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# Modelling Possibility of Short-Term Forecasting of Market Parameters for Portfolio Selection*

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We discuss modelling possibility of short-term forecasting for market parameters in the portfolio selection problems. We suggest a continuous time financial market model and a discrete time market model featuring this possibility. For these models, optimal portfolio selection problem has an optimal quasi-myopic solution. Computationally, the problem is reduced to a stochastic optimal control problem with delay in the plant equation. This allowed to quantify the degree of non-myopicness for a given utility function.

*Key Words*: Market models; Portfolio selection; Forecasting; Myopic strategies.
*JEL Classification Numbers*: C52, C53, G11.

## 1. INTRODUCTION

We consider the problems of optimal portfolio strategy selection under uncertainty. The goal is to derive a model and problem setting that takes into account the consequences of possibility or, respectively, impossibility of forecasting of market parameters. By the obvious reasons, the problem of forecasting for financial series always attracted significant attention. This problem is related to the open problem of validation of the so-called *technical analysis* methods that offer trading strategies based on historical observations. There are many different strategies suggested in the framework of technical analysis. In Hsu and Kuan (2005) it was mentioned that there are more than 18,326 different empirical trading rules being used in practice. However, the question remains open if the main hypothesis of technical analysis is correct. This hypothesis suggests that it is possible to make a statistically reliable forecast for future stock price movements using recent prices, and, finally, to find "winning" in statistical sense trading

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strategies. However, the dependence from the past (if any) is extremely weak for the stock prices, and this dependence is difficult to catch by usual statistical methods. Statistical studies of historical prices made as early as in 1933 didn't support the hypothesis that there is significant dependence from the past and predictability for the stock prices; see the discussion and the bibliography in Chapter 2, pp. 37-38, from Shiryaev (1999). This is the reason why the most common and mainstream model for the stock prices is the random walk or its modifications, in the accordance with the Efficient Market Hypothesis that suggests that a market is arbitrage free; see, e.g., Timmermann and Granger (2004).

Recently, new efforts were devoted to this problem, and some signs of possible presence of statistically significant dependence from the past were found; see, e.g., Hsu and Kuan (2005), Lorenzoni *et al* (2007), Lo *et al* (2000), Dokuchaev (2012d). In particular, it was shown in Lorenzoni *et al* (2007) that, for certain models of the stock price evolution, there is a statistically significant informational content in some patterns from technical analysis. This gives a hope that at least the evolution of the parameters of the price distributions can be forecasted. Under this assumption, the optimal choice of a portfolio strategy requires a forecast of market parameters to ensure the best choice defined by probabilistic criterions.

In the classical single period Markowitz setting, one step predictability was assumed for the first two moments of returns (Markowitz (1959)). For a stochastic diffusion model, this approach can be reformulated as the following: the dynamic of the stock prices is defined by an unpredictable noise process (for example, a Brownian motion or a jump noise process with some standard distribution law) and by a less unpredictable parameter process (for instance, the volatility process) that defines the probability distributions of the state vector. In this paper, we consider forecasting of the parameter process only; forecasting of the prices will not be considered.

Clearly, the forecast errors are inevitable due to the immanent unpredictability of the real world. If the forecast is not the best possible one (in a statistical/probabilistic sense), then the strategy selected is not optimal. Therefore, the optimality of the decision can be affected by the hypothesis and as well as by the forecast errors. Clearly, the impact of the forecast error can be reduced via improvement of the forecasting technique. However, this will not lead to complete elimination of the forecast error. The reason for this is that, in the mainstream framework, a forecast of the future parameters (and, therefore, the future distributions of the prices) is usually obtained by using an evolution model or prior distribution hypothesis and by collecting the historical data. On the other hand, the information collected for past historical data is not necessarily helpful to predict the future scenarios. Therefore, there is a problem of reducing the impact of the forecast error caused by the inevitable imperfection of the model (prior distribution hypothesis).

For the mainstream stochastic optimization and control methods, the evolution law for the future probability distributions is assumed to be known. For instance, the optimal estimate of the drift of the Brownian motion is actually using the hypothesis that the underlying process is a Brownian motion. Another example: the solution of the classical stationary linear-quadratic control problems assumes that the system coefficients are constant in time. An error in the prior hypothesis leads to the non-optimality of the forecast error and non-optimality of the strategy. Unfortunately, a hypothesis about the future distributions is more difficult to justify for the financial markets than in the natural sciences, biostatistics, or engineering, due to a lack of causality and lesser stability. Respectively, it is difficult to justify application of an optimization method based on a given evolution model. Since the choice of the distribution hypothesis is not very reliable, a decision maker has to take the possibility of the related forecast error into account. The optimal forecast can rarely be achieved; therefore, the perfect optimality appears to not be feasible in practice. It is why some special methods are required for the financial models to deal with this limited predictability. However, the existing theory is more oriented on obtaining optimal strategies even if it requires quite strong hypothesis about the probability distributions.

