ANNALS OF ECONOMICS AND FINANCE **16-1**, 95–114 (2015)

Robustness of Stability to Cost of Carrying Money in a Matching Model of Money

Pidong Huang*

Department of Economics, Korea University, Seoul, South Korea
E-mail: pidonghuang@korea.ac.kr

This paper studies stability of monetary steady states in a Trejos-Wright random matching model of money with the money holding set $\{0, 1, 2\}$ and a cost of carrying money. Three steady states are generic: pure-strategy full-support steady state; mixed-strategy full-support steady state; and non-full-support steady state. Stability analysis shows that both full-support steady states are stable and determinate. The non-full-support steady state is stable and indeterminate if there is a sufficiently small positive carrying cost. The non-full-support steady state becomes unstable if the carrying cost is zero.

Key Words: Random matching model; Monetary steady state; Instability; Determinacy.
JEL Classification Numbers: C62, C78, E40.

1. INTRODUCTION

Trejos and Wright (1995) show the existence of a monetary steady state in a random matching model under the assumption that an agent's money holding is in $\{0, 1\}$. In the same model, for consumer take-it-or-leave-it offers and for money holdings in $\{0, 1, \cdots, B\}$, Lee, Wallace and Zhu (2005) show that a full-support monetary steady state with a strictly increasing, concave value function is robust when a small carrying cost for money is introduced. By way of a variant of the neutrality argument, the result also implies the robustness of the non-full-support steady states in which all agents treat bundles of money, each bundle being $B/l \in \mathbb{N}$ units, as the smallest unit held and traded.

* The author especially thank Neil Wallace for his guidance and ecouragement. I am also grateful to the participants of the Cornell-Penn State macro workshop for helpful comments and discussion.

1529-7373/2015
All rights of reproduction in any form reserved.

Among the questions that Lee, Wallace and Zhu (2005)'s existence result leaves open are the following. First, are their full-support steady states unique? Second, do their steady states have a unique optimal strategy (pure-strategy) or multiple optimal strategies (mixed-strategy)? Third, are full-support steady states stable? Fourth, are the non-full-support steady states stable? The smallest set of money holdings for which these questions arise is $\{0, 1, 2\}$, the smallest set for which the distribution of money holdings over people depends on the trades that are made. For this set, Huang and Igarashi (2015) address all but the first questions when the carry cost is zero. This paper provides robust analysis on these existence and stability properties when the cost of carrying money is introduced.

Under a condition that is weaker than Lee, Wallace and Zhu (2005)'s sufficient conditions, a full-support steady state exists. Both pure-strategy and mixed-strategy full-support steady states exist generically and any full-support steady state is stable. We also show that any full-support steady state is determinate, a missing result in Huang and Igarashi (2015).

The non-full-support steady state, which necessarily has the support of $\{0, 2\}$, is unstable, but becomes stable and indeterminate if a small positive cost of carrying money is introduced. Lomeli and Temzelides (2002) shows in the $B = 1$ case that the non-monetary steady state is stable and indeterminate, and that this property is robust to the introduction of a small carrying cost. This result and ours suggest that the dynamic equilibria around the non-full-support steady in the $B = 2$ economy is not a simple "translation" of dynamic equilibria of those steady states in the $B = 1$ case. The two-unit bound, although restrictive, at least, provides conjecture for the general case.

2. THE ZHU (2003) MODEL

The model is that of Zhu (2003), where a small carry cost is introduced. Time is discrete, dated as $t \geq 0$. There is a non-atomic unit measure of infinitely-lived agents, and a consumption good that is perfectly divisible and perishable. Each agent maximizes the discounted sum of expected utility with a discount factor $\beta \in (0, 1)$. Utility in a period is $u(c)-q$, where $c \in \mathbb{R}_+$ is the amount of good consumed and $q \in \mathbb{R}_+$ is the amount of good produced. $u : \mathbb{R}_+ \to \mathbb{R}$ is continuously differentiable, strictly increasing and strictly concave. Also, $u(0) = 0$, $u'(\infty) = 0$ and $u'(0)$ is sufficiently large but finite. These assumptions imply that there is a unique $\bar{x} > 0$ such that $u(\bar{x}) = \bar{x}$.

There exists a fixed stock of indivisible money that is perfectly durable and that can potentially serve as a medium of exchange. Storing each unit of money in each period incurs a disutility $\gamma > 0$, that in the model we assume to be sufficiently small. There is a bound on individual money holdings, denoted $B \in \mathbb{N}$, so the individual money-holding set is $\mathbb{B} \equiv \{0, 1, \cdots, B\}$. Let $m \in (0, 1)$ denote the per capita stock of money divided by the bound on individual money holdings so that the per capita stock is Bm.

In each period, agents are randomly matched in pairs. With probability $1/N$, where $N \geq 2$, an agent is a consumer (producer) and the partner is a producer (consumer). Such meetings are called single-coincidence meetings. With probability $1 - 2/N$, the match is a no-coincidence meeting.[1] In all meetings, the agents' money holdings are observable, but any other information about an agent's trading history is private.

Consider a date-t single-coincidence meeting between a consumer (potential buyer) with i units of money (pre-trade) and a producer (potential seller) with j units of money (pre-trade), an (i, j)-*meeting*. If $i > 0$ and $j < B$, the meeting is called a *trade meeting*. In trade meetings, the consumer makes a take-it-or-leave-it offer. (There are no lotteries.) The producer accepts or rejects the offer. If the producer rejects it, both sides leave the meeting and go on to the next date.

For each $k \in \mathbb{B}$, let w_k^t be the expected discounted value of holding k units of money prior to date-t matching. Using w_k^t, the consumer's problem in an (i, j)-meeting is

$$\max_{p \in \Gamma(i,j), q \in \mathbb{R}_+} \{u(q) + \beta w_{i-p}^{t+1}\} \tag{1}$$

$$\text{s.t.} \ -q + \beta w_{j+p}^{t+1} \geq \beta w_j^{t+1}, \tag{2}$$

where $\Gamma(i, j) \equiv \{p \in \mathbb{B} | p \leq \min\{i, B - j\}\}$ is the set of feasible payments. As (2) holds with equality in the solution, the consumer's problem reduces to

$$f^t(i, j) \equiv \max_{p \in \Gamma(i,j)} \{u\left(\beta w_{j+p}^{t+1} - \beta w_j^{t+1}\right) + \beta w_{i-p}^{t+1}\}$$

$$p^t(i, j) = \operatorname*{argmax}_{p \in \Gamma(i,j)} \{u\left(\beta w_{j+p}^{t+1} - \beta w_j^{t+1}\right) + \beta w_{i-p}^{t+1}\}. \tag{3}$$

Because the solution set $p^t(i, j)$ may be multi-valued, Zhu introduces randomization. The behavior λ^t assign probability $\lambda^t(p; i, j)$ to trading p units of money in (i, j)-meeting. The distribution $\lambda^t(\cdot; i, j)$ is the (consumer's) optimal strategy if it has the support in $p^t(i, j)$.

