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# Optimal Consumption and Investment with a Wealth-Dependent Time-Varying Investment Opportunity*

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We study an optimization problem of an investor in which there is a better investment opportunity when he is rich than when he is poor. We model the betterment of the investment opportunity by considering an exogenously specified wealth threshold such that the investor's investment opportunity is better when his wealth is above the threshold than when it is below the threshold. We derive a closed form solution for the optimal consumption and investment strategies by using a dynamic programming method, and investigate the effects of the potential investment opportunity changes on the optimal strategies.

*Key Words*: Consumption; Investment; Investment opportunity set; Threshold wealth level; Bellman equation.
*JEL Classification Numbers*: E21, G11.

## 1. INTRODUCTION

We study an optimization problem of an investor in which there is a better investment opportunity when he is rich than when he is poor. It is well-established in the literature that major participants in the stock market are wealthy households while majority of poor households have no equity holdings (See e.g., Mankiew and Zeldes (1991), Carroll (2002), Inkman et

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al. (2011)). When studying limited participation and enlargement of the investment opportunity set, researchers have introduced information acquisition cost and an endogenous wealth threshold, at which an investor pays the cost to enlarge the set (Gomes and Michaelides (2005), Shim (2011)). However, we model the betterment of the investment opportunity by considering an exogenously specified wealth threshold such that the investor's investment opportunity is better when his wealth is above the threshold than when it is below the threshold. Justification for the assumption can be provided by institutional aspects. For example, US Rule 144A stipulates that unregistered securities can be traded only by qualified institutional investors. Commonly, the qualified investors' net worth is above a certain threshold, even though the threshold is not explicitly specified. Furthermore, there exist minimum required investments to enroll in certain hedge funds. Even though the required minimum does not imply a fixed wealth threshold, we can regard the exogenous threshold as a first order approximation to the existence of such restrictions.

Choi et al. (2003) have also studied a model in which there exists an exogenous wealth threshold such that the investment opportunity gets better once the investor's wealth exceeds the threshold level. Their model has the unrealistic aspect that the investment opportunity set stays constant once after being enlarged, regardless investment outcomes. In reality, investors tend to be excluded from certain investments when their net worth shrinks. Information economics has discovered ample reasons for such exclusion (see e.g. Jaffee & Russell (1976), Keeton (1979), and Stiglitz & Weiss (1981)). Consider the situation, e.g., an investor has discovered a new investment opportunity, which requires a certain amount of initial investment. Then the ability to exploit the investment opportunity depends on a loan which often has a specific down payment requirement. Furthermore, the loan may not be renewed if the investor's net worth shrinks below a certain level. We introduce this realistic feature by assuming that the investment opportunity returns to the smaller one when wealth falls below the threshold level.

We derive a closed form solution for the optimal strategies and discuss their properties. We first show that optimal consumption is a continuous function of wealth for all wealth levels. Despite the existence of an exogenous threshold across which the investment opportunity undergoes a noticeable change, the investor does not make a discrete adjustment to his optimal consumption. This is in sharp contrast to the result in Choi et al. (2003). In their model the change in the investment opportunity is permanent without returning to the smaller one, thus the investor makes a discrete adjustment of consumption at the threshold as an optimal response to the permanent change. However, in our model the change in the investment opportunity is only temporary with the possibility of the opportunity returning to the smaller one, thus the investor does not make a discrete adjustment and optimal consumption is continuous for all wealth levels.

We show that the investor consumes less with the wealth-dependent investment opportunity set than he would without it. When the investor's wealth is below the threshold level, he tries to accumulate his wealth fast enough by reducing consumption to reach the threshold wealth level to have an access to the better investment opportunity. When his wealth is above the threshold level, he reduces consumption because of the risk of losing the better investment opportunity. Furthermore, we show that such effect becomes stronger as the investor's wealth gets closer to the threshold level, while the effect is negligible when his wealth is sufficiently far from it.

The marginal propensity to consume out of wealth exhibits a rather interesting behavior. It is defined as the ratio of the incremental change in consumption to a small increase in wealth. It is a strictly decreasing function of wealth for all wealth levels except at the threshold level where it makes an upward jump. When wealth is below the threshold, the marginal propensity to consume is smaller than it is in the absence of potential investment opportunity changes. When wealth exceeds the threshold level, the marginal propensity to consume is greater than it is in the absence of potential investment opportunity changes. The marginal propensity to consume approaches its value in the absence of potential investment opportunity changes either as wealth approaches 0 or as it gets arbitrarily large. Thus, even though the investor does not make a discrete adjustment in optimal consumption, he makes a discrete adjustment in its marginal propensity.

We show that the investor takes more (resp. less) risk at wealth below (resp. above) the threshold level than he would in the absence of potential investment opportunity changes. When the investor's wealth is below the threshold level, he increases the expected growth rate of his wealth by taking more risk and thereby taking advantage of risk premia in the risky assets to reach the threshold wealth level fast enough, while, when his wealth is above it, he reduces his risk taking because of the risk of wealth falling below the threshold level and losing the better investment opportunity. Furthermore, we show that such effect becomes stronger as the investor's wealth gets closer to the threshold level, while the effect is negligible when his wealth is sufficiently far from it.

Implicit in the previous discussion of the investor's risk taking behavior is his revealed coefficient of relative risk aversion. Remarkably, it is a strictly decreasing function of wealth except at the threshold level, where it makes an upward jump. The investor makes a discrete adjustment in his revealed relative risk aversion at the threshold. As a consequence of the behavior of revealed relative risk aversion, the investor's marginal propensity to invest in the risky assets, defined as the ratio of the incremental change in his investment in the risky assets to a small increase in wealth, gets larger (resp. smaller) as wealth increases at wealth below (resp. above) the threshold level. The marginal propensity approaches its value in the absence of potential investment opportunity changes as wealth becomes either arbitrarily small or arbitrarily large.

There has been an increasing interest in the consumption and investment behavior of the rich (see e.g., Carroll (2002), Wachter & Yogo (2010)). We make a contribution to the literature by showing theoretically that the consumption and portfolio selection of a moderately rich person can be quite different from predictions of the traditional models or of more recent models (e.g., Merton (1969), Wachter & Yogo (2010)). There have been investigations of enlargement of an investor's investment opportunity set. For example, Gomes and Michaelides (2005) have studied a discrete-time model where investors with Epstein-Zin preferences choose the time to enter the stock market by paying a fixed cost, and Shim (2011) has considered a continuous-time consumption and investment problem, in which the investor can enlarge the investment opportunity by paying a cost of information gathering. The enlargement of investment opportunity in reality, however, often depends on the investor's net worth staying a certain level, which is beyond his control. We investigate this aspect of reality.

The paper proceeds as follows. Section 2 sets up the consumption and investment problem. Section 3 derives the closed form solutions for the value function, optimal consumption and portfolio strategies. Section 4 discusses the properties of optimal strategies. Section 5 concludes.

## 2. THE MODEL

We consider a financial market in which an investor's investment opportunity depends on his wealth level. There are one riskless asset and $m+n$ risky assets in the market. We assume that the risk-free rate is a constant, $r > 0$, and the price, $p_0(t)$, of the riskless asset follows a deterministic process

$$dp_0(t) = p_0(t)rdt, \quad p_0(0) = p_0.$$

The price, $p_j(t)$, of the $j$-th risky asset, as commonly assumed in the literature (see e.g., Merton (1969), Merton (1971), Karatzas et al. (1986), Choi et al. (2003), etc.) follows a geometric Brownian motion

$$dp_j(t) = p_j(t)\big\{\alpha_j dt + \sum_{k=1}^{m+n} \sigma_{jk} dw_k(t)\big\}, \quad p_j(0) = p_j, \quad j = 1, \dots, m+n,$$

where $(\mathbf{w}(t))_{t=0}^{\infty} = ((w_1(t), \ldots, w_{m+n}(t)))_{t=0}^{\infty}$ is an $m+n$ dimensional standard Brownian motion defined on the underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $(\mathcal{F}_t)_{t=0}^{\infty}$ be the augmentation under $\mathbb{P}$ of the natural filtration generated by the standard Brownian motion $(\mathbf{w}(t))_{t=0}^{\infty}$. The market parameters, $\alpha_j$'s and $\sigma_{jk}$'s for $j, k \in \{1, \ldots, m+n\}$, are assumed to be constants. We will use notation $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_{m+n})$. We assume that matrix $D = (\sigma_{ij})_{i,j=1}^{m+n}$ is nonsingular so that the symmetric matrix, $\Sigma = DD^{\top}$, is positive definite. Let $D_m$ denote the first $m \times (m+n)$ submatrix of $D$, and let $\Sigma_m = D_m D_m^{\top}$ so that the symmetric matrix $\Sigma_m$ is the first $m$ by $m$ submatrix of $\Sigma$ and also positive definite. Note that $\Sigma^{-1}$ and $\Sigma_m^{-1}$ are also positive definite.