As was mentioned above, the optimality of the decision can be affected by the errors in the hypothesis and the errors in the forecast (the former are also inevitable even if the hypothesis is correct). Any reduction of the impact of this error is extremely important for decision making; the decisions could be more robust with respect to the forecast errors. The goal of this paper is to explore some possibilities to reduce the impact of the forecast error caused by the error in the choice of the hypothesis.

The classical approach is the maximin setting when the best strategy is selected for the worst case scenario. This approach is also called "robust performance" approach. Usually, it leads to a more conservative approach than the optimal one, and, therefore, the selected strategy may be underperforming. Rather than this, the decision theory and stochastic optimal control theory focusing on minimization of the dependence of a prior distribution hypothesis have yet to be developed. Therefore, it is timely to look for new approaches to deal with the unpredictability and the impact of forecast errors.

One of promising special tools is related to the so-called "myopic strategies". The term "myopia" may have negative meaning when it is applied to strategies, meaning inability to forecast. We apply this term to strategies that can be reasonable or even optimal and such that they do not need the future market scenarios. More precisely, we say that the optimal strategy is myopic if it does not require to know the future distributions of market parameters. In this case, the impact of the forecast error will be minimal. Therefore, it is important to be able to detect situations where the optimal strategy is myopic and where the forecast error have a mild impact: it may save valuable resources.

The financial market models have a very interesting and attractive feature: some optimal strategies for them can be myopic. Optimal myopic strategies were first introduced for continuous time optimal portfolio problems where the prices are stochastic Ito processes (Merton (1969)). So far, optimality for myopic strategies was established for special utility functions, including $U(x) = \ln x$ and $U(x) = q^{-1}x^q$, where $q < 1$, $q \neq 0$; see Mossin (1968), Samuelson (1969), Hakansson (1971), Pliska (1997), Dokuchaev (2010b). For the optimal myopicness phenomena, the answers have yet to be found why myopicness can be achieved for optimal financial decisions but not for a typical stochastic control problem. It would be timely to investigate the phenomenon of optimality of myopic strategies and to extend the use of these strategies. So far, the main efforts in the literature were directed on finding financial models where the optimal strategies are myopic.

In this paper, we suggest models with relaxed versions of myopicness: for these models, the parameter processes for the prices distributions are predicable on some short time horizon. Therefore, the corresponding optimal strategies can be regarding as "almost myopic": they require short-term predictability. (We mean predictability of the parameter processes for the prices distributions rather than the predicability of the prices). If an optimal strategy is myopic, then the predictability of the model parameters does not improve the performance of this strategy. Therefore, the dependence of the performance on the predicability horizon can be used as a measure on "non-myopicness" of the market model. The paper suggests models helping to explore this features.

We accept the following principles.

(I) A smooth enough process can be short-term forecasted.

(II) A stochastic model with a given parameter process accompanied by a noise process can be approximated by models where the parameter processes are smooth enough. These smooth parameter processes can be made arbitrarily close to the original parameter process in the uniform metric, and such that the distributions of the corresponding new state processes approximate the distributions of the state process from the original model.

We illustrate below how these principles can be justified and where they lead for the most common stochastic market models: for the continuous time diffusion market model and for the discrete time model. For both cases, we suggested models where optimal portfolio selection problem has an optimal quasi-myopic solution with short term forecasting. Computationally, the problem is reduced to a stochastic optimal control problem with a delay in the plant equation. This allowed to quantify the degree of non-myopicness for a given utility function. For continuous time diffusion market models and for discrete time stochastic market models, we suggest quantification of non-myopicness, i.e., some criterions that help to estimate the dependence of a stochastic market model from a possibility of short term forecasting.

## 2. CONTINUOUS TIME MARKET MODELS WITH SHORT TERM FORECASTING

We consider first a continuous time market model, where the market dynamic is described by stochastic differential equations. The randomness is presented in these equations in two ways: as a Wiener process (or cumulative white noise, or Brownian motion) and as the randomness/uncetainty of the coefficients (market parameters) that describes the correlations with the past, non-Markov properties, and unpredictability of the future price distributions.

### 2.1. The model

Let us consider the following stripped to the bone example that allows to illustrate the decision problems related to forecasting and myopicness and show that these problems can be reduced to challenging but still solvable mathematical problems. (More general models can be found in, e.g., Karatzas and Shreve (1998)).