For each $z \in \mathbb{B}$, let π_z^t denote the fraction of agents holding z units of money at the start of period t, so that π^t is a probability distribution over

[1]One foundation is that there are N types of agents and N types of consumption goods, that type-n agents can only produce type-n goods and only consume type-$(n+1)$ goods only, and that the money is symmetrically distributed across the types.

$\mathbb{B}$ with mean Bm. Given strategy λ^t, the law of motion for π^{t+1} can be expressed as

$$\pi_z^{t+1} = \frac{N-2}{N}\pi_z^t + \frac{2}{N}\sum_{i=0}^{B}\sum_{j=0}^{B}\pi_i^t\pi_j^t \frac{\lambda^t(i-z;i,j)+\lambda^t(z-j;i,j)}{2}. \tag{4}$$

The second term of (4) tells us who in the single-coincidence meetings will end up with z units: consumers who originally had i units and spent $i-z$ units, and producers who originally had j units and acquired $z-j$ units.

The value function w^t satisfies the Bellman equation

$$w_i^t = \frac{N-1}{N}\beta w_i^{t+1} + \frac{1}{N}\sum_{j=0}^{B}\pi_j^t f^t(i,j) - \gamma i, \tag{5}$$

The first term on the r.h.s. corresponds to either entering a no-coincidence meeting or becoming a producer who is indifferent between trading or not. When $i=0$, equation (5) reduces to $w_0^t = \beta w_0^{t+1}$, so the only nonexplosive case is $w_0^t = 0, \forall t$. For this reason, we focus on equilibria in which the value from owning no money is always zero and let $w^t \equiv (w_1^t, \cdots, w_B^t)$. Finally, we allow the free disposal of money and consider equilibria in which agents are not willing to throw away money. That is, the value function must be nondecreasing in every period:

$$w_i^t \geq w_{i-1}^t, \text{ for } i = 1, \cdots B, \text{ and } w_0^t = 0. \tag{6}$$

DEFINITION 2.1. Given π^0, an *equilibrium* is a sequence $\{(\pi^t, w^t)\}_{t=0}^{\infty}$ that satisfies (3)-(6). A tuple (π, w) is a *monetary steady state* if $(\pi^t, w^t) = (\pi, w)$ for $t \geq 0$ is an equilibrium and $w \neq 0$. Pure-strategy steady states are those for which (3) has a unique solution. Other steady states are called mixed-strategy steady states.

3. STEADY STATES FOR $\mathbb{B} = \{0, 1, 2\}$

We consider the simplest economy where the dynamics in money-holding distribution could occur: an economy with $\mathbb{B} = \{0, 1, 2\}$. Three monetary steady states are generic. The first one is a steady state with the support of $\{0, 2\}$, called a non-full-support steady state. They can be constructed by treating the bundle of two units of money as one unit as in Trejos and Wright (1995), the case $B = 1$. Paying two units in a $(2, 0)$-meeting and one unit in other trade meetings is the optimal strategy at the steady state. The value of holding one unit, w_1, is the smaller of the two solutions to the

Bellman equation (7), while w_2 is the largest of the two solutions to (8).

$$w_1 = \frac{N+m-1}{N}\beta w_1 + \frac{1-m}{N}u(\beta w_1) - \gamma \tag{7}$$
$$w_2 = \frac{N+m-1}{N}\beta w_2 + \frac{1-m}{N}u(\beta w_2) - 2\gamma \tag{8}$$

When γ is strictly greater than zero but sufficiently small, w_1 is strictly positive and the value function is strictly increasing. When $\gamma = 0$, the steady state has a step value function. This property is similar to the observation in an overlapping generation model, where the non-monetary steady state equilibrium is eliminated by the cost of carrying money, and will play a role in altering the stability of the non-full-support steady state.

The other two are the only full-support steady states in $B = 2$ economy. One has paying one unit in all meetings as the unique optimal and, thus, is pure-strategy. The other one has agents indifferent between one-unit payment and two-unit payment in a $(2,0)$-meeting, and is mixed-strategy. The following says that all three steady states are generic.

PROPOSITION 1. *Generically, all pure- and mixed-strategy steady states, and non-full-support steady state exist.*

Huang and Igarashi (2015) shows the existence of the three steady states for an economy with $\mathbb{B} = \{0, 1, 2\}$ and $\gamma = 0$. Proposition 1 implies that such an existence is robust when a small cost of carrying money is introduced.

The proof of this and all other proofs appear in Section 5. The proof finds a pure-strategy steady state for a β sufficiently close to one, and a mixed-strategy steady state for a β of intermediate value. Intuitively, for a β sufficiently close to one, each unit of money tends to have value large enough, so that in a $(2,0)$-meeting, paying all two units is too much and becomes suboptimal. As β decreases from one, the value of money is depreciated and it is more likely that one unit of money is not enough to induce the optimal production in a $(2,0)$-meeting. The critical point corresponds to a β of intermediate magnitude. The proof has the details of finding different steady states for different β.

4. STABILITY

Our stability criterion is asymtoptic stability.

DEFINITION 4.1. A steady state (π, w) is locally stable if there is a neighborhood of π such that for any initial distribution in the neighborhood, there is an equilibrium path such that $(\pi^t, w^t) \to (\pi, w)$. A locally stable steady state is determinate, if for each initial distribution in this neighborhood, there is only one equilibrium that converges to it.

This definition of stability only requires convergence of some equilibria, not all equilibria. This is because there are always equilibria that do not converge to a given monetary steady state. In particular, a non-monetary equilibrium always exists from any initial condition.

Notice that the above definition of local stability implies that the valued-money steady state in the Trejos-Wright $\{0,1\}$ model is stable, because there is no 'neighborhood' of the steady state. Also, for that model, the only non-explosive path converging to that steady state is the one in which the value of money remains constant, which implies determinacy of that steady state. The following is our stability results for the $\{0,1,2\}$ economy.