Let $\boldsymbol{\pi}_t := (\pi_{1,t}, \ldots, \pi_{m+n,t})$ be the row vector representing dollar amounts invested in the risky assets at time $t$ and $c_t$ be the investor's consumption rate at time $t$. The consumption rate process, $\mathbf{c} = (c_t)_{t=0}^{\infty}$, is a nonnegative process adapted to $(\mathcal{F}_t)_{t=0}^{\infty}$ and satisfies $\int_0^t c_s \, ds < \infty$, for all $0 \le t < \infty$ a.s. The portfolio process, $\boldsymbol{\pi} = (\boldsymbol{\pi}_t)_{t=0}^{\infty}$, is adapted to $(\mathcal{F}_t)_{t=0}^{\infty}$ and satisfies $\int_0^t \|\boldsymbol{\pi}_s\|^2 \, ds < \infty$, for all $0 \le t < \infty$ a.s. The investor's wealth process, $x_t$, with initial wealth $x_0 = x \ge 0$ evolves according to

$$dx_t = (\boldsymbol{\alpha} - r\mathbf{1}_{m+n})\boldsymbol{\pi}_t^{\top} \, dt + (rx_t - c_t) \, dt + \boldsymbol{\pi}_t D \, d\mathbf{w}^{\top}(t), \quad 0 \le t < \infty, \tag{1}$$

where $\mathbf{1}_{m+n} = (1, \ldots, 1)$ is the $(m+n)$-row vector of ones. A special feature of this paper is the assumption that the investor's investment opportunity set depends on his wealth. There exists an exogenously given threshold wealth level $z > 0$ below which the investor does not have access to the last $n$ risky assets, however, above which he has an access to the full array of assets. That is, the portfolio process, $\boldsymbol{\pi} = (\boldsymbol{\pi}_t)_{t=0}^{\infty}$, should satisfy

$$\pi_{m+1,t} = \cdots = \pi_{m+n,t} = 0 \quad \text{if} \quad x_t < z \quad \text{for} \quad t \ge 0. \tag{2}$$

Condition (2) is different from condition (3) in Choi et al. (2003) in the following sense: the investor under condition (2) loses the acquired better investment opportunity if his wealth falls below the threshold level after exceeding it, while the investor under condition (3) in Choi et al. (2003) does not lose the once acquired investment opportunity although his wealth falls below the threshold level.

To preclude an arbitrage opportunity we assume that the investor faces the following nonnegative wealth constraint (see e.g., Dybvig & Huang (1988) and Karatzas & Shreve (1998)):

$$x_t \ge 0, \quad \text{for all } t \ge 0 \text{ a.s.} \tag{3}$$

Given initial wealth $x_0 = x \ge 0$, a pair of controls, $(\mathbf{c}, \boldsymbol{\pi})$, satisfying conditions in the above including (2) and (3) is said to be admissible at $x$. Let $A(x)$ denote the set of admissible controls at $x \ge 0$.

We assume that the investor has an infinite horizon, derives utility from a stream of consumption, and exhibits constant relative risk aversion with the coefficient of relative risk aversion being equal to $\gamma$ so that his utility function is given by the following[1]

$$u := E\Big[\int_0^\infty e^{-\beta t} U(c_t)\, dt\Big], \qquad U(c) = \frac{c^{1-\gamma}}{1-\gamma}, \quad 0 < \gamma \neq 1.$$

where $\beta$ is the investor's subjective discount rate. Thus, the investor's optimization problem is to maximize, for a given $x_0 = x \geq 0$,

$$V_{(\mathbf{c},\boldsymbol{\pi})}(x) := E\Big[\int_0^\infty e^{-\beta t} \frac{c_t^{1-\gamma}}{1-\gamma}\, dt\Big] \tag{4}$$

over all $(\mathbf{c},\boldsymbol{\pi}) \in A(x)$. If $x_0 = x = 0$, then the optimization problem becomes trivial. Therefore, we assume that $x_0 = x > 0$.

The value function(indirect utility), $V^*(x)$, is defined by

$$V^*(x) = \sup\{V_{(\mathbf{c},\boldsymbol{\pi})}(x) : (\mathbf{c},\boldsymbol{\pi}) \in A(x)\} \text{ for } x > 0.$$

Let

$$\kappa_1 := \frac{1}{2}(\tilde{\boldsymbol{\alpha}} - r\mathbf{1}_m)\Sigma_m^{-1}(\tilde{\boldsymbol{\alpha}} - r\mathbf{1}_m)^\top \text{ and } \kappa_2 := \frac{1}{2}(\boldsymbol{\alpha} - r\mathbf{1}_{m+n})\Sigma^{-1}(\boldsymbol{\alpha} - r\mathbf{1}_{m+n})^\top.$$

where $\mathbf{1}_m$ is the $m$-row vector of ones and $\tilde{\boldsymbol{\alpha}}$ denotes the $m$-row vector consisting of the first $m$ components of $\boldsymbol{\alpha}$. Since $\Sigma_m^{-1}$ is positive definite, $\kappa_1 > 0$, unless $\tilde{\boldsymbol{\alpha}} - r\mathbf{1}_m$ is the zero vector. By Remark 2.1 in Shim (2011), $\kappa_2 \geq \kappa_1$. If $\kappa_2 = \kappa_1$, then the optimization problem gives the same value function as that in the absence of constraint (2). Therefore, we consider the case where

$$\kappa_2 > \kappa_1 > 0. \tag{5}$$

In order to make the problem well-posed (see e.g. Merton (1969), Karatzas et al. (1986), Choi et al. (2003)) we assume

$$K_1 := r + \frac{\beta - r}{\gamma} + \frac{\gamma - 1}{\gamma^2}\kappa_1 > 0 \text{ and } K_2 := r + \frac{\beta - r}{\gamma} + \frac{\gamma - 1}{\gamma^2}\kappa_2 > 0. \tag{6}$$

Let $\lambda_- < -1$ be the negative solution of the quadratic equation of $\lambda$, $\kappa_1\lambda^2 - (r - \beta - \kappa_1)\lambda - r = 0$, and let $\eta_+ > 0$ the positive solution of the quadratic equation of $\eta$, $\kappa_2\eta^2 - (r - \beta - \kappa_2)\eta - r = 0$.

[1] $U(c) = \log c$ corresponds to the case where $\gamma = 1$. We do not consider this case in this paper, but results similar to the ones for the case $\gamma \neq 1$ can be obtained for this case too.

*Remark 2.1.* As mentioned in Choi et al. (2003), Choi & Shim (2006), $K_1 > 0$ if and only if $1 + \gamma\lambda_- < 0$.

## 3. DERIVATION OF THE SOLUTION

We define constants $b$, $B_1$, and $B_2$ by

$$b := \frac{(1-\gamma)(\eta_+ - \lambda_-)K_1K_2}{(1-\gamma)(\eta_+ - \lambda_-)K_2 + (1+\gamma\eta_+)(1+\lambda_-)(K_2 - K_1)}z, \tag{7}$$

$$B_1 := b^{\gamma\lambda_-}(z - \frac{b}{K_1}), \tag{8}$$

and

$$B_2 := b^{\gamma\eta_+}(z - \frac{b}{K_2}). \tag{9}$$

Lemma 1. *$b > 0$, $B_1 > 0$, and $B_2 > 0$.*

*Proof.* Calculation shows that

$$B_1 = b^{\gamma\lambda_-}\frac{(1+\gamma\eta_+)(1+\lambda_-)(K_2 - K_1)}{(1-\gamma)(\eta_+ - \lambda_-)K_2 + (1+\gamma\eta_+)(1+\lambda_-)(K_2 - K_1)}z$$

and

$$B_2 = b^{\gamma\eta_+}\frac{(1+\eta_+)(1+\gamma\lambda_-)(K_2 - K_1)}{(1-\gamma)(\eta_+ - \lambda_-)K_2 + (1+\gamma\eta_+)(1+\lambda_-)(K_2 - K_1)}z.$$

By (5) and (6), $K_1 > K_2 > 0$ (resp. $0 < K_1 < K_2$) if $0 < \gamma < 1$ (resp. $\gamma > 1$), which together with inequality $1 + \lambda_- < 0$ and Remark 2.1 proves the lemma. ∎

Define functions $X_i(c)$ on $(0, \infty)$ for $i = 1, 2$ by

$$X_1(c) := B_1 c^{-\gamma\lambda_-} + \frac{c}{K_1} \quad \text{and} \quad X_2(c) := B_2 c^{-\gamma\eta_+} + \frac{c}{K_2}. \tag{10}$$

By (8) and (9), we have

$$X_1(b) = z = X_2(b). \tag{11}$$

Since $B_1 > 0$ by Lemma 1, function $X_1(\cdot)$ is strictly increasing and maps $(0, \infty)$ onto $(0, \infty)$ so that its inverse function, $C_1(\cdot)$, exists, is strictly increasing and maps $(0, \infty)$ onto $(0, \infty)$.