Consider the model of a securities market consisting of a risk free bond or bank account with the price $B(t)$, $t \geq 0$, and a risky stock with price $S(t)$, $t \geq 0$. The prices of the stocks evolve as

$$dS(t) = S(t)\left(a(t)dt + \sigma(t)dw(t)\right), \quad t > 0, \tag{1}$$

where $w(t)$ is a Wiener process, $a(t)$ is a random appreciation rate, $\sigma(t)$ is a random volatility coefficient. The initial price $S(0) > 0$ is a given deterministic constant. The price of the bond evolves as

$$dB(t) = r(t)B(t)dt,$$

where $B(0)$ is a given constant, $r(t)$ is a short rate process that is assumed to be a non-negative random process.

We assume that $w(\cdot)$ is a standard Wiener process on a given standard probability space $(\Omega, \mathcal{F}, \mathbf{P})$, where $\Omega$ is a set of elementary events, $\mathcal{F}$ is a complete $\sigma$-algebra of events, and $\mathbf{P}$ is a probability measure.

Let $\{\mathcal{F}_t\}_{t\geq 0}$ be a filtration generated by the currently observable data (i.e., this can be interpreted as information flow generated by observations). We assume that $\mathcal{F}_t$ is independent of $\{w(t_2) - w(t_1)\}_{t_2\geq t_1\geq t}$, and $\mathcal{F}_0$ is trivial, i.e., it is the $\mathbf{P}$-augmentation of the set $\{\emptyset, \Omega\}$.

We assume that the process $(S(t), \sigma(t))$ is $\mathcal{F}_t$-adapted. In particular, this means that the process $(S(t), \sigma(t))$ is currently observable.

*Strategies and wealth*

The rules for the operations of the agents on the market define the class of admissible strategies where the optimization problems have to be solved.

Let $X(0) > 0$ be the initial wealth at time $t = 0$ and let $X(t)$ be the wealth at time $t > 0$.

We assume that the wealth $X(t)$ at time $t \in [0, T]$ is

$$X(t) = \beta(t)B(t) + \gamma(t)S(t). \tag{2}$$

Here $\beta(t)$ is the quantity of the bond portfolio, $\gamma(t)$ is the quantity of the stock portfolio, $t \geq 0$. The pair $(\beta(\cdot), \gamma(\cdot))$ describes the state of the bond-stocks securities portfolio at time $t$. Each of these pairs is called a strategy.

A pair $(\beta(\cdot), \gamma(\cdot))$ is said to be an admissible strategy if the processes $\beta(t)$ and $\gamma(t)$ are progressively measurable with respect to the filtration $\mathcal{F}_t$.

In particular, the agents are not supposed to know the future (i.e., the strategies have to be adapted to the flow of current market information). This is why forecasting is involved.

In addition, we require that

$$\mathbf{E}\int_0^T \left(\beta(t)^2 B(t)^2 + S(t)^2\gamma(t)^2\right) dt < +\infty.$$

A pair $(\beta(\cdot), \gamma(\cdot))$ is said to be an admissible self-financing strategy, if

$$dX(t) = \beta(t)dB(t) + \gamma(t)dS(t).$$

Under this condition, the process $\gamma(t)$ alone defines the strategy.

Let $U(x)$ be a non-decreasing concave function such that $\max(0, U(x)) \leq \text{const} \cdot (|x| + 1)$ and $U(X(0)) < -\infty$. Consider the following problem:

$$\text{Maximize} \quad \mathbf{E}U(X(T)) \quad \text{over admissible self-financing strategies.} \tag{3}$$

The selection of function $U$ is defined by the risk preferences.

In a typical situation, the optimal strategy can be represented in the form

$$\gamma(t) = g(t, S(t), X(t)),$$

where the function $g$ depends on $U$ and on the distribution law of the process $\mu|_{[0,T]}$, where $\mu(t) = (r(t), a(t), \sigma(t))$.

A problem arises: how to find an optimal strategy in a setting where the evolution law for the vector $\mu(t)$ is unknown.

### 2.2. Myopic strategies

In the model where the vector process $\mu(t) = (r(t), a(t), \sigma(t))$ is deterministic (i.e., known), the optimal strategy can be found as a deterministic function of current flow of observable data for a quite general selection of $U$. This function depends on the values of $\mu(t)|_{t\in[0,T]}$. Therefore, the same formulas cannot be applied pathwise for the paths of random processes $\mu(t)$.

For the continuous time portfolio selection problem, optimal myopic strategies can be described as strategies such that

$$\gamma(t) = f(t, S(t), \mu(t), X(t)),$$

where the function $f$ depends on $U$ only. Examples of optimal myopic strategies were first introduced in Merton (1969). Currently, optimality for myopic strategies was established for special utility functions, including $U(x) = \ln x$ and $U(x) = q^{-1}x^q$, where $q < 1$, $q \neq 0$. This is true when the evolution of $\mu(t)$ is defined by an Ito equation and $\mu$ is independent of $w(\cdot)$; see, e.g., Brennan (1998).