Proposition 2. *Generically, both full-support steady states are locally stable and determinate, while the non-full-support steady state is locally stable and indeterminate. If the cost of carrying money is eliminated, the non-full-support steady state becomes unstable.*

The proof starts from a first-order difference equation in (π_1^t, w_1^t, w_2^t) that is derived from the $B = 2$ version of (4)-(5). In this system, only π_1^t has an exogenous initial value and is a 'predetermined' variable. Proving the stability of the pure-strategy steady is standard (see Lucas et. al (1989)). We show that the stable manifold is one-dimensional.

The stability analysis of the mixed-strategy steady state considers a dynamical system with a control variable. The proof find an equilibrium that jumps onto the steady state at $t = 1$, if the initial distribution is sufficiently close to the steady state distribution. The convergent path is unique, because the indifference condition between trading one unit and two units in a $(2,0)$-meeting is not preserved by the Bellman equation off the steady state generically.

Proving the stability of the non-full-support steady state is not standard. The idea is borrowed from Huang and Igarashi (2014). There is a unique feature of any convergent path. The convergence of π_1^t is slow, because there is no inflow into holdings of 1 unit and because the outflow, which comes from (1,1)-meetings, approaches zero as the frequency of such meetings goes to zero. This implies that the dominant root that determines the speed of convergence is equal to one.

As Huang and Igarashi (2014) shows, when $\gamma = 0$, the steady state is on the boundary of the state space. Hence, even if the system appears to be convergent, we have to check that the convergence is such that π_1^t, $w_1^t \geq 0$, using the eigenvector that corresponds to the dominant unit root, and this condition fails. Such properties do not occur when $\gamma > 0$ is sufficiently small and hence the instability of the non-full-support steady state is not robust when the carrying cost is introduced.

The non-monetary steady state in the overlapping generation model is also on the boundary of the state space. The presence of a postive cost of carrying money does not alter its stability properties, because the value of money always approaches the steady state value from $w_1^t \geq 0$.

5. PROOFS

Before turning to the proofs, we set out some steady state consequences that we use in the proofs. It is helpful to express π_0 and π_2 in terms of π_1 using $\sum \pi_i = 1$ and $\sum i\pi_i = Bm$:

$$\begin{aligned}(\pi_0, \pi_2) &= (1 - m - \frac{\pi_1}{2}, m - \frac{\pi_1}{2}) \\ \text{where } \pi_1 &\in \Pi \equiv [0, 2\min\{m, 1-m\}]. \end{aligned} \tag{9}$$

Throughout this paper, the dependence of π on π_1 is kept implicit to simplify the notations. The steady-state law of motion implies

$$(\pi_1)^2 \lambda(1; 1, 1) = \pi_0 \pi_2 \lambda(1; 2, 0), \tag{10}$$

which equates outflows from holdings of 1 (the lefthand side) to inflows into holdings of 1 (the righthand side). The Bellman equations are

$$\begin{aligned} w_1 = \frac{n - 1 + \pi_2}{n} \beta w_1 + \frac{\pi_0}{n} \max[u(\beta w_1), \beta w_1] \\ + \frac{\pi_1}{n} \max[u(\beta w_2 - \beta w_1), \beta w_1] - \gamma, \quad \text{and} \end{aligned} \tag{11}$$

$$\begin{aligned} w_2 = \frac{n - 1 + \pi_2}{n} \beta w_2 + \frac{\pi_1}{n} \max[\{u(\beta w_2 - \beta w_1) + \beta w_1\}, \beta w_2] \\ + \frac{\pi_0}{n} \max[u(\beta w_2), \{u(\beta w_1) + \beta w_1\}, \beta w_2] - 2\gamma. \end{aligned} \tag{12}$$

Lemma 1 could be viewed as an existence result of mixed-strategy steady state.

LEMMA 1. *Let*

$$\delta_{\pi_1} \equiv \frac{(1 - \pi_2)\beta}{[n(1 - \beta) + (1 - \pi_2)\beta]}^{2} \tag{13}$$

[2]The subscript in δ_{π_1} emphasizes the dependence on π_1.

$$\pi_1^* \equiv (\sqrt{1 + 12m(1-m)} - 1)/3. \tag{14}$$

and

$$x = \frac{\delta_{\pi_1}}{1-\pi_2}[\pi_0 u(x) + \pi_1 u\left(\delta_{\pi_1} x - \frac{\delta_{\pi_1}}{1-\pi_2} n\gamma\right) - n\gamma] \equiv f(x, \pi_1). \tag{15}$$

Suppose $f_x(0,0) > 1 > f_x(0, \pi_1^*)$ *when* $\gamma = 0$. *There exists a sufficiently small* $\gamma > 0$ *such that a mixed-strategy full-support steady state exists.*

The proof uses the intermediate value theorem. The process involves checking whether some strict inequality conditions hold at both end of an interval, corresponding to $\lambda(1;2,0) = 0$ and $\lambda(1;2,0) = 1$. Then we show that all these strict inequalities are robust to the introduction of a small carrying cost.

Proposition 1 of Huang and Igarashi (2015) provides a necessary and sufficient condition for the existence of full-support steady states in the model with $\gamma = 0$. Our proposition 1 says that both pure-strategy and mixed-strategy steady states are robust to the introduction of carrying cost.

The proof starts out by considering $\gamma = 0$. To find the pure-strategy steady state for this case, we consider a β sufficiently close to one. We guess the pure strategy of the steady state. With this strategy, we can solve the Bellman equation and the law of motion, and then check whether the solutions imply the optimality of the guessed strategy. This process is mathematically equivalent to checking certain strict inequalities. The same set of strict inequalities still hold, when $\gamma > 0$ is sufficiently small. To show the existence of a mixed-strategy steady state, we find a suitable β of intermediate value so that the strict inequatity condition in Lemma 1 holds.

Proof. [Proof of Proposition 1] We are going to construct a pure-strategy steady state with a one-unit payment being optimal in all trade meetings, and a mixed-strategy steady state with a two-unit payment also being optimal in $(2,0)$-meeting. The corresponding inequalities are

$$\begin{aligned}
&(1,1)\text{-meeting } u(\beta w_2 - \beta w_1) &>& \ \beta w_1 && (16)\\
&(1,0)\text{-meeting } \qquad u(\beta w_1) &>& \ \beta w_1 && (17)\\
&(2,1)\text{-meeting } u(\beta w_2 - \beta w_1) &>& \ \beta w_2 - \beta w_1 && (18)\\
&(2,0)\text{-meeting } u(\beta w_1) + \beta w_1 &\geq& \ u(\beta w_2) && (19)\\
&\qquad\qquad \& \ u(\beta w_1) + \beta w_1 &>& \ \beta w_2. && (20)
\end{aligned}$$

Our pure-strategy steady state has strict inequality in (19), while the mixed-strategy steady state has equality in (19). With these inequalities, the Bellman equation (11)-(12) is equivalent to (15) and (13) with

$$x = \beta w_1 \tag{21}$$

$$\delta_{\pi_1} x - \frac{\delta_{\pi_1}}{1-\pi_2} n\gamma = \beta w_2 - \beta w_1. \tag{22}$$

The pure strategy steady state has $\pi_1 = \pi_1^*$. Let $\beta \to 1$ and $\gamma = 0$. (15)[3] approaches $x = u(x)$. The l.h.s. and r.h.s. of (19) approaches $u(x)+x$ and $u(2x)$, respectively. Concavity of u implies

$$u(2x) < u(x) + x.$$

(19) holds with strict inequality in the limit. Claim A.1 implies (16)-(18) and (20). Overall, the positive solution to (15)-(13) satisfies (16)-(20) with strict inequality in (19).