LEMMA 2. *$X_2'(b) > 0$ if and only if $K_1+(1+\gamma\lambda_-)\frac{\eta_+(1+\eta_+)}{\gamma(\eta_+-\lambda_-)}(\kappa_2-\kappa_1) > 0$.*

*Proof.* By using (10), (11), (7), and (6), we can calculate that

$$X_2'(b) = \frac{1}{K_1 K_2}\Big[K_1 + (1+\gamma\lambda_-)\frac{\eta_+(1+\eta_+)}{\gamma(\eta_+ - \lambda_-)}(\kappa_2 - \kappa_1)\Big],$$

from which the lemma follows. ∎

Thus, we will make the following assumption throughout the paper:

ASSUMPTION 1.

$$K_1 + (1+\gamma\lambda_-)\frac{\eta_+(1+\eta_+)}{\gamma(\eta_+ - \lambda_-)}(\kappa_2 - \kappa_1) > 0.$$

Since $B_2 > 0$ by Lemma 1, the derivative $X_2'(c) = -\gamma\eta_+ B_2 c^{-\gamma\eta_+ - 1} + \frac{1}{K_2}$ is strictly increasing with $X_2'(0) = \lim_{c\downarrow 0} X_2'(c) = -\infty$ and $\lim_{c\uparrow\infty} X_2'(c) = \frac{1}{K_2} > 0$. Since $X_2'(b) > 0$ by Lemma 2 and Assumption 1, $X_2'(c) > 0$ for $c \geq b$. Therefore, if we restrict the domain to $[b,\infty)$, then $X_2(\cdot)$ is strictly increasing and maps $[b,\infty)$ onto $[z,\infty)$ by (11) so that its inverse function, $C_2(\cdot)$, exists, is strictly increasing and maps $[z,\infty)$ onto $[b,\infty)$. Note that (11) is equivalent to

$$C_1(z) = b = C_2(z). \tag{12}$$

Define functions $J_i(c)$ on $(0,\infty)$ for $i = 1, 2$ by

$$\begin{aligned} J_1(c) &:= \frac{\lambda_-}{\rho_-} B_1 c^{-\gamma\rho_-} + \frac{c^{1-\gamma}}{(1-\gamma)K_1} \quad \text{and} \\ J_2(c) &:= \frac{\eta_+}{\nu_+} B_2 c^{-\gamma\nu_+} + \frac{c^{1-\gamma}}{(1-\gamma)K_2}, \end{aligned} \tag{13}$$

where $\rho_- := 1 + \lambda_- < 0$ and $\nu_+ := 1 + \eta_+ > 1$. Define function $V(x)$ by

$$V(x) = \begin{cases} V_1(x) := J_1(C_1(x)) & \text{for } 0 < x < z, \\ V_2(x) := J_2(C_2(x)) & \text{for } x \geq z. \end{cases} \tag{14}$$

LEMMA 3. *$V(x)$ defined by (14) is strictly increasing and strictly concave. It is continuously differentiable for $x > 0$ and twice continuously differentiable for $x \in (0,\infty) \setminus \{z\}$.*

*Proof.* By using (7), (8), and (9), we can show that $J_1(b) = J_2(b)$ which implies $V_1(z) = V_2(z)$ by (12). Thus, $V(x)$ is continuous at $x = z$. For $i = 1, 2$, differentiating each $V_i$ gives

$$V_i'(x) = J_i'(C_i(x))C_i'(x) = \frac{J_i'(C_i(x))}{X_i'(C_i(x))} = (C_i(x))^{-\gamma} > 0. \tag{15}$$

Therefore, $V(x)$ is strictly increasing for $x > 0$. Since $V_1'(z) = (C_1(z))^{-\gamma} = b^{-\gamma} = (C_2(z))^{-\gamma} = V_2'(z)$ by (12), $V(x)$ is continuously differentiable for $x > 0$. Since, for $i = 1, 2$,

$$V_i''(x) = -\gamma(C_i(x))^{-\gamma-1}C_i'(x) < 0 \tag{16}$$

in its domain, $V(x)$ is strictly concave for $x > 0$. ▮

Lemma 4. *$V(x)$ defined by (14) satisfies the Bellman equation:*

$$\begin{aligned} &\beta V(x) \\ = &\max_{c\geq 0, \boldsymbol{\pi}\in\Theta(x)} \{(\boldsymbol{\alpha} - r\mathbf{1}_{m+n})\boldsymbol{\pi}^\top V'(x) + (rx - c)V'(x) + \frac{1}{2}\boldsymbol{\pi}\Sigma\boldsymbol{\pi}^\top V''(x) + \frac{c^{1-\gamma}}{1-\gamma}\}, \end{aligned} \tag{17}$$

*where* $\Theta(x) := \left[ \begin{array}{l} \{(\pi_1, ..., \pi_{m+n}) \in \mathbb{R}^{m+n} : \pi_{m+1} = \cdots = \pi_{m+n} = 0\} \text{ if } 0 < x < z, \\ \mathbb{R}^{m+n} \text{ if } x \geq z. \end{array} \right.$

*The righthand side is maximized when*

$$\begin{cases} c = C_1(x), \quad \boldsymbol{\pi} = -\frac{V_1'(x)}{V_1''(x)}S, \text{ if } 0 < x < z, \\ c = C_2(x), \quad \boldsymbol{\pi} = -\frac{V_2'(x)}{V_2''(x)}(\boldsymbol{\alpha} - r\mathbf{1}_{m+n})\Sigma^{-1}, \text{ if } x \geq z, \end{cases} \tag{18}$$

*where $S$ denotes the $(m+n)$-row vector whose first $m$ components are equal to those of $(\tilde{\boldsymbol{\alpha}} - r\mathbf{1}_m)\Sigma_m^{-1}$ and whose last $n$ components are all equal to zero.*

*Proof.* We can prove the lemma in the case where $x \geq z$ by using (15) and (16), similarly to the proof of Theorem 9.1 in Karatzas et al. (1986). When $0 < x < z$, the Bellman equation (4) is equivalent to $\beta V(x) = \max_{c\geq 0, \tilde{\boldsymbol{\pi}}\in\mathbb{R}^m} \{(\tilde{\boldsymbol{\alpha}} - r\mathbf{1}_m)\tilde{\boldsymbol{\pi}}^\top V'(x) + (rx - c)V'(x) + \frac{1}{2}\tilde{\boldsymbol{\pi}}\Sigma_m\tilde{\boldsymbol{\pi}}^\top V''(x) + \frac{c^{1-\gamma}}{1-\gamma}\}$. Therefore, we can also prove the lemma in the case where $0 < x < z$ by using (15) and (16), similarly to the proof of Theorem 9.1 in Karatzas et al. (1986). ▮

Theorem 1. *With initial wealth $x_0 = x > 0$, the value function of the optimization problem is equal to $V(x)$ defined by (14), that is, $V^*(x) = V(x)$. An optimal strategy in $A(x)$ is provided by $(\bar{\mathbf{c}}, \bar{\boldsymbol{\pi}})$ defined as follows:*

$$\begin{cases} \bar{c}_t = C_1(x_t), \quad \bar{\boldsymbol{\pi}}_t &= -\frac{V'(x_t)}{V''(x_t)}S = -\frac{V_1'(x_t)}{V_1''(x_t)}S, \text{ if } 0 < x_t < z, \\ \bar{c}_t = C_2(x_t), \quad \bar{\boldsymbol{\pi}}_t &= -\frac{V'(x_t)}{V''(x_t)}(\boldsymbol{\alpha} - r\mathbf{1}_{m+n})\Sigma^{-1} \\ &= -\frac{V_2'(x_t)}{V_2''(x_t)}(\boldsymbol{\alpha} - r\mathbf{1}_{m+n})\Sigma^{-1}, \text{ if } x_t \geq z, \end{cases} \tag{19}$$