If this vector is observable but not predictable, then this approach is not working for more general $U$. Respectively, a major setback with this approach is that the myopicness of the optimal strategy vanishes after a small change of the problem setting.

Example 2.1. The optimal strategy is not myopic for an utility $U(x) = \log x + 0.001\sqrt{x}$, even when the evolution of $\mu(t)$ is defined by an Ito equation and $\mu$ is independent of $w(\cdot)$.

Example 2.2. Assume that the time horizon $T$ is fixed but the initial investment time $\tau$ has to be selected among Markov times (stopping times) such that $\tau \in [0, T]$ a.s. Let $X_{\tau+T}$ be the corresponding terminal wealth. The goal is to maximize $\mathbf{E}U(X_{\tau+T})$ over $\tau$ and over admissible strategies (it can be noted that it would be a novel setting in the portfolio selection theory). In this setting, the optimal strategy is not myopic even with $U(x) = \log x$, in contrast with the classical setting with given starting time $\tau = 0$.

### 2.3. Quasi-myopic strategies based on short term predictability of $(r, a, \sigma)$

Let us discuss the application of principles (I)-(II) from Section 1 to the continuous time diffusion market model. Principle (I) is well known and is actually used in all forecasting methods.

Let us justify the application of principle (II) to continuous time market models. It is known that a continuous time process can be transferred into a band-limited predictable process via an ideal low-pass filter. Similarly, a process becomes predictable on the infinite horizon if it is smoothed by a Gaussian filter, i.e., by a convolution with a kernel similar to $\exp(-t^2)$. These smoothing filters are not causal; moreover, the distance of the set of these ideal low-pass time invariant filters from the set of all causal filters is positive; see Almira and Romero (2008).

A Gaussian filter can be replaced by a filter with an exponential decrease of energy on higher frequencies. These smoothing filters are also not causal. However, they allow arbitrarily close approximation by causal filters that require only past historical observations; see Dokuchaev (2012a). The output of a process transferred with this kernel is a process that is predictable on some finite horizon (or short-time predictable); see Dokuchaev (2010a).

Consider now a stochastic market model with a parameter process $\mu(t)$. We suggest to consider a model where $\mu(t)$ is replaced by a predicable process that is close to $\mu(t)$. It would be unreasonable to expect predictability of stock prices, so the process $w(t)$ is still a Wiener process. It was shown in Dokuchaev (2012c) that, since the approximation of $\mu(t)$ can be arbitrarily precise (pathwise), the distributions of the stock prices in the new model can be made arbitrarily close to the distribution of the prices in the original model. In addition, the new model still feature the desired unpredictability of the prices. Therefore, many commonly used financial models can be effectively approximated by statistical indistinguishable models where the reference processes $\mu(t)$ can be locally predicted on the short time, without loosing the fundamental no-arbitrage market properties.

We suggest below models that take into account principles (I)-(II).

### 2.4. The model with short-term forecasting

Let $y(t)$ be some parameter process that is $\mathcal{F}_t$-adapted (i.e., currently observable). We assume that

$$dy(t) = h(y(t), t) + b(y(t), t)dw(t) + \widehat{b}(y(t), t)d\widehat{w}(t). \tag{4}$$

Here $h, b, \widehat{b}$ are some real valued functions defined on $\mathbf{R}^2$, $\widehat{w}(t)$ is a Wiener process that is independent of $w(t)$. The case when $y(t)$ is just the process $\mu(t) = (r(t), a(t), \sigma(t))$ is not excluded.

Proposition 1. *Assume that, for some $\delta > 0$,*

$$r(t) = \rho(y(t-\delta), t), \quad a(t) = f(y(t-\delta), t), \quad \sigma(t) = g(y(t-\delta), t),$$

*where $\rho, f, g$ are some real valued functions defined on $\mathbf{R}^2$ i.e.,*

$$\begin{aligned} dS(t) &= S(t)\left(f(y(t-\delta), t)dt + g(y(t-\delta), t)dw(t)\right), \\ dB(t) &= B(t)\rho(y(t-\delta), t)dt. \end{aligned} \tag{5}$$

*Then the market model features the following properties:*

*(i)At any time $t$, the market parameters $r(t), a(t), \sigma(t)$ can be forecasted without error on the time horizon $[t, t+\delta]$.*

*(ii)The forecast error for forecasting of $r(\tau), a(\tau), \sigma(\tau)$ at time $\tau > t+\delta$ is increasing with $\tau$ and with the size of the coefficients $b|_{s\in[t,\tau-\delta)}$ and $\widehat{b}|_{s\in[t,\tau-\delta)}$ in equation (4).*