Then $\gamma > 0$ is introduced. For a β sufficiently close to one and a sufficiently small $\gamma > 0$, the continuity implies that (16)-(20) still hold with strict inequality in (19). We have a pure-strategy full-support steady state.

Then we use Lemma 1 to find a mixed-strategy steady state. One condition in the lemma is $f_x(0, \pi_1^*) < 1$ when $\gamma = 0$, which is equivalent to

$$u'(0) < \frac{1-\pi_2^*}{\delta_{\pi_1^*}(\pi_0^* + \pi_1^* \delta_{\pi_1^*})} \equiv J_{\beta\pi_1^*}. \tag{23}$$

Another condition in the lemma is $f_x(0,0) > 1$ when $\gamma = 0$, equivalent to

$$u'(0) > \frac{1}{\delta_0} = J_{\beta 0}. \tag{24}$$

As β increases from zero toward one, both $\delta_{\pi_1^*}$ and δ_0 increase from zero to one, and hence $J_{\beta\pi_1^*}$ and $J_{\beta 0}$ decrease from $+\infty$ to 1.

Some algebra gives

$$J_{\beta\pi_1^*} - J_{\beta 0} = \frac{n(1-\beta)}{\beta} \cdot \frac{\pi_1^* n(1-\beta) + \beta\pi_1^*\pi_0^*}{[\pi_0^* n(1-\beta) + \beta(1-\pi_2^*)^2](2-2m)} > 0.$$

Therefore, we can find a β^* of intermediate value so that the open interval $(J_{\beta 0}, J_{\beta\pi_1^*})$ contains $u'(0)$. Lemma 1 applies.

Finally, the non-full-support steady state has $\lambda(2;2,0) = 1$ and Bellman equation becomes (7) and (8). It is not hard to see that for sufficiently large $u'(0)$ and sufficiently small $\gamma > 0$, each of these two equations has two solutions. Let the steady state w_1 be the smaller solution to (7) and w_2 be the

[3] It is not difficult to ensure that a positive solution to (15) exists; $u'(0) > 1$ for instance.

larger solution to (8). And it is not hard to check that (w_1, w_2) implies the optimality of the strategy for this steady state. Hence a non-full-support steady state exists. ∎

Then, the remaining are devoted to the stability analysis. The following lemma specifies the strategy near the steady state so that the difference equation can be derived from the Bellman equation and Law of motion. It says that the optimal trades resemble the steady-state trades near the steady state.

LEMMA 2. *Along any equilibrium convergent to any of the three steady states, trading one unit is strictly preferred in* $(1,0)$-, $(1,1)$- *&* $(2,1)$-*meetings. For* $(2,0)$-*meeting, we have:*

i) in the pure strategy steady state, one-unit payment is strictly preferred.

ii) in the non-full-support steady state, two-unit payment is strictly preferred.

iii) in the mixed-strategy steady state, the agents are indifferent between one- and two-unit payment.

The proof of Proposition 2 linearizes the dynamical system. The eigenvalue of the linearized system tells us the speed of convergence, and the associated eigenvector tells us the limiting behavior of state variables.[4] The stability analysis on the pure-strategy steady state and mixed-strategy steady state involves matrix computations.

Because randomization occurs in $(2,0)$-meeting, the linearized system for the mixed strategy is a dynamical system with a control variable, the trade in this meeting. The analysis involves finding a sequence of such variable so that the state variable converges and satisfies the indifference condition in Lemma 2.

Proof (Proof of Proposition 2). Lemma 2 gives all the strategies for all steady states. We can construct a dynamical system from the law of motion and the Bellman equation under each strategy. The following is the common form of the three dynamical systems:

$$\begin{aligned}
\pi_1^{t+1} &= \pi_1^t - \frac{2(\pi_1^t)^2}{n} + \frac{2}{n}\left(1 - m - \frac{\pi_1^t}{2}\right)\left(m - \frac{\pi_1^t}{2}\right)\eta^t && (25)\\
w_1^t &= \frac{n-1+\pi_2^t}{n}\beta w_1^{t+1} + \frac{\pi_0^t}{n}u(\beta w_1^{t+1}) + \frac{\pi_1^t}{n}u(\beta w_2^{t+1} - \beta w_1^{t+1}) - \gamma && (26)\\
w_2^t &= \frac{n-1+\pi_2^t}{n}\beta w_2^{t+1} + \frac{\pi_0^t}{n}\max[u(\beta w_1^{t+1}) + \beta w_1^{t+1}, u(\beta w_2^{t+1})]\\
&\quad + \frac{\pi_1^t}{n}[u(\beta w_2^{t+1} - \beta w_1^{t+1}) + \beta w_1^{t+1}] - 2\gamma, && (27)
\end{aligned}$$

[4]The subsection "dominant eigenvector" in Luenberger (1979) says that trajectory will be parallel to the eigenspace spanned by the dominant eigenvectors.

where $\eta^t = \lambda^t(1;2,0)$. Denote (25) by $\pi_1^{t+1} = \Phi(\eta^t, \pi_1^t) : [0,1] \times \Pi \to \Pi$ and (26)-(27) by $w^t = \phi(\eta^t, \pi_1^t, w^{t+1}) : [0,1] \times \Pi \times W \to W$, where $w^t \equiv (w_1^t, w_2^t)$ and $W \equiv \{(w_1, w_2) | 0 \leq w_1 \leq w_2\}$. In the vicinity of each of the steady states, we implicitly solve w^{t+1} as a function of (η^t, π_1^t, w^t) to obtain $w^{t+1} = \Psi(\eta^t, \pi_1^t, w^t) : \Pi \times W \to W$. The joint system is

$$\begin{pmatrix} \pi_1^{t+1} \\ w^{t+1} \end{pmatrix} = \begin{pmatrix} \Phi(\eta^t, \pi_1^t) \\ \Psi(\eta^t, \pi_1^t, w^t) \end{pmatrix}. \tag{28}$$