*where $S$ is the same as in Lemma 4.*

*Proof.* Similarly to Karatzas et al. (1986) or Choi & Shim (2006) (that is, by using Itô's rule), we can find the stochastic differential equation for $C_1(x_t)$ in (19) and show that $x_t > 0$ for all $t \geq 0$ with strategy (19). Thus, strategy (19) is in $A(x)$. By Lemma 3 and (17) in Lemma 4, and by using the generalized Itô rule, we can show, similarly to Choi & Shim (2006), that $V(x) \geq V_{(\mathbf{c},\boldsymbol{\pi})}(x)$ for arbitrary $(\mathbf{c},\boldsymbol{\pi}) \in A(x)$. Thus, we have

$$V(x) \geq V^*(x). \tag{20}$$

Now we consider the strategy $(\bar{\mathbf{c}},\bar{\boldsymbol{\pi}})$ given by (19) with the corresponding wealth process $(x_t)_{t\geq 0}$. Let $\bar{S}_n = \inf\{t \geq 0 : \int_0^t \|\bar{\boldsymbol{\pi}}_s\|^2\, ds = n\}$ and let $\xi \in (0,x)$ and let $T = \bar{S}_n \wedge n \wedge \inf\{t \geq 0 : x_t = \xi\}$. Then, $T \uparrow \infty$ as $n \uparrow \infty$ and $\xi \downarrow 0$. By (17) and (18) in Lemma 4, and by using the generalized Itô rule, we can derive that

$$E\Big[\int_0^T e^{-\beta t}\frac{\bar{c}_t^{1-\gamma}}{1-\gamma}dt\Big] = -E\Big[e^{-\beta T}V(x_T)\Big] + V(x). \tag{21}$$

When $\gamma > 1$, we can show that $V(x_T) < 0$ so that, by (21),

$$E\Big[\int_0^T e^{-\beta t}\frac{\bar{c}_t^{1-\gamma}}{1-\gamma}dt\Big] \geq V(x). \tag{22}$$

When $0 < \gamma < 1$, we can show that $0 < \int_0^\infty E\Big[e^{-\beta t}\frac{\bar{c}_t^{1-\gamma}}{1-\gamma}\Big]dt < \infty$ by using Fubini's Theorem, (20), and the fact that $(\bar{\mathbf{c}},\bar{\boldsymbol{\pi}}) \in A(x)$. Therefore, if $0 < \gamma < 1$, then we have

$$\lim_{t\uparrow\infty} E\Big[e^{-\beta t}\frac{\bar{c}_t^{1-\gamma}}{1-\gamma}\Big] = 0. \tag{23}$$

For $0 < \gamma < 1$, we can show, by using (14), that

$$E\Big[e^{-\beta T}V(x_T)\Big] \leq e^{-\beta T}\Big(\frac{\lambda_-}{\rho_-}B_1 b^{-\gamma\rho_-} + \frac{\eta_+}{\nu_+}B_2 b^{-\gamma\nu_+}\Big) + \frac{1}{K_2}E\Big[e^{-\beta T}\frac{\bar{c}_T^{1-\gamma}}{1-\gamma}\Big],$$

which, together with (21), implies

$$\begin{aligned} E\Big[\int_0^T e^{-\beta t}\frac{\bar{c}_t^{1-\gamma}}{1-\gamma}dt\Big] \;\geq\; & -e^{-\beta T}\Big(\frac{\lambda_-}{\rho_-}B_1 b^{-\gamma\rho_-} + \frac{\eta_+}{\nu_+}B_2 b^{-\gamma\nu_+}\Big) \\ & -\frac{1}{K_2}E\Big[e^{-\beta T}\frac{\bar{c}_T^{1-\gamma}}{1-\gamma}\Big] + V(x). \end{aligned} \tag{24}$$

Let $T \uparrow \infty$ by letting $n \uparrow \infty$ and $\xi \downarrow 0$ in (22) and (23), respectively, apply the monotone convergence theorem to the left-hand sides of them, respectively, and use (23) for the right-hand side of (23). Then we get

$$V_{(\bar{\mathbf{c}},\bar{\boldsymbol{\pi}})}(x) = E\Big[\int_0^\infty e^{-\beta t}\frac{\bar{c}_t^{1-\gamma}}{1-\gamma}dt\Big] \geq V(x). \tag{25}$$

By (20), (25), and the fact that $V^*(x) \geq V_{(\bar{\mathbf{c}},\bar{\boldsymbol{\pi}})}(x)$ since $(\bar{\mathbf{c}},\bar{\boldsymbol{\pi}}) \in A(x)$, we get $V^*(x) = V_{(\bar{\mathbf{c}},\bar{\boldsymbol{\pi}})}(x) = V(x)$. ▮

## 4. PROPERTIES OF THE SOLUTION

In this section, we will derive and discuss properties of the optimal strategy in Theorem 1. We start with the following remark which discusses an important difference between the optimal consumption policy in our model and Choi et al.'s

*Remark 4.1.* If the (enlarged) investment opportunity set stays constant after the investor's wealth exceeds the threshold level $z$ although his wealth falls below $z$ as assumed by Choi et al. (2003), then the consumption jump of positive size occurs at the first time when the investor's wealth exceeds the threshold level(see Lemma 1 in Choi et al. (2003)). However, in our model, the consumption jump does not occur by (12). That is, $\bar{c}_t$ is continuous as a function of $x_t$.

As mentioned in Section 1 and 2, our model is different from Choi et al.'s, since the investor in our model loses the acquired better investment opportunity if his wealth falls below the threshold level after exceeding it, while the investor in Choi et al. (2003) does not lose the once acquired investment opportunity although his wealth falls below the threshold level. A prominent feature of Choi et al.'s model is the existence of a jump in the optimal consumption rate at the threshold wealth level. Intuitively, the investor makes a discrete adjustment of his consumption as an optimal response to the permanent change in the investment opportunity. In contrast to their model, optimal consumption is a continuous function of wealth and does not exhibit a jump in our model. This is because the investor does not make a discrete adjustment of his consumption after enlargement of his investment opportunity since it is not permanent and the investment opportunity returns to the smaller one if his wealth falls below the threshold level.

The marginal propensity to consume out of wealth (MPC) is defined as the ratio of the incremental change in consumption to a small increase in wealth, i.e., MPC $:= \frac{dc_t}{dx_t}$. Next we would like to define the marginal propensity to invest in the risky assets out of wealth (MPIR) as a measure proportional to the ratio of the incremental change in his risky asset holdings to a small increase in wealth. We will call $-\frac{V''(x)}{V'(x)}$ the revealed coefficient of absolute risk aversion and its reciprocal, $-\frac{V'(x)}{V''(x)}$, risk tolerance. By (19) the investor's risky asset holdings is proportional to his risk tolerance. Therefore, we can define MPIR $:= \frac{d}{dx}\Big( -\frac{V'(x)}{V''(x)}\Big)$.