*Proof.* Since $y(t)$ is observable, the value

$$\mu(\tau) = (r(\tau), a(\tau), \sigma((\tau)) = (\rho(y(\tau-\delta), t), f(y(\tau-\delta), t), g(y(\tau-\delta), t))$$

is known without error at time $t$ for all $\tau \in [t, t+\delta]$. Then statement (i) follows. Let us prove statement (ii). We assume that forecasting is based on Kalman filters; see, e.g.,Dokuchaev (2005). The forecast error for $y(\tau)$ is generated by the uncertainty on the interval $[t, \tau-\delta]$ only. This uncertainty is defined by the size of the non-zero diffusion coefficients in equation (4). Then statement (ii) follows. ▌

The classical model with non-random and known $\mu(t)$ can be described as a special case of model (3)–(5) with $\delta = T$. A model with stochastic

$\mu(t)$ that does not allow an error-free forecast on any time interval can be described as a special case of model (3)–(5) with $\delta = 0$. There are many publications devoted to these two extreme cases.

It can be noted that analysis of system (3)–(5) does not lead to difficult solvability and regularity issues, since the delay term is not presented in a closed loop equation. In the literature, many solvability and regularity results were obtained for more general and complicated stochastic delay equations, including equations for market models; see, e.g., Arriojas *et al* (1998), Ivanov *et al* (1998), Mao *et al* (1998), Stoica (2004), Luong and Dokuchaev (2014), and the bibliography therein. The first market model with a stochastic delay equation for the prices was introduced and investigated in Stoica (2004).

Currently, there are many methods for dealing with optimal control problem (3)–(5) for $\delta \in (0, T)$ in the framework of the optimal stochastic control for systems with delay; see, e.g., Chen and Wu (2010), Elsanousi *et al* (2000), Larssen (2002), Larssen and Risebro (2003), Øksendal and Sulem (2001).

Using the model with delay, we are able now to analyze the impact of narrowing the set of admissible strategies to the set of myopic strategies only. This would have the same effect as letting $\mathrm{Var} y(t+\varepsilon) \to +\infty$ for any $\varepsilon > 0$. It happens when $\|b|_{[t,T]}\| \to +\infty$ and $\|\widehat{b}|_{[t,T]}\| \to +\infty$, where $\|\cdot\|$ is a norm for the corresponding functions. This leads to the limit case of model (3)–(5) with $\delta = 0$. In this case, the limit optimal $\gamma(t)$ can be found as the solution of a static problem

$$\text{Maximize} \quad U_x'(X(t))\gamma a + \frac{1}{2} U_{xx}''(X(t))\gamma^2\sigma^2 \quad \text{over} \quad \gamma,$$

where $a = f(y(t), t)$ and $\sigma = g(y(t), t)$, given some regularity of $U$.

### 2.5. Quantification of the impact of myopicness

Consider a set of utility functions $U(\cdot)$ such that the optimal strategies for them are not myopic. It is clear intuitively that this “non-myopicness” depends on the selection of the utility function. One can expect that, for instance, that the optimal strategy is “less non-myopic” for $U(x) = \log x + 0.001\sqrt{x}$ than for $U(x) = \log x + \sqrt{x}$. The model suggested above leads to a method of quantification of this property and classification of utility functions with respect to this “non-myopicness”.

Let a utility function $U(\cdot)$ be given. For a set of parameters $\delta \ge 0$, consider a family of models $M_\delta$ introduced above with the delay $\delta$. Let $J_U(\delta)$ be the optimal value of the performance criterion given $\delta$, i.e., $J_U(\delta) =$

$\sup_\gamma \mathbf{E}U(X(T))$. Assume that, for all these models, the process $\mu(\cdot)$ is independent on $w(\cdot)$ (i.e., $b \equiv 0$), and that the distribution of the process $\mu(\cdot)$ is the same (i.e., is independent on $\delta$). In addition, assume that the right-hand side derivative

$$D_\delta^+ J_U(\delta) = \lim_{\varepsilon \to 0+} \varepsilon^{-1}[J_U(\delta+\varepsilon) - J_U(\delta)]$$

is defined at $\delta = 0$.

PROPOSITION 2. *For a given utility function, "non-myopicness" of the corresponding optimal strategy can be characterized by the value*

$$D_\delta^+ J_U(0). \tag{6}$$

*Proof.* By the assumptions, the distribution of the process $(S(t), B(t))$ is independent on $\delta$ representing a time horizon where error free forecasting of $\mu$ is possible. It suffices to observe that $D_\delta^+ J_U(\delta) \ge 0$ for for all $\delta \ge 0$ and all utilities, and $D_\delta^+ J_U(\delta) = 0$ for all $\delta \ge 0$ for $U(x) = \log x$ and $U(x) = q^{-1}x^q$, where $q < 1$, $q \neq 0$. Hence (6) achieves its minimal value (zero) for utilities for which the optimal strategy is myopic. ∎