The three linearized systems, associated with different steady states respectively, share the same form:

$$\begin{pmatrix} \Delta\pi_1^{t+1} \\ \Delta w^{t+1} \end{pmatrix} = A \begin{pmatrix} \Delta\pi_1^t \\ \Delta w^t \end{pmatrix} + \begin{pmatrix} \Phi_\eta \\ 0 \end{pmatrix} \Delta\eta^t, \tag{29}$$

where

$$A \equiv \begin{bmatrix} \Phi_\pi & O \\ -(\phi_w)^{-1}\phi_\pi & (\phi_w)^{-1} \end{bmatrix}. \tag{30}$$

This matrix is generically invertible, confirming that applying the implicit function theorem to solve w^{t+1} is valid. Along any convergent path, we approximate the system (28) by its linearized version (29).

Note that $\Delta\eta^t = 0$ near the pure-strategy steady state. Lemma 3 implies that A for the pure-strategy full-support steady state has only one eigenvalue less than one in absolute value. Therefore the pure-strategy steady state has a one-dimensional stable manifold. Because we have one initial condition (or restriction), this steady state is locally stable and determinate.

Next, we consider the mixed-strategy steady state. It is going to be shown that the Bellman equation (26)-(27) is not consistent generically with the indifference condition (A.6), whose linearized version is

$$[u'(\beta w_1^*) + 1]\Delta w_1^t = u'(\beta w_2^*)\Delta w_2^t. \tag{31}$$

In what follows, we view the linearized system (29) as a dynamical system with a single input variable, $\Delta\eta^t$. We will find a sequence $\{\Delta\eta^t\}_0^\infty$ such that the equilibrium path satisfies (31) for all $t \geq 1$.

Let $\Delta\eta(z)$, $\Delta\pi_1(z)$ and $\Delta w(z)$ be the z-transforms[5] of $\{\Delta\eta^t\}_0^\infty$, $\{\Delta\pi_1^t\}_0^\infty$ and $\{\Delta w^t\}_0^\infty$ respectively. By (29), $\Delta\pi_1(z)$ and $\Delta w(z)$ are functions of $\Delta\eta(z)$:

$$\begin{pmatrix} \Delta\pi_1(z) \\ \Delta w(z) \end{pmatrix} = [Iz - A]^{-1}[\begin{pmatrix} \Phi_\eta \\ 0 \end{pmatrix} \Delta\eta(z) + \begin{pmatrix} \Delta\pi_1^0 \\ \Delta w^0 \end{pmatrix} z]. \tag{32}$$

[5]The z-transform of a sequence of numbers $\{y_t\}$ is $Y(z) = \sum_{t=0}^\infty \frac{y_t}{z^t}$. Please refer to Subsection 8.2-8.4 in Luenberger (1979) for detailed discussion.

Applying z-transform to (31) for $t \geq 1$ and then substituting (32) into it, we have

$$\begin{aligned}&\begin{pmatrix} 0 & u'(\beta w_1^*)+1 & -u'(\beta w_2^*) \end{pmatrix}[Iz-A]^{-1}[\begin{pmatrix} \Phi_\eta \\ 0 \end{pmatrix}\Delta\eta(z)+\begin{pmatrix} \Delta\pi_1^0 \\ \Delta w^0 \end{pmatrix}z] \\ = \; & [u'(\beta w_1^*)+1]\Delta w_1^0 - u'(\beta w_2^*)\Delta w_2^0. \end{aligned} \tag{33}$$

Lemma 4 implies that (33) holds as an identity only if $\Delta\eta(z) = \Delta\eta^0$ and hence $\Delta\eta^t = 0$ for all $t \geq 1$. Therefore the only possible convergent path must jump into the steady state at $t = 1$. By (32), the path has $\binom{\Delta\pi_1^0}{\Delta w^0} = -A^{-1}\binom{\Phi_\eta}{0}\Delta\eta^0$, where $\Delta\pi_1^0$ is given by the initial condition, and $\Delta\eta^t = \Delta\pi_1^t = \Delta w_1^t = \Delta w_2^t = 0$ for all $t \geq 1$. This path satisfies (29) and (31), and thus it is an equilibrium. The mixed-strategy steady state is locally stable and determinate generically.

Finally we consider the non-full-support steady state, which has $\eta^t = 0$. By Lemma 5, the matrix A has a unit eigenvalue, due to the law of motion. In the context of the discrete-time dynamical system theory, the unit root is a "border" case in which the higher-order terms should be examined. In our case, the higher-order term seems to imply unit-root convergence (i.e., Figure 1). Lemma 5 implies a two dimensional stable manifold. When $\gamma > 0$ is sufficiently small, the initial condition on π_1^0 imposes one restriction on the convergent paths, and reduces the degree of freedom by one. The steady state is locally stable and indeterminate.

When $\gamma = 0$, this steady state is on the boundary of the state space $\Pi \times W$, which makes it necessary to explicitly study the limiting behavior by looking at the eigenspace of (30)[6]. It turns out that keeping the entire convergent path in the space becomes problematic. In particular, unit root convergence implies that the convergent trajectory of $(\pi_1^t, w_1^t, w_2^t - w_2)$ will be parallel to the eigenspace spanned by the eigenspace of (A.11) associated with the unit eigenvalue[7]. By Lemma 5, π_1^t and w_1^t will eventually have different signs, contradicting $\pi_1^t, w_1^t > 0$ for all t. Therefore, the steady state is not stable. ▌

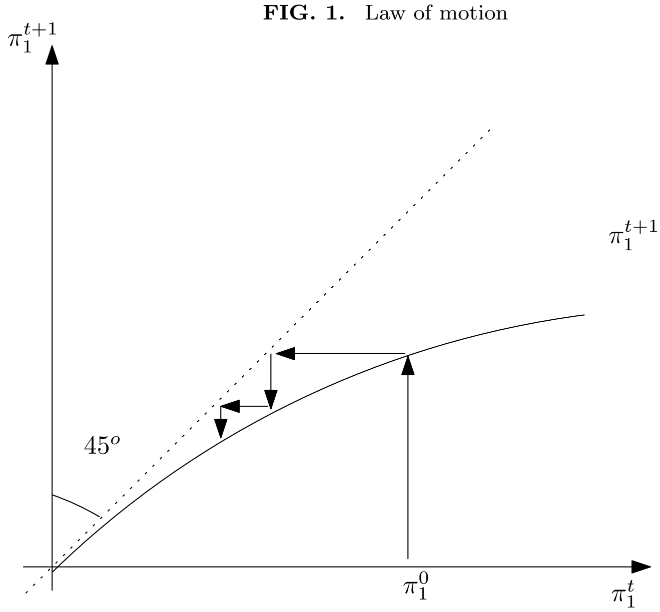
6. CONCLUDING REMARKS

We show that both the pure-strategy and mixed-strategy full-support steady states are generic. They are stable and determinate generically. The non-full-support steady state has a strictly increasing value function

[6]Note that this analysis is not needed for the pure-strategy full-support steady state because that steady state is in the interior of $\Pi \times W$.