If the investment opportunity set consisted constantly of the riskless asset and the first $m$ risky assets without change, then the value function, say $M_1(x)$, would be given by

$$M_1(x) = \frac{K_1^{-\gamma}}{1-\gamma} x^{1-\gamma}$$

and the optimal consumption/investment strategy, say $(\mathbf{c}^m, \boldsymbol{\pi}^m)$, would be

$$\begin{aligned} c_t^m &= K_1 x_t, \\ \boldsymbol{\pi}_t^m &= -\frac{M_1'(x_t)}{M_1''(x_t)} S = \frac{x_t}{\gamma} S, \ \text{for}\, t \geq 0, \end{aligned}$$

as in Merton (1969) or Karatzas et al. (1986). In this case, the corresponding MPC, $\frac{dc_t^m}{dx_t}$, and the MPIR, $\frac{d}{dx_t}\Big( -\frac{M_1'(x_t)}{M_1''(x_t)}\Big)$ (the corresponding risk tolerance is equal to $-\frac{M_1'(x_t)}{M_1''(x_t)} = \frac{x_t}{\gamma}$), are constant for all wealth levels:

$$\frac{dc_t^m}{dx_t} = K_1 \quad \text{and} \quad \frac{d}{dx_t}\Big( -\frac{M_1'(x_t)}{M_1''(x_t)}\Big) = \frac{1}{\gamma} \quad \text{for} \quad x_t \geq 0.$$

If the investment opportunity set consisted constantly of the riskless asset and the whole $m+n$ risky assets without change, then the value function, say $M_2(x)$, would be given by

$$M_2(x) = \frac{K_2^{-\gamma}}{1-\gamma} x^{1-\gamma}$$

and the optimal consumption/investment strategy, say $(\mathbf{c}^{m+n}, \boldsymbol{\pi}^{m+n})$, would be

$$c_t^{m+n} = K_2 x_t, \quad \boldsymbol{\pi}_t^{m+n} = -\frac{M_2'(x_t)}{M_2''(x_t)} (\boldsymbol{\alpha} - r\mathbf{1}_{m+n})\Sigma^{-1} = \frac{x_t}{\gamma}(\boldsymbol{\alpha} - r\mathbf{1}_{m+n})\Sigma^{-1}, \ \text{for}\, t \geq 0,$$

as in Merton (1969) or Karatzas et al. (1986). In this case we have

$$\text{MPC} = \frac{dc_t^{m+n}}{dx_t} = K_2 \quad \text{and} \quad \text{MPIR} = \frac{d}{dx_t}\Big( -\frac{M_2'(x_t)}{M_2''(x_t)}\Big) = \frac{1}{\gamma} \quad \text{for} \quad x_t \geq 0.$$

Proposition 1 and Proposition 2 illustrate the effects of the change in the investment opportunity on the optimal consumption strategy.

PROPOSITION 1.

*(i)The investor consumes less than he would if the investment opportunity set did not change across $z$. That is,*

$$\begin{cases} \bar{c}_t = C_1(x_t) < K_1 x_t = c_t^m, & \textit{for } 0 < x_t < z, \\ \bar{c}_t = C_2(x_t) < K_2 x_t = c_t^{m+n}, & \textit{for } x_t \geq z. \end{cases}$$

*(ii)The difference between optimal consumption rates, $c_t^m - \bar{c}_t = K_1 x_t - C_1(x_t)$, is an increasing function of $x_t$ for $0 < x_t < z$ and approaches $0$ as $x_t \downarrow 0$. The difference, $c_t^{m+n} - \bar{c}_t = K_2 x_t - C_2(x_t)$, is a decreasing function of $x_t$ for $x_t \geq z$ and approaches $0$ as $x_t \uparrow \infty$.*

*Proof.* For $0 < x_t < z$, we have

$$\begin{aligned} c_t^m - \bar{c}_t &= K_1 x_t - C_1(x_t) \\ &= K_1 X_1(C_1(x_t)) - C_1(x_t) \\ &= K_1 \Big( B_1 (C_1(x_t))^{-\gamma\lambda_-} + \frac{C_1(x_t)}{K_1} \Big) - C_1(x_t) \\ &= K_1 B_1 (C_1(x_t))^{-\gamma\lambda_-}, \end{aligned}$$

where the second equality comes from the fact that $X_1$ is the inverse function of $C_1$, and the third from (10). Therefore, since $-\gamma\lambda_- > 0$ and $B_1 > 0$ as stated in Lemma 1, $c_t^m - \bar{c}_t$ is positive(that is, $\bar{c}_t < c_t^m$) and an increasing function of $x_t$ for $0 < x_t < z$, and approaches 0 as $x_t \downarrow 0$. For $x_t \geq z$, similarly we have

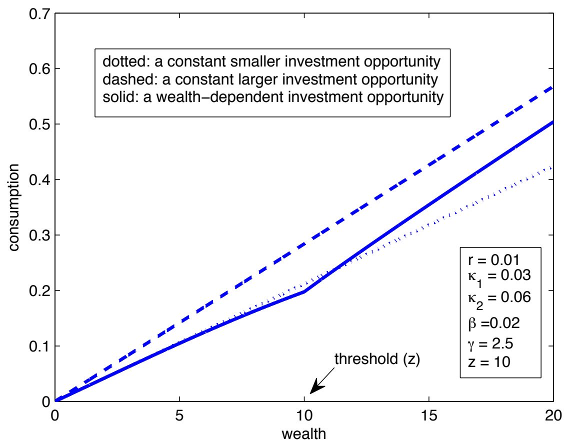
$$\begin{aligned} c_t^{m+n} - \bar{c}_t &= K_2 x_t - C_2(x_t) \\ &= K_2 X_2(C_2(x_t)) - C_2(x_t) \\ &= K_2 \Big( B_2 (C_2(x_t))^{-\gamma\eta_+} + \frac{C_2(x_t)}{K_2} \Big) - C_2(x_t) \\ &= K_2 B_2 (C_2(x_t))^{-\gamma\eta_+}. \end{aligned}$$

Therefore, since $-\gamma\eta_+ < 0$ and $B_2 > 0$ as stated in Lemma 1, $c_t^{m+n} - \bar{c}_t$ is positive(that is, $\bar{c}_t < c_t^{m+n}$) and a decreasing function of $x_t$ for $x_t \geq z$, and approaches 0 as $x_t \uparrow \infty$. ▮

Proposition 1(i) states that the investor consumes less with the wealth-dependent investment opportunity set than he would with a constant investment opportunity set. When the investor's wealth is below the threshold level, he tries to accumulate his wealth fast enough by reducing consumption to reach the threshold wealth level and to have an access to the better investment opportunity. When his wealth is above the threshold level, he reduces consumption because of the risk of losing the better investment opportunity. Proposition 1(ii) states that such effect becomes stronger as the investor's wealth gets closer to the threshold level, while the effect is negligible when his wealth is sufficiently far from it. Figure 1 illustrates these properties of optimal consumption.

*Remark 4.2.* The statement that $c_t^m - \bar{c}_t = K_1 x_t - C_1(x_t)$ is an increasing function of $x_t$ for $0 < x_t < z$ and $c_t^{m+n} - \bar{c}_t = K_2 x_t - C_2(x_t)$ is a decreasing function of $x_t$ for $x_t \geq z$ in Proposition 1 (ii), is equivalent to the statement that the MPC, $\frac{d\bar{c}_t}{dx_t}$, is smaller(resp. greater) at the wealth level below(resp. above) $z$ than it is in the absence of potential investment opportunity changes, that is, $\frac{d\bar{c}_t}{dx_t} = C_1'(x_t) < K_1 = \frac{dc_t^m}{dx_t}$ for $0 < x_t < z$ and $\frac{d\bar{c}_t}{dx_t} = C_2'(x_t) > K_2 = \frac{dc_t^{m+n}}{dx_t}$ for $x_t \geq z$.

**FIG. 1.** Consumption




Proposition 2 discusses properties of the MPC.

PROPOSITION 2. *The MPC,* $\frac{d\bar{c}_t}{dx_t}$*, is a strictly decreasing function of wealth* $x_t$ *both for* $0 < x_t < z$ *and for* $x_t > z$ *with*

$$\lim_{x_t \uparrow z} \frac{d\bar{c}_t}{dx_t} = C_1'(z) < C_2'(z) = \lim_{x_t \downarrow z} \frac{d\bar{c}_t}{dx_t}, \tag{26}$$

*and it satisfies*

$$\lim_{x_t \downarrow 0} \frac{d\bar{c}_t}{dx_t} = \lim_{x_t \downarrow 0} C_1'(x_t) = K_1 \quad \textit{and} \quad \lim_{x_t \uparrow \infty} \frac{d\bar{c}_t}{dx_t} = \lim_{x_t \uparrow \infty} C_2'(x_t) = K_2.$$

*Proof.* By (10) and the fact that $X_1$ is the inverse function of $C_1$, we have

$$C_1'(x_t) = \frac{1}{X_1'(C_1(x_t))} = \frac{1}{-\gamma\lambda_- B_1 (C_1(x_t))^{-\gamma\lambda_- -1} + \frac{1}{K_1}}. \tag{27}$$

Since $-\gamma\lambda_- - 1 > 0$ by Remark 2.1, we have

$$\lim_{x_t \downarrow 0} \frac{d\bar{c}_t}{dx_t} = \lim_{x_t \downarrow 0} C_1'(x_t) = K_1.$$