### 2.6. Additional opportunities: forecasting of a single scalar parameter

In the setting described above, we assumed for simplicity a single stock market model and an one dimensional process $y(t)$. This model can be extended to the case of a multi-stock market. In this case, the process $\mu(t)$ can be of a high dimension, as well as the parameter process $y(t)$. However, the following useful fact takes a place: the general models require to forecast a single scalar parameter $\Theta = \int_0^T |\theta(t)|^2 dt$ only (Dokuchaev and Haussmann (2001)). Here $\theta(t) = \sigma(t)^{-1}(a(t) - r(t)\mathbf{1})$ is the market price of risk vector process, $\mathbf{1} = (1, \dots, 1)^\top \in \mathbf{R}^n$, $|\cdot|$ is the Euclidean norm, $a(t)$ is the vector of the appreciation rates for $n$ stocks, $\sigma(t)$ is the volatility matrix for the vector of stock prices.

Therefore, multi-dimensional forecasting could be avoided in some cases and replaced by forecasting of a single scalar parameter. This is possible even for a market with a large number of stocks, for general type utilities and for random parameters. It could be interesting to develop a setting with delay that can take into account this feature.

It can be also noted that the problem of forecasting of $\Theta$ can be further reduced to the selection of a new time scale and an optimal stopping time among Brownian stopping times in random interval defined by a time scale that transform this process into a Brownian motion. This "whitening" time scale can be selected as is suggested by Dambis–Dubins–Schwarz Theorem (see, e.g., Revuz and Yor (1999)). In this approach, the uncertainty of the market parameters is transformed into the uncertainty of the terminal time.

## 3. DISCRETE TIME MARKET MODELS WITH SHORT-TERM FORECASTING

### 3.1. The model

Let us consider a model of a market consisting of the bond or bank account with price $B_t$ and stocks with prices $S_{t,k}$, $t = 0, 1, 2, \ldots$, $k = 1, \ldots, n$, where $n \geq 1$ is the number of stocks. The initial prices $S_{0,k} > 0$ and $B_0 > 0$ are given non-random variables.

We consider discounted stock prices $\widetilde{S}_{t,k} \triangleq R_t^{-1} S_{t,k}$, $k > 1$, with $\widetilde{S}_{0,k} \triangleq S_0$ and $R_t \triangleq B_t/B_0$. We assume that

$$\begin{aligned} \widetilde{S}_{t,k} &= \widetilde{S}_{t-1,k}(1 + \xi_{t,k}), \\ B_t &= \rho_t B_{t-1}, \qquad\qquad t \geq 1, \quad k = 1, \ldots, n. \end{aligned} \tag{7}$$

Here $\xi_{t,k}$ and $\rho_t$ are random variables. We assume that $\xi_{t,k} > -1$ and $\rho_t \geq 1$ for all $t, k$. Clearly,

$$R_t \triangleq \prod_{m=1}^{t} \rho_m, \qquad S_{t,k} = \rho_t S_{t-1,k}(1 + \xi_{t,k}), \qquad t \geq 1, \quad k = 1, \ldots, n.$$

In this setting, the single period risk-free return is $\rho_t - 1$, and the single period return for the $k$th stock is $\rho_t - 1 + \rho_t \xi_{k,t}$.

Let $\mathcal{F}_t$ be the filtration generated by the flow of observable data, i.e., by the process $(\xi_{t,k}, \rho_t)$.

Let $X_0 = 1$ be the initial wealth of an investor at time $t = 0$, and let $X_t$ be the wealth at time $t \geq 0$. We set that

$$X_t = \beta_t B_t + \sum_k \gamma_{t,k} S_{t,k}, \tag{8}$$

where $\beta_t$ is the quantity of the bond portfolio and where $\gamma_t = (\gamma_{t,1}, \ldots, \gamma_{t,n})$ is the vector describing the quantities if the shares for the particular stocks in the stock portfolio. The pair $(\beta_t, \gamma_t)$ describes the state of the bond-stocks securities portfolio at time $t \geq 0$. We call the sequences of these pairs portfolio strategies.

We consider the problem of trading or choosing a portfolio strategy. Some constraints will be imposed on current operations in the market.

A portfolio strategy $\{(\beta_t, \gamma_t)\}$ is said to be admissible if the process $(\beta_t, \gamma_t)$ is adapted to the filtration $\mathcal{F}_t$ and the following condition of self-financing is satisfied: for all $t \geq 0$,

$$X_{t+1} - X_t = \beta_t \left(B_{t+1} - B_t\right) + \sum_{k=1}^{N} \gamma_{t,k}^{\top} \left(S_{t+1,k} - S_{t,k}\right).$$

We do not impose additional conditions on strategies such as transaction costs, bid-ask gap, restrictions on short selling; furthermore, we assume that shares are divisible arbitrarily, and that the current prices are available at the time of transactions without delay. In any case, we shall ignore these difficulties here.