[7]See Subsection "Dominant Eigenvector" on page 165 of Luenberger (1979).

FIG. 1. Law of motion



if there is a sufficiently small carrying cost, namely $\gamma > 0$. Such a feature alters its stability.

Given our result, several reasonable conjectures could be made. For a higher bound, the existence of both types of full-support steady state is generic. For values of parameters that lead to lower values of money (i.e., high n, low β and high m), randomizations may occur. When $\gamma > 0$ is sufficiently small, the stability of a non-full-support steady state will depend on the stability of the associated full-support steady state.[8]

APPENDIX A

Claim A.1. Suppose $\gamma = 0$. For any $\pi_1 \in (0, \pi_1^*]$, the solution to equations (15)-(13) satisfies (16)-(18) and (20).

Proof. Suppose by way of contradiction that (16) does not hold:

$$u(\beta w_2 - \beta w_1) = u\left(\frac{(1-\pi_2)\beta}{n(1-\beta)+(1-\pi_2)\beta}\beta w_1\right) \leq \beta w_1.$$

[8]Huang and Igarashi (2014) is an attempt to generalise the instability of non-full-support steady states (Proposition 2) to a general bound case when $\gamma = 0$.

Then, we have

$$\begin{aligned}\beta w_1 &< \frac{\pi_0\beta}{n(1-\beta)+\pi_0\beta}u(\beta w_1)\\ &< u\left(\frac{\pi_0\beta}{n(1-\beta)+\pi_0\beta}\beta w_1\right)\\ &< u\left(\frac{(1-\pi_2)\beta}{n(1-\beta)+(1-\pi_2)\beta}\beta w_1\right)=u(\beta w_2-\beta w_1),\end{aligned}$$

where substituting the supposition into (15) gives the first inequality, and $u(0)=0$ and strict concavity of u imply the second. This is a contradiction and thus (16) should hold.

Inequality (20) follows from

$$\begin{aligned}u(\beta w_1) &> u(\beta w_2-\beta w_1)\\ &> \beta w_1\\ &> \beta w_2-\beta w_1,\end{aligned}$$

where the first and the third inequalities are by (13) and the second is (16).

Suppose by way of contradiction that (17) does not hold: $u(\beta w_1)\le\beta w_1$. Then (16) implies $\beta w_2-\beta w_1>\beta w_1$. Combining this with (20) gives $u(\beta w_1)>\beta w_1$, which is a contradiction.

Suppose by contradiction that (18) does not hold: $u(\beta w_2-\beta w_1)\le\beta w_2-\beta w_1$. Then (20) implies $\beta w_2-\beta w_1\le\beta w_1$. But (16) and supposition imply $\beta w_2-\beta w_1>\beta w_1$, which is a contradiction. ▮

Proof (Proof of Lemma 1). Start with $\gamma=0$. Because $f_x(0,0)>1>f_x(0,\pi_1^*)$, the intermediate value theorem implies that there exists $\bar\pi_1\in(0,\pi_1^*)$ such that $1=f_x(0,\bar\pi_1)$ and $1<f_x(0,\pi_1)$ for all $\pi_1\in[0,\bar\pi_1)$. With $\lim_{x\to\infty}f_x(x,\pi_1)=0$, there is a unique positive solution, denoted by x_{π_1}, to (15) for each $\pi_1\in[0,\bar\pi_1)$. (13) defines δ_{π_1} as a function of π_1.

A necessary condition for the existence of x_0 is $f_x(x_0,0)<1$, which in turn implies

$$u'(x_0)<\frac{n(1-\beta)}{\beta(1-m)}+1=\frac{1}{\delta_0}. \tag{A.1}$$

Then by the mean value theorem, we have

$$\begin{aligned}u(x_0)+x_0-u(x_0(1+\delta)) &= x_0-u'(\xi)\delta_0x_0,\quad \xi\in(x_0,x_0(1+\delta))\\ &> x_0-u'(x_0)\delta_0x_0\\ &> 0,\end{aligned} \tag{A.2}$$

where the second inequality follows from (A.1). Therefore (19) holds with strict inequality when $\pi_1 = 0$.

Then $1 = f_x(0, \bar{\pi}_1)$ implies

$$\begin{aligned} &\frac{n(1-\beta)+\beta(1-m+\frac{\bar{\pi}_1}{2})}{\beta(1-m+\frac{\bar{\pi}_1}{2})} \\ = &\left[\frac{1-m-\frac{\bar{\pi}_1}{2}}{1-m+\frac{\bar{\pi}_1}{2}} + \frac{\beta\bar{\pi}_1}{n(1-\beta)+\beta(1-m+\frac{\bar{\pi}_1}{2})}\right] u'(0), \end{aligned} \tag{A.3}$$

where the coefficient of $u'(0)$ is proven to be smaller than one for any $n > 0$. This implies

$$u'(0) > \frac{n(1-\beta)+\beta(1-m+\frac{\bar{\pi}_1}{2})}{\beta(1-m+\frac{\bar{\pi}_1}{2})}. \tag{A.4}$$

Thus

$$\begin{aligned} &\frac{u(x_{\pi_1}) + x_{\pi_1} - u(x_{\pi_1}(1+\delta_{\pi_1}))}{x_{\pi_1}} < 1 - u'(x_{\pi_1}(1+\delta_{\pi_1}))\delta_{\pi_1} \\ \to\ &1 - u'(0)\frac{\beta(1-m+\frac{\bar{\pi}_1}{2})}{n(1-\beta)+\beta(1-m+\frac{\bar{\pi}_1}{2})} < 0, \text{ as } \pi_1 \to \bar{\pi}_1, \end{aligned} \tag{A.5}$$

where the first inequality follows from the concavity of u and the limit uses $\lim_{\pi_1 \to \bar{\pi}_1} x_{\pi_1} = 0$. Therefore, we have $u(x_{\pi_1}) + x_{\pi_1} < u(x_{\pi_1}(1+\delta))$, for π_1 sufficiently close to $\bar{\pi}_1$. This inequality and (A.2) for $\pi_1 = 0$ are strict, and by continuity, a sufficiently small $\bar{\gamma} > 0$ exists so that they still hold for any $\gamma \in [0, \bar{\gamma}]$. Then for each γ, the intermediate value theorem can be applied, and a $\pi_1^{**} \in (0, \bar{\pi}_1)$ exists such that

$$u(x_{\pi_1^{**}}) + x_{\pi_1^{**}} = u(x_{\pi_1^{**}}(1+\delta_{\pi_1^{**}})). \tag{A.6}$$