Similarly, we have

$$C_2'(x_t) = \frac{1}{X_2'(C_2(x_t))} = \frac{1}{-\gamma\eta_+ B_2 (C_2(x_t))^{-\gamma\eta_+ -1} + \frac{1}{K_2}}. \tag{28}$$

Since $-\gamma\eta_+ - 1 < 0$, we have

$$\lim_{x_t \uparrow \infty} \frac{d\bar{c}_t}{dx_t} = \lim_{x_t \uparrow \infty} C_2'(x_t) = K_2.$$

Since $X_i$ is the inverse function of $C_i$ for $i = 0, 1$, we have

$$X_i'(C_i(x_t))C_i'(x_t) = 1 \,\text{and}\, X_i''(C_i(x_t))(C_i'(x_t))^2 + X_i'(C_i(x_t))C_i''(x_t) = 0$$

so that

$$C_i''(x_t) = -\frac{X_i''(C_i(x_t))(C_i'(x_t))^2}{X_i'(C_i(x_t))}. \tag{29}$$

By (10), Lemma 1 and Remark 2.1 we have

$$X_1''(c) = (\gamma\lambda_- + 1)\gamma\lambda_- B_1 c^{-\gamma\lambda_- -2} > 0. \tag{30}$$

By (10) and Lemma 1 we have

$$X_2''(c) = (\gamma\eta_+ + 1)\gamma\eta_+ B_2 c^{-\gamma\eta_+ -2} > 0. \tag{31}$$

Since $X_i$ and $C_i$ are strictly increasing functions, (29), (30), and (31) imply

$$C_1''(x_t) < 0 \;\text{ for }\; 0 < x_t < z \;\text{ and }\; C_2''(x_t) < 0 \;\text{ for }\; x_t > z.$$

By (27), (28) and (12) we have

$$C_1'(z) = \frac{1}{-\gamma\lambda_- B_1 b^{-\gamma\lambda_- -1} + \frac{1}{K_1}} \quad \text{and} \quad C_2'(z) = \frac{1}{-\gamma\eta_+ B_2 b^{-\gamma\eta_+ -1} + \frac{1}{K_2}}.$$

By (10) and (11) we have

$$-\gamma\lambda_- B_1 b^{-\gamma\lambda_- -1} + \frac{1}{K_1} = -\gamma\lambda_- b^{-1} X_1(b) + \frac{1+\gamma\lambda_-}{K_1} = -\gamma\lambda_- b^{-1} z + \frac{1+\gamma\lambda_-}{K_1}$$

and

$$-\gamma\eta_+ B_2 b^{-\gamma\eta_+ -1} + \frac{1}{K_2} = -\gamma\eta_+ b^{-1} X_2(b) + \frac{1+\gamma\eta_+}{K_2} = -\gamma\eta_+ b^{-1} z + \frac{1+\gamma\eta_+}{K_2}.$$

By using (7) we can calculate that

$$-\gamma\lambda_- b^{-1} z + \frac{1+\gamma\lambda_-}{K_1} - \Big(-\gamma\eta_+ b^{-1} z + \frac{1+\gamma\eta_+}{K_2}\Big) = (1+\gamma\eta_+)\frac{1+\gamma\lambda_-}{1-\gamma}\Big(\frac{1}{K_1} - \frac{1}{K_2}\Big),$$

which is positive since $1+\gamma\lambda_- < 0$ by Remark 2.1, and $K_1 > K_2 > 0$(resp. $0 < K_1 < K_2$) if $0 < \gamma < 1$(resp. $\gamma > 1$) by (5) and (6). Therefore,

$$-\gamma\lambda_- B_1 b^{-\gamma\lambda_- -1} + \frac{1}{K_1} > -\gamma\eta_+ B_2 b^{-\gamma\eta_+ -1} + \frac{1}{K_2},$$

which implies $C_1'(z) < C_2'(z)$. ∎

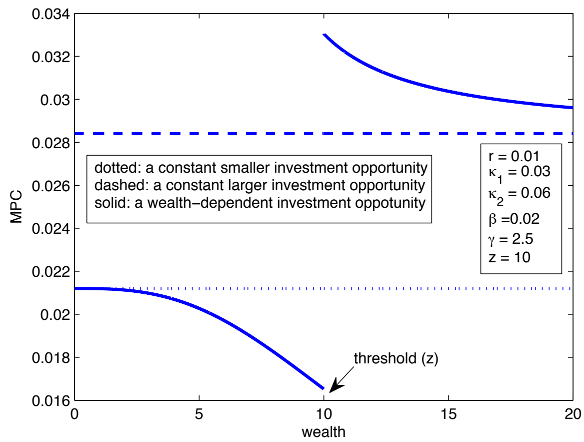
Figure 2 illustrates Proposition 2. When wealth is below the threshold, the MPC is smaller than it is in the absence of potential investment opportunities, decreases as wealth increases. It makes a jump when wealth reaches the threshold level. When wealth exceeds the threshold level, the MPC is greater than it is in the absence of potential investment opportunities, decreases as wealth increases. The MPC approaches its value in the absence of potential investment opportunity changes either as wealth approaches 0 or as it gets arbitrarily large, i.e., as $x_t \downarrow 0$ or as $x_t \uparrow \infty$.

Proposition 2 implies a downward jump in the investor's risk tolerance implied by the value function at the threshold wealth level, which we state as a remark. The remark will be useful to understand the investor's optimal investment strategy.

*Remark 4.3.* Note that

$$-\frac{V_1'(z)}{V_1''(z)} = \frac{C_1(z)}{\gamma C_1'(z)} > \frac{C_2(z)}{\gamma C_2'(z)} = -\frac{V_2'(z)}{V_2''(z)},$$

**FIG. 2.** MPC




where the inequality comes from (12) and (26), and the equalities from (15) and (16). Thus, there is a downward jump in his risk tolerance at the threshold wealth level.

In the remainder of this section we will state and prove Proposition 3, Corollary 1, and Proposition 4 which illustrate the effects of the potential investment opportunity changes on the optimal investment strategy.

PROPOSITION 3.

*(i)The investor takes more(resp. less) risk at the wealth level below(resp. above) $z$ than he would in the absence of potential investment opportunity changes. That is, the risk tolerance satisfies*

$$\begin{cases} -\frac{V_1'(x_t)}{V_1''(x_t)} > \frac{x_t}{\gamma} = -\frac{M_1'(x_t)}{M_1''(x_t)}, & \textit{for } 0 < x_t < z, \\ -\frac{V_2'(x_t)}{V_2''(x_t)} < \frac{x_t}{\gamma} = -\frac{M_2'(x_t)}{M_2''(x_t)}, & \textit{for } x_t \geq z. \end{cases}$$

*(ii)The difference,* $-\frac{V_1'(x_t)}{V_1''(x_t)} - \left(-\frac{M_1'(x_t)}{M_1''(x_t)}\right) = -\frac{V_1'(x_t)}{V_1''(x_t)} - \frac{x_t}{\gamma}$*, is an increasing function of* $x_t$ *for* $0 < x_t < z$ *and approaches* $0$ *as* $x_t \downarrow 0$*. The difference,* $-\frac{M_2'(x_t)}{M_2''(x_t)} - \left(-\frac{V_2'(x_t)}{V_2''(x_t)}\right) = \frac{x_t}{\gamma} - \left(-\frac{V_2'(x_t)}{V_2''(x_t)}\right)$*, is a decreasing function of* $x_t$ *for* $x_t \geq z$ *and approaches* $0$ *as* $x_t \uparrow \infty$*.*

*Proof.* For $0 < x_t < z$, we have

$$
\begin{aligned}
-\frac{V_1'(x_t)}{V_1''(x_t)} + \frac{M_1'(x_t)}{M_1''(x_t)} &= \frac{1}{\gamma}\Big[\frac{C_1(x_t)}{C_1'(x_t)} - x_t\Big] \\
&= \frac{1}{\gamma}\Big[C_1(x_t)X_1'(C_1(x_t)) - X_1(C_1(x_t))\Big] \\
&= -\frac{1}{\gamma}(1+\gamma\lambda_-)B_1(C_1(x_t))^{-\gamma\lambda_-},
\end{aligned}
$$

where the first equality comes from (15) and (16), the second from the fact that $X_1$ is the inverse function of $C_1$, and the last from (10). Therefore, since $-\gamma\lambda_- > 0$, $-(1+\gamma\lambda_-) > 0$ by Remark 2.1, and $B_1 > 0$ by Lemma 1, we conclude that $-\frac{V_1'(x_t)}{V_1''(x_t)} + \frac{M_1'(x_t)}{M_1''(x_t)}$ is positive(that is, $-\frac{V_1'(x_t)}{V_1''(x_t)} > -\frac{M_1'(x_t)}{M_1''(x_t)}$) and an increasing function of $x_t$ for $0 < x_t < z$, and approaches 0 as $x_t \downarrow 0$.

