there can be only two points of intersection of $q_1(c,\cdot)$ and, thus, of $q(c,\cdot)$ with the horizontal axis.

A.4. PROOF OF LEMMA 2

From the proof of proposition 3 we know that along the BGP the following relation holds:

$$0=c-c^{-\beta/(1-\beta+\gamma)}\upsilon(1-\tau)(1-\alpha)-\rho-\rho b$$

Implicitly differentiating gives:

$$\frac{dc}{db} = \frac{\rho}{1 + (\beta/(1-\beta+\gamma))\, c^{-(\beta/(1-\beta+\gamma))-1}\, \upsilon\,(1-\tau)(1-\alpha)} > 0$$

Since a higher value of c implies a lower long-run growth rate, the lemma is proven.

A.5. THE HOPF BIFURCATION THEOREM

Here we present the Hopf bifurcation theorem as it can be found in Guckenheimer and Holmes (1983), pp. 151-52.

Theorem A.3 *Suppose that the system* $\dot{x} = G(x, \omega), x \in \mathbb{R}^n,\ \omega \in \mathbb{R}$ *has an equilibrium* (x_0, ω_0)*, at which the following properties are satisfied:*

(i) $D_x\, G(x_0, \omega_0)$ *has a simple pair of purely imaginary eigenvalues and no other eigenvalues with zero real parts.*

Then (i) implies that there is a smooth curve of equilibria $(x(\omega),\ \omega)$ *with* $x(\omega_0) = x_0$*. The eigenvalues* $\lambda(\omega),\ \bar{\lambda}(\omega)$ *of* $D_x\, G(x(\omega), \omega_0)$ *which are imaginary at* $\omega = \omega_0$ *vary smoothly with* ω*. If, moreover,*

$$\frac{d}{d\omega} Re(\lambda(\omega)) = d_1 \neq 0, \text{ für } \omega = \omega_0,$$

then there is a unique three-dimensional center manifold passing through (x_0, ω_0) *in* $\mathbb{R}^n \times \mathbb{R}$ *and a smooth system of coordinates (preserving the planes for* $\omega = const.$*) for which the Taylor expansion of degree 3 on the center manifold is given by*

$$\dot{x}_1 = [d_1\omega + \beta_1(x_1^2 + x_2^2)]x_1 - [d_4 + d_2\omega + d_3(x_1^2 + x_2^2)]x_2,$$

$$\dot{x}_2 = [d_4 + d_2\omega + d_3(x_1^2 + x_2^2)]x_1 + [d_1\omega + \beta_1(x_1^2 + x_2^2)]x_2.$$

If $\beta_1 \neq 0$*, there is a surface of periodic solutions in the center manifold which has quadratic tangency with the eigenspace of* $\lambda(\omega_0),\ \bar{\lambda}(\omega_0)$ *agreeing to second order with the paraboloid* $\omega = -(\beta_1/d_1)(x_1^2 + x_2^2)$*. If* $\beta_1 < 0$*, then these periodic solutions are stable limit cycles, while if* $\beta_1 > 0$*, the solutions are repelling.*

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