For the trivial "keep-only-bonds" portfolio strategy, the portfolio contains only the bonds, $\gamma_t \equiv 0$, and the corresponding wealth is $X_t \equiv \beta_0 B_t \equiv \prod_{m=1}^{t} \rho_m$.

The process $\widetilde{X}_t \triangleq R_t^{-1} X_t$ is called the discounted wealth.

Proposition 3. *Let $\{X_t\}$ be a sequence, and let the sequence $\{(\beta_t, \gamma_t)\}$ be an admissible portfolio strategy, where $\beta_t = (X_t - \gamma_t^{\top} S_t) B_t^{-1}$. Then the process $\widetilde{X}_t$ evolves as*

$$\widetilde{X}_{t+1} - \widetilde{X}_t = \sum_{k=1}^{N} \gamma_{t,k} (\widetilde{S}_{t+1,k} - \widetilde{S}_{t,k}), \quad t \geq 0.$$

*Proof of Proposition 3* is standard (see, e.g., Pliska (1997)).

It follows from Proposition 3 that the sequence $\{\gamma_t\}$ alone suffices to specify admissible portfolio strategy $\{(\beta_t, \gamma_t)\}$.

Let $U(\cdot)$ be again a non-decreasing function such described in Section 2. We may state our general problem as follows: Find an admissible strategy $\{\gamma_t\}$ which solves the following optimization problem:

$$\text{Maximize} \quad \mathbf{E} U(X_T) \quad \text{over} \quad \{\gamma_t\}. \tag{9}$$

### 3.2. Myopic and quasi-myopic strategies

The real market prices are presented as time series, so the discrete time models are more natural than the continuous time models. Unfortunately, these models are more difficult for analytical study. There are only few special cases when the discrete time optimal portfolio problems allow explicit solution and when optimal strategies are myopic; see Mossin (1968), Hakansson (1971), Pliska (1997). Even power utilities do not allow myopic optimal solutions: in Hakansson (1971) it was shown s that the optimal strategy is not myopic for $U(x) = \sqrt{x}$ if returns evolve as a Markov process. It was found later that optimal strategies can still be myopic for power utilities in a discrete time setting for more restrictive assumptions about correlations (Dokuchaev (2007b,2010b)).

A possible direction for the research would be to introduce relaxed versions of myopicness and consider short-term predictability only, as was suggested above for the continuous time market model.

It does not make sense to assume that the exact value of $(\xi_{t+1,k}, \rho_{t+1})$ is known at time $t$ without error. Remind that, for continuous time market, we considered predicability of market parameters $(r(t), a(t), \sigma(t))$ only. For the discrete time market model, there is no a direct analog of the appreciation rate and the volatility. Therefore, a market model with short term forecasting has to be constructed differently.

Let as assume that $\xi_{t,k}$ and $\rho_t$ are expressed via some reference process $\{Y_s\}_{s<t}$ and some discrete time white noise $w_t$. The evolution of the process $Y_t$ has some regularity and describes the evolution of the distributions of $\xi_{t,k}$ and $\rho_t$.

We will apply non-parametric spectral methods to describe "smoothness" and predictability of discrete time processes.

We accept the following principles.

(i) A smooth enough discrete-time process $Y_t$ can be short-term forecasted. This class includes processes such that their Z-transform is vanishing with a certain rate at $e^{i\omega}$ as $\omega \to \pi$.

(ii) A stochastic model with a given parameter process $Y_t$ accompanied by a noise process $w_t$ can be approximated by models where the parameter processes are such as described in (i). These "smooth" parameter processes can be made arbitrarily close to the original parameter process such that the distributions of the corresponding new state processes approximate the distributions of the state process from the original model.

Principle (i) can be justified as the following. It was shown in Dokuchaev (2012b), Dokuchaev (2012c). that a discrete time process $\widehat{x}(t)$ is predictable if, for some $c > 0$ and $q > 1$,

$$\sup_{\omega\in[-\pi,\pi]} |\widehat{X}(e^{i\omega})| \exp \frac{c}{[e^{i\omega}+1|^q} < +\infty, \tag{10}$$

where $\widehat{X}$ is the Z-transform of $\widehat{x}$, i.e.,

$$\widehat{X}(z) = \sum_{t=-\infty}^{\infty} \widehat{x}(t) z^{-t}, \quad z \in \mathbf{C}.$$

It can be noted that frequency method are commonly used for financial time series; see, e.g., Pollock (2012).