For $x_t \geq z$, similarly we have

$$
\begin{aligned}
-\frac{M_2'(x_t)}{M_2''(x_t)} + \frac{V_2'(x_t)}{V_2''(x_t)} &= \frac{1}{\gamma}\Big[x_t - \frac{C_2(x_t)}{C_2'(x_t)}\Big] \\
&= \frac{1}{\gamma}\Big[X_2(C_2(x_t)) - C_2(x_t)X_2'(C_2(x_t))\Big] \\
&= \frac{1}{\gamma}(1+\gamma\eta_+)B_2(C_2(x_t))^{-\gamma\eta_+}.
\end{aligned}
$$

Therefore, since $-\gamma\eta_+ < 0$, $1+\gamma\eta_+ > 0$, and $B_2 > 0$ by Lemma 1, we conclude that $-\frac{M_2'(x_t)}{M_2''(x_t)} + \frac{V_2'(x_t)}{V_2''(x_t)}$ is positive(that is, $-\frac{V_2'(x_t)}{V_2''(x_t)} < -\frac{M_2'(x_t)}{M_2''(x_t)}$) and a decreasing function of $x_t$ for $x_t \geq z$, and approaches 0 as $x_t \uparrow \infty$. ∎

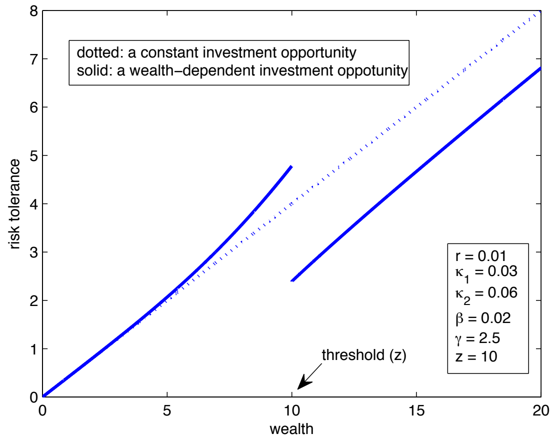
An intuitive explanation for Proposition 3 can be given as follows: when the investor's wealth is below the threshold level, he increases the expected growth rate of his wealth by taking more risk and thereby taking advantage of the risk premia in the risky assets to reach the threshold wealth level fast enough, while, when his wealth is above it, he reduces his risk taking because of the risk of wealth falling below the threshold level and losing the better investment opportunity. Proposition 3(ii) states that such effect becomes stronger as the investor's wealth gets closer to the threshold level, while the effect is negligible when his wealth is sufficiently far from it. Figure 3 illustrates these properties of optimal risky asset holdings.

We now state and prove Corollary 1 where (i) is a direct consequence of Proposition 3(i).

Corollary 1.

*(i) When the investor's wealth is below(resp. above) the threshold level $z$, the revealed coefficient of relative risk aversion implied by the value func-*

**FIG. 3.** Risk Tolerance




*tion,* $-\frac{xV''(x)}{V'(x)}$*, is smaller(resp. greater) than it is in the absence of potential investment opportunity changes. That is,*

$$\begin{cases} -\frac{xV_1''(x)}{V_1'(x)} < \gamma = -\frac{xM_1''(x)}{M_1'(x)} & \text{for } 0 < x < z, \\ -\frac{xV_2''(x)}{V_2'(x)} > \gamma = -\frac{xM_2''(x)}{M_2'(x)} & \text{for } x \geq z. \end{cases}$$

*(ii)The revealed coefficient of relative risk aversion implied by the value function is a decreasing function of* $x$ *for all wealth levels except at the threshold level where it makes an upward jump. It approaches* $\gamma$ *either as* $x \downarrow 0$ *or as* $x \uparrow \infty$*.*

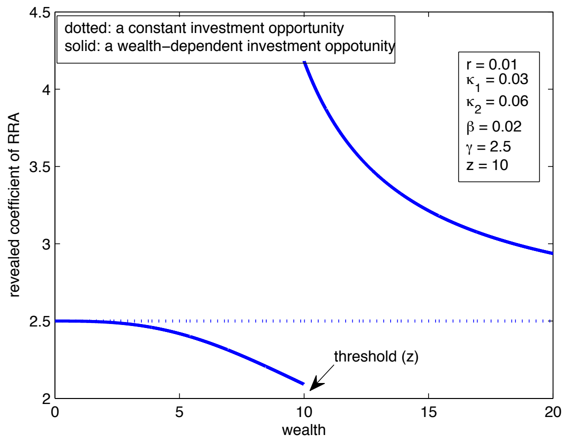
*Proof.* Assertion (i) in the corollary follows from (i) in Proposition 3. We will prove (ii). For $0 < x < z$, by using (15), (16), the inverse relationship beween $X_1$ and $C_1$, and (10), we can derive that

$$-\frac{xV_1''(x)}{V_1'(x)} = \frac{1}{-\lambda_-}\Big[1 + \frac{\frac{-\gamma\lambda_- - 1}{K_1}}{-\gamma\lambda_- B_1(C_1(x))^{-\gamma\lambda_- - 1} + \frac{1}{K_1}}\Big].$$

Since $\lambda_- < 0$ and $-\gamma\lambda_- - 1 > 0$ by Remark 2.1, $-\frac{xV_1''(x)}{V_1'(x)}$ is a decreasing function of $x$ for $0 < x < z$ and approaches $\gamma$ as $x \downarrow 0$. Since $\frac{x}{\gamma} - \Big(-\frac{V_2'(x)}{V_2''(x)}\Big) = x\Big[\frac{1}{\gamma} - \Big(-\frac{V_2'(x)}{xV_2''(x)}\Big)\Big] > 0$ is a decreasing function of $x$ for $x \geq z$ and approaches 0 as $x \uparrow \infty$ by Proposition 3, $\frac{1}{\gamma} - \Big(-\frac{V_2'(x)}{xV_2''(x)}\Big) > 0$ is a decreasing function of $x$ for $x \geq z$ and approaches 0 as $x \uparrow \infty$, which implies that $-\frac{xV_2''(x)}{V_2'(x)}$ is a decreasing function of $x$ for $x \geq z$ and approaches $\gamma$ as $x \uparrow \infty$. The upward jump at the threshold wealth level is shown by Remark 4.3. ▮

Figure 4 illustrates Corollary 1. At wealth below the threshold level the investor becomes more aggressive as his wealth increases, however, when his wealth crosses the threshold he suddenly becomes very conservative being afraid of losing the just acquired better investment opportunity and his risk aversion is far greater than it would be in the absence of potential investment opportunity changes and gets less and less conservative as his wealth increases. As his wealth becomes either very small or very large, the revealed coefficient of relative risk aversion approaches $\gamma$, the coefficient in the absence of potential investment opportunity changes.

**FIG. 4.** Revealed Coefficient of Relative Risk Aversion




*Remark 4.4.* The statement that $-\frac{V_1'(x_t)}{V_1''(x_t)} - \left(-\frac{M_1'(x_t)}{M_1''(x_t)}\right) = -\frac{V_1'(x_t)}{V_1''(x_t)} - \frac{x_t}{\gamma}$ $\left(-\frac{M_2'(x_t)}{M_2''(x_t)} - \left(-\frac{V_2'(x_t)}{V_2''(x_t)}\right) = \frac{x_t}{\gamma} - \left(-\frac{V_2'(x_t)}{resp. V_2''(x_t)}\right)\right)$ is an increasing (resp. decreasing) function of $x_t$ for $0 < x_t < z$ (resp. $x_t \geq z$) in Proposition 3(ii) is equivalent to the statement that the MPIR, $\frac{d}{dx_t}\left(-\frac{V_1'(x_t)}{V_1''(x_t)}\right)$ (resp. $\frac{d}{dx_t}\left(-\frac{V_2'(x_t)}{V_2''(x_t)}\right)$), is greater at the wealth level below (resp. above) the threshold than it is in the absence of potential investment opportunity changes, that is, $\frac{d}{dx_t}\left(-\frac{V_1'(x_t)}{V_1''(x_t)}\right) > \frac{1}{\gamma} = \frac{d}{dx_t}\left(-\frac{M_1'(x_t)}{M_1''(x_t)}\right)$ for $0 < x_t < z$ and $\frac{d}{dx_t}\left(-\frac{V_2'(x_t)}{V_2''(x_t)}\right) > \frac{1}{\gamma} = \frac{d}{dx_t}\left(-\frac{M_2'(x_t)}{M_2''(x_t)}\right)$ for $x_t \geq z$.