For $\gamma = 0$, Claim A.1 implies that $(\pi_1^{**}, x_{\pi_1^{**}})$ satisfies (16)-(18) and (20), all of which are strict inequalities. Because $(\pi_1^{**}, x_{\pi_1^{**}})$ depends on $\gamma \in [0, \bar{\gamma}]$ continuously, a sufficiently small $\gamma > 0$ exists such that (16)-(18) and (20) still hold. By the definition of π_1^{**}, (19) holds with equality. Thus a mixed-strategy steady state exists. ▌

Proof (Proof of Lemma 2). For the pure-strategy steady states, trading one unit in all trade meetings is the strictly preferred strategy at the steady state, so it is also optimal in its neighborhood.

For the mixed-strategy steady state, similar statement implies trading one unit being the unique optimal in $(2,1)$-, $(1,0)$- and $(1,1)$- meetings. Our existence proof shows that generically this steady state involves the complete randomization over one-unit and two-unit payments in a $(2,0)$-meeting. Along any convergent paths starting from a sufficiently nearby neighborhood, complete randomization and, hence, associated indifference condition (see (A.6)) must also occur in $(2,0)$-meeting eventually, because π^t will jump out of any sufficiently small neighborhood otherwise.

We can also pin down the optimal trading strategy that is constantly played along a path that starts and remains near the non-full-support steady state ($\pi_1^0 \neq 0$), if any such path exists. As is shown in the proof of Proposition 1, trading one unit is strictly preferred in $(1,1)$- and $(2,1)$-meetings, and paying two units is strictly preferred in $(2,0)$-meetings at (π, w). When $\gamma > 0$, trading one unit is strictly preferred in $(1,0)$-meetings at the steady state and hence it is also the unique optimal in its neighborhood.

When $\gamma = 0$, the same strategy becomes weakly preferred at the steady state but it will be the unique optimal in the neighborhood. The following argument shows that $\lambda^t(1;1,0) = 1$ should be the case for all $t \geq 0$. Because the economy is close to but not equal to (π, w), we have $\pi_1^t > 0$ for all $t \geq 0$ so (5) implies $w_1^t > 0$ for all $t > 0$, because there is always a positive probability that a consumer with one unit meets a producer with one unit and the consumer can get a positive amount of utility from such a meeting. A sufficiently large $u'(0)$ implies $u(x) > x$ for a sufficiently small $x > 0$ and therefore $u(\beta w_1^t) > \beta w_1^t$ holds all along the path. So in $(1,0)$-meetings, paying one unit is strictly preferred along the path. ▌

The following two lemmas are straightforward matrix computations.

Lemma 3. *Jacobian A evaluated at each full-support steady state has one eigenvalue strictly less than one, and two strictly greater than one in absolute value.*

Proof. Straightforward differentiation leads to

$$\Phi_\pi(\eta, \pi_1) = 1 - \frac{[4-\eta]\pi_1 + \eta}{n} \tag{A.7}$$

$$\phi_w = \begin{pmatrix} \frac{n-1+\pi_2}{n}\beta + \frac{\pi_0}{n}\beta u'(\beta w_1) - \frac{\pi_1}{n}\beta u'(\beta \Delta w) & \frac{\pi_1}{n}\beta u'(\beta \Delta w) \\ \frac{1-\pi_2}{n}\beta + \frac{\pi_0}{n}\beta u'(\beta w_1) - \frac{\pi_1}{n}\beta u'(\beta \Delta w) & \frac{n-1+\pi_2}{n}\beta + \frac{\pi_1}{n}\beta u'(\beta \Delta w) \end{pmatrix}, \tag{A.8}$$

where $\Delta w \equiv w_2 - w_1$. Because the top-right submatrix of A is a zero matrix, one eigenvalue is given by (A.7), which is smaller than one for both full-support steady states, and the other two eigenvalues are those of $(\phi_w)^{-1}$, which are the reciprocals of the eigenvalues of ϕ_w. In what follows, we are going to show that the eigenvalues of ϕ_w are smaller than one in absolute value.

Note that the slope of the r.h.s. of (15) at the steady state βw_1 should be smaller than the slope of the l.h.s. Therefore we have

$$\frac{n(1-\beta)+(1-\pi_2)\beta}{\beta} > \pi_0 u'(\beta w_1) + \pi_1 \frac{(1-\pi_2)\beta}{n(1-\beta)+(1-\pi_2)\beta} u'(\beta\Delta w). \tag{A.9}$$

The eigenvalues of a general 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ are given by

$$\eta_+, \eta_- = \frac{a+d \pm \sqrt{(a-d)^2+4bc}}{2}.$$

Because

$$\begin{aligned}
& (a-d)^2+4bc \\
= & \left[\frac{\pi_0}{n}\beta u'(\beta w_1) - 2\frac{\pi_1}{n}\beta u'(\beta\Delta w)\right]^2 \\
& +4\left[\frac{1-\pi_2}{n}\beta + \frac{\pi_0}{n}\beta u'(\beta w_1) - \frac{\pi_1}{n}\beta u'(\beta\Delta w)\right]\frac{\pi_1}{n}\beta u'(\beta\Delta w) \\
= & \left[\frac{\pi_0}{n}\beta u'(\beta w_1)\right]^2 + 4\frac{1-\pi_2}{n}\beta\frac{\pi_1}{n}\beta u'(\beta\Delta w) > 0,
\end{aligned}$$

both eigenvalues are real. They are smaller than one in absolute value if and only if $a+d<2$ and $(1-a)(1-d)-bc>0$. Checking these conditions for (A.8) gives

$$\begin{aligned}
& 1-a+1-d \\
= & 2\left(1-\frac{n-1+\pi_2}{n}\beta\right) - \frac{\pi_0}{n}\beta u'(\beta w_1) \\
> & 2\frac{n(1-\beta)+(1-\pi_2)\beta}{n} - \frac{\pi_0}{n}\beta u'(\beta w_1) - \frac{\pi_1}{n}\frac{(1-\pi_2)\beta}{n(1-\beta)+(1-\pi_2)\beta}\beta u'(\beta\Delta w) \\
> & \frac{n(1-\beta)+(1-\pi_2)\beta}{n} > 0,
\end{aligned}$$