Let us provide some reasons for principle (ii). Consider a discrete time parameter process $Y_t$ that drives the distribution of the prices for a discrete time market model. It can be seen from (10) that a discrete time process is predictable if it is an output of an ideal low-pass filter. Clearly, an output of this filter can be made arbitrarily close to the output process; at the same time, this output is a predictable process. Therefore, if one transforms the parameter process using this filter, then the transformed process will be predicable; the distribution of the price process $S_t$ can be made arbitrarily close to the distribution of the original process. This leads to the conclusion that a market model by a can be replaced by a statistical indistinguishable no-arbitrage model with predictable parameters. In can be noted that a related result was obtained in Dokuchaev (2014).

For discrete time market, we suggest a model with the short-term predictability of parameters that can be described as the following: there exists $\delta \in \{1, 2, \ldots\}$ such that, for all $m = 1, \ldots, \delta$, the following holds.

*For* $(\xi_{t+m,k}, \rho_{t+m})$*, the conditional distribution given* $\mathcal{F}_{t+m-1}$ *is the same as the conditional distribution given* $\mathcal{F}_t$*.* (11)

The number $\delta - 1$ represents the predictability horizon. For a single period optimal portfolio selection problem, there is no a non-zero predictability horizon. This is the case of the classical Markovitz setting. Let us explain why it is a reasonable model. To select an optimal strategy at time $t$ with the purpose of maximization of the wealth at time $t+T$, one has to use the historical observations as well as the probability distributions of the entire vector $\{\xi_{t+m,k}, \rho_{t+m}\}_{m=1}^{T}$ (for the general case where the optimal strategy is not myopic). If (11) holds, then the price evolution law is known at time $t$ for the next $\delta$ time periods, i.e., for $t+1, \ldots, t+\delta$. For the more distant future, the evolution law is defined by a particular realization of the future price values that are unknown at time $t$.

The case where $\delta = 1$ corresponds to the model without predictability. This is equivalent to the case in the continuous time setting described above with $\delta = 0$. This is not surprising, since the continuous time model can be considered as the limit case of the discrete time model with the physical time between instances $t_k$ and $t_{k+1}$ converges to zero.

Let us consider a special case where the predictability of the evolution law is ensured by the predictability of some parameter process similar to the process $(r(t), a(t), \sigma(t))$ for the continuous time diffusion market model. Let as assume that

$$\xi_{t,k} = a_{t,k}(Y_{t-\delta}, w_t), \quad \rho_t = b_{t,k}(Y_{t-\delta}, w_t),$$

where $Y_t = \{(S_m, B_m)\}_{m \leq t}$ represents the historical observations, $w_t$ is a discrete time white noise process with values in $\mathbf{R}^N$, and $a_{t,k} : \mathbf{R}^{2\tau} \times \mathbf{R}^N \to \mathbf{R}$ and $b_t : \mathbf{R}^{2\tau} \times \mathbf{R}^N \to \mathbf{R}$ are some known functions. In particular, $w_t$ are mutually independent and have the same distribution; this distribution is assumed to be known. In this case, the marked model can be again described via discrete time equations with delay

$$\begin{aligned} \widetilde{S}_{t,k} &= (1 + a_{t,k}(Y_{t-\delta}, w_t))\widetilde{S}_{t-1,k}, \\ B_t &= b_{t,k}(Y_{t-\delta}, w_t)B_{t-1}, \qquad\qquad t \geq 1, \quad k = 1, \ldots, n. \end{aligned} \tag{12}$$

Respectively, optimal portfolio selection problem is in fact a stochastic control problem for a discrete time system with delay.

Similarly to Proposition 2, we can use this model to estimate utility functions with respect to "non-myopicness" of optimal strategies. For a family of utilities $U(\cdot)$, let $J_U(\delta) = \sup_\gamma \mathbf{E}U(X_T)$ be the optimal values calculated for two models, model $M_1$ and model $M_0$, such as described above; model $M_1$ with $\delta = 1$ and model $M_0$ with $\delta = 0$. In addition, we require that this two models are constructed such that the distribution of the vector $\{Y_{t-\delta}\}$ is the same for both models, after substitution of the corresponding $\delta$. This values are defined for market models with $\delta$ as a parameter. In discrete time case, it will require to compare for different utilities the differences $J_U(1) - J_U(0)$.

## 4. CONCLUSIONS

A possible way to reduce the impact of forecast errors is to use so-called "myopic strategies" that can be reasonable or even optimal and such that they do not use future market scenarios. For some financial market models the optimal strategies can be myopic. Therefore, it is important to be able to detect situations where the optimal strategy is myopic and the forecast error does not have a big impact: it may save valuable resources.

The paper suggests models with optimal "almost myopic" strategies that require short-term predictability of a reference/parameter processes; these models accommodate the possibility of forecasting for the parameters of stock price evolution such as the volatility. We suggest a quantification of "non-myopicness" of a given utility and criterion that estimates the dependence of a stochastic market model from a possibility of short term forecasting.

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