Proposition 4 states that the MPIR is a strictly increasing(resp. decreasing) function of wealth at wealth below(resp. above) the threshold level and it approaches its value in the absence of potential investment opportunity changes as wealth becomes either arbitrarily small or arbitrarily large.

PROPOSITION 4. *The MPIR,* $\frac{d}{dx_t}\Big(-\frac{V_1'(x_t)}{V_1''(x_t)}\Big)$*, is a strictly increasing function of wealth* $x_t$ *for* $0 < x_t < z$ *and the MPIR,* $\frac{d}{dx_t}\Big(-\frac{V_2'(x_t)}{V_2''(x_t)}\Big)$*, is a strictly decreasing function of wealth* $x_t$ *for* $x_t \geq z$*, with*

$$\lim_{x_t\downarrow 0}\frac{d}{dx_t}\Big(-\frac{V_1'(x_t)}{V_1''(x_t)}\Big)=\frac{1}{\gamma}=\frac{d}{dx_t}\Big(-\frac{M_1'(x_t)}{M_1''(x_t)}\Big)$$

*and*

$$\lim_{x_t\uparrow \infty}\frac{d}{dx_t}\Big(-\frac{V_1'(x_t)}{V_1''(x_t)}\Big)=\frac{1}{\gamma}=\frac{d}{dx_t}\Big(-\frac{M_1'(x_t)}{M_1''(x_t)}\Big).$$

*Proof.* By (15) and (16) we have

$$-\frac{V_i'(x_t)}{V_i''(x_t)}=\frac{C_i(x_t)}{\gamma C_i'(x_t)}\quad\text{for}\quad i=1,2.$$

Since $X_i$ is the inverse function of $C_i$ for $i = 1, 2$, we have

$$\begin{aligned}\frac{d}{dx}\Big(\frac{C_i(x_t)}{C_i'(x_t)}\Big) &= \frac{d}{dx}\Big(C_i(x_t)X_i'(C_i(x_t))\Big)\\ &= C_i'(x_t)X_i'(C_i(x_t))+C_i(x_t)X_i''(C_i(x_t))C_i'(x_t)\\ &= 1+C_i(x_t)X_i''(C_i(x_t))C_i'(x_t). \qquad (32)\end{aligned}$$

Therefore, by (27), (30), we have

$$\begin{aligned}&\frac{d}{dx_t}\Big(-\frac{V_1'(x_t)}{V_1''(x_t)}\Big)\\ =\ &\frac{1}{\gamma}[1+C_1(x_t)X_1''(C_1(x_t))C_1'(x_t)]\\ =\ &\frac{1}{\gamma}\Big[1+\frac{(\gamma\lambda_-+1)\gamma\lambda_-B_1(C_1(x_t))^{-\gamma\lambda_--1}}{-\gamma\lambda_-B_1(C_1(x_t))^{-\gamma\lambda_--1}+\frac{1}{K_1}}\Big]\\ =\ &\frac{1}{\gamma}\Big[1+\frac{(-\gamma\lambda_--1)\Big[-\gamma\lambda_-B_1(C_1(x_t))^{-\gamma\lambda_--1}+\frac{1}{K_1}\Big]+\frac{\gamma\lambda_-+1}{K_1}}{-\gamma\lambda_-B_1(C_1(x_t))^{-\gamma\lambda_--1}+\frac{1}{K_1}}\Big]\\ =\ &\frac{1}{\gamma}\Big[-\gamma\lambda_--\frac{\frac{-\gamma\lambda_--1}{K_1}}{-\gamma\lambda_-B_1(C_1(x_t))^{-\gamma\lambda_--1}+\frac{1}{K_1}}\Big],\end{aligned}$$

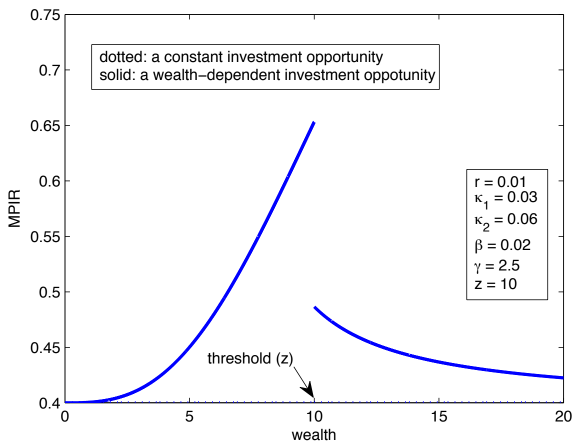
which, by Remark 2.1 and Lemma 1, is an increasing function of $x_t$ for $0 < x_t < z$ and approaches $\frac{1}{\gamma}$ as $x_t \downarrow 0$. Similarly, we have

$$
\begin{aligned}
& \frac{d}{dx_t}\Big( - \frac{V_2'(x_t)}{V_2''(x_t)} \Big) \\
= \; & \frac{1}{\gamma}[1 + C_2(x_t) X_2''(C_2(x_t)) C_2'(x_t)] \\
= \; & \frac{1}{\gamma}\Big[ 1 + \frac{(\gamma\eta_+ + 1)\gamma\eta_+ B_2(C_2(x_t))^{-\gamma\eta_+ - 1}}{-\gamma\eta_+ B_2(C_2(x_t))^{-\gamma\eta_+ - 1} + \frac{1}{K_2}} \Big] \\
= \; & \frac{1}{\gamma}\Big[ 1 + \frac{(-\gamma\eta_+ - 1)\Big[ -\gamma\eta_+ B_2(C_2(x_t))^{-\gamma\eta_+ - 1} + \frac{1}{K_2} \Big] + \frac{\gamma\eta_+ + 1}{K_2}}{-\gamma\eta_+ B_2(C_2(x_t))^{-\gamma\eta_+ - 1} + \frac{1}{K_2}} \Big] \\
= \; & \frac{1}{\gamma}\Big[ -\gamma\eta_+ + \frac{\frac{\gamma\eta_+ + 1}{K_2}}{-\gamma\eta_+ B_2(C_2(x_t))^{-\gamma\eta_+ - 1} + \frac{1}{K_2}} \Big],
\end{aligned}
$$

which, by Lemma 1, is a decreasing function of $x_t$ for $x_t \geq z$ and approaches $\frac{1}{\gamma}$ as $x_t \uparrow \infty$. ▮

According to Proposition 4, the incremental change in the investor's risky asset holdings gets larger(resp. smaller) as wealth increases at wealth below(resp. above) the threshold level. The incremental change approaches its value in the absence of potential investment opportunity changes as wealth becomes either arbitrarily small or arbitrarily large. Figure 5 illustrates these properties of the MPIR.

**FIG. 5.** MPIR



## 5. CONCLUSION

We have investigated an optimal consumption and investment problem in which the investor has a better investment opportunity when his wealth is above an exogenously given threshold level than he does when his wealth is below it. We have derived a closed form solution for the optimal consumption and investment strategies by using a dynamic programming method, and investigated the effects of the potential investment opportunity changes on them. We have shown that the investor consumes less with the wealth-dependent investment opportunity set than he would without it, and that he takes more(resp. less) risk at wealth below(resp. above) the threshold level than he would in the absence of potential investment opportunity changes. Furthermore, we have shown that such effects of the potential investment opportunity changes on the optimal consumption and investment strategies become stronger as the investor's wealth gets closer to the threshold level, while the effects are negligible when his wealth is sufficiently far from it.

The consumption and investment behavior of the investor we have derived is different from predictions either from traditional models (e.g.. Merton (1971)) or from recent model with decreasing relative risk aversion (e.g., Wachter & Yogo (2010)). It will be interesting to see if one can identify such a behavior from the data. This is left for future research.

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