$$
\begin{aligned}
& (1-a)(1-d)-bc \\
= \; & \left(1-\frac{n-1+\pi_2}{n}\beta-\frac{\pi_0}{n}\beta u'(\beta w_1)+\frac{\pi_1}{n}\beta u'(\beta\Delta w)\right)\left(1-\frac{n-1+\pi_2}{n}\beta-\frac{\pi_1}{n}\beta u'(\beta\Delta w)\right) \\
& -\frac{\pi_1}{n}\beta u'(\beta\Delta w)\left[\frac{1-\pi_2}{n}\beta+\frac{\pi_0}{n}\beta u'(\beta w_1)-\frac{\pi_1}{n}\beta u'(\beta\Delta w)\right] \\
= \; & \frac{(n(1-\beta)+(1-\pi_2)\beta)\beta}{n^2}\times \\
& \qquad \left(\frac{n(1-\beta)+(1-\pi_2)\beta}{\beta}-\pi_0 u'(\beta w_1)-\pi_1\frac{(1-\pi_2)\beta}{n(1-\beta)+(1-\pi_2)\beta}u'(\beta\Delta w)\right) \\
> \; & 0,
\end{aligned}
$$

where the last inequalities of the above two conditions follow from (A.9). Therefore, the eigenvalues of $(\phi_w)^{-1}$ are greater than one in absolute value ▌

Lemma 4. *$\begin{pmatrix} 0 & u'(\beta w_1^*)+1 & -u'(\beta w_2^*) \end{pmatrix}[Iz-A]^{-1}\binom{\Phi_\eta}{0}$ can be expressed as a rational function. The denominator is a polynomial of degree three. The numerator is of degree two and it has an constant term generically.*

Proof. The rational function is

$$
\begin{pmatrix} 0 & u'(\beta w_1^*)+1 & -u'(\beta w_2^*) \end{pmatrix}\operatorname{adj}[Iz-A]\begin{pmatrix}\Phi_\eta \\ 0\end{pmatrix}/\left|Iz-A\right|.
$$

The denominator is equal to $(z-\Phi_\pi)\left|Iz-\phi_w^{-1}\right|$, a polynomial of degree three. By looking into the adjoint of matrix $Iz-A$ and carrying out the multiplication, we can show that the numerater is of degree two and that the constant term is equal to $\begin{pmatrix} u'(\beta w_1^*)+1 & -u'(\beta w_2^*) \end{pmatrix}\phi_\pi *(\Phi_\eta\left|\phi_w^{-1}\right|)$. This constant is not equal to zero generically, because we can always change $u'(\beta w_1^*)$ and $u'(\beta w_2^*)$ arbitrarily without changing the values of u evaluated at a finite number of points and hence fixing ϕ_π. ▌

Lemma 5. *Jacobian A evaluated at the non-full-support steady state has a unit eigenvalue that is given by the law of motion, an eigenvalue strictly less than one and an eigenvalue strictly greater than one. The first two entries of any eigenvector associated with the unit eigenvalue have different signs.*

Proof. Equation (A.7) computes the unit eigenvalue of the law of motion, which is also an eigenvalue of A. Then we compute

$$\phi_\pi = \begin{bmatrix} r \\ s \end{bmatrix} \equiv \begin{bmatrix} \frac{1}{n}u(\beta w_2 - \beta w_1) - \frac{1}{2n}[u(\beta w_1) + \beta w_1] \\ \frac{1}{n}[u(\beta w_2 - \beta w_1) + \beta w_1] - \frac{1}{2n}[u(\beta w_2) + \beta w_2] \end{bmatrix} > 0$$

and

$$\phi_w = \begin{bmatrix} a' & 0 \\ 0 & d' \end{bmatrix} \equiv \tag{A.10}$$

$$\begin{bmatrix} \frac{(n-1+m)\beta}{n} + \frac{1-m}{n}\beta u^{'}(\beta w_1) & 0 \\ 0 & \frac{(n-1+m)\beta}{n} + \frac{1-m}{n}u^{'}(\beta w_2)\beta \end{bmatrix}.$$

Because w_2 is the biggest positive solution to (12), $d' \in (0,1)$ holds. And because w_1 is the smallest positive solution to (11), $a' > 1$ holds. We have

$$A^\lambda = \begin{bmatrix} 1 & 0 & 0 \\ -r/a' & 1/a' & 0 \\ -s/d' & 0 & 1/d' \end{bmatrix}. \tag{A.11}$$

The associated eigenvector, which constitutes the base of the space, has the form

$$\begin{bmatrix} 1 \\ \frac{-r}{a'-1} \\ \frac{s}{1-d'} \end{bmatrix}.$$

Then we have $\frac{-r}{a'-1} < 0$. ∎

REFERENCES

Green, Edward, and Ruilin Zhou, 2002. Dynamic monetary equilibrium in a random matching economy.*Econometrica* **70**, 929-969.

Huang, Pidong, and Yoske Igarashi, 2014. The instability of some non-full-support steady states in a random matching model of money. *Journal of Mathematical Economics* **Volume 55**, 177-185.

Huang, Pidong, and Yoske Igarashi, 2015. Trejos-Wright with a 2-unit bound: Existence and stability of monetary steady states. *Mathematical Social Sciences* **Volume 73**, 55-62.

Lee, Manjong, Neil Wallace, and Tao Zhu, 2005. Notes and comments modeling denomincation structures.*Econometrica* **73**, 949-960.

Lomeli, Hector, and Ted Temzelides, 2002. Discrete time dynamics in a random matching monetary model. *Economic Theory* **20**, 259-269.

Luenberger, David, 1979. Introduction to Dynamic Systems: Theory, Models, and Applications. John Wiley & Sons, Inc.

Lucas, Robert ,Edward Prescott, and Nancy Stocky, 1989. Recursive Methods in Economic Dynamics. Harvard University Press

Trejos, Alberto, and Randall Wright, 1995. Search, bargaining, money and prices. *Journal of Political Economy* **103**, 118-141.

Wallace, Neil, and Tao Zhu, 2004. A commodity-money refinement in matching models. *Journal of Economic Theory* **117**, 246-258.

Zhu, Tao, 2003, Existence of a monetary steady state in a matching model: indivisible money. *Journal of Economic Theory* **112**, 307-324.