Corporate Tax Policy and Long-Run Capital Formation: The Role of Irreversibility and Fixed Costs

Jianjun Miao[*]

This paper presents an analytically tractable continuous-time general equilibrium model with investment irreversibility and fixed adjustment costs. In the model, there is a continuum of firms that are subject to idiosyncratic shocks to capital. Although the presence of investment frictions lowers consumer welfare, it may raise or reduce the long-run average capital stock, depending on the degree of idiosyncratic uncertainty. An increase in this uncertainty may raise equilibrium aggregate capital, but reduce welfare. An unexpected permanent change in the corporate income tax rate affects the investment trigger and target values, and hence the size and rate of capital adjustment. Following this tax policy, the percentage changes in equilibrium quantities are larger when fixed adjustment costs are larger. These changes are significantly smaller in a general equilibrium model than in a partial equilibrium model.

Key Words: Irreversibility; Fixed costs; Heterogeneity; Tax policy; General equilibrium.

JEL Classification Numbers: D92, E22, E62.

1. INTRODUCTION

Recent empirical studies have documented that nonconvexities of microeconomic capital adjustment and irreversibility of investment are widespread phenomena. Examining a 17-year sample of large, continuing US manufacturing plants, Doms and Dunne (1998) find that typically more than half of a plant's cumulative investment occurs in a single episode. In addition, they find that long periods of relatively small changes are interrupted by investment spikes. Using the Longitudinal Research Database, Cooper and Haltiwanger (2006) find that about 8 percent of observations entail an investment rate near zero. These observations of inaction are comple-

---

[*] Department of Economics, Boston University, 270 Bay State Road, Boston MA 02215. Email: miaoj@bu.edu. Part of this research was completed when Miao was visiting the Hong Kong University of Science and Technology in 2007. The hospitality of this university is greatefully acknowledged. First version: May 2008.

mented by periods of rather intensive adjustment of the capital stock. Both studies indicate the existence of nonconvexities of adjustment costs. Using data from the equipment auction of an aerospace firm, Ramey and Shapiro (2001) find that machine tools sell at about 31 percent of their purchase value while structural equipment sell at only 5 percent, suggesting a high degree of investment irreversibility.

Given these pieces of evidence, an important question is whether micro-level nonconvexities and irreversibility matter for aggregate macroeconomic behavior. Some researchers (e.g., Bachmann et al. (2008), Gourio and Kashyap (2007), Khan and Thomas (2003, 2008), and Thomas (2002)) study the implications of lumpy investment for business cycles in general equilibrium, while Veracierto (2002) examine the implications of irreversibility.[1] In this paper, I instead focus on the role of nonconvexities and irreversibility in long-run capital accumulation and their implications for tax policy. Aggregation of micro-level behavior of heterogenous firms requires researchers to analyze the cross-sectional distribution of firms over state variables. Because this distribution is hard to characterize, researchers typically use complicated numerical methods.

By contrast, in this paper, I build an analytically tractable continuous-time general equilibrium model. The model features a continuum of firms, a representative household, and a government. Firms pay corporate income taxes. Tax revenues are rebated to the household in a lump-sum manner. A fraction of firms do not face any investment frictions, while the remaining firms are subject to fixed adjustment costs and investment irreversibility. I focus on a long-run stationary equilibrium for the economy. For this purpose, I abstract from aggregate uncertainty and assume that uncertainty is idiosyncratic so that a law of large number can be applied. In order to derive a closed-form solution for the stationary distribution of firms, I assume that idiosyncratic (Brownian motion) shocks are specific to capital so that firms are heterogeneous in capital only. As a result, a single state variable (capital) describes a firm's decision problem. This feature allows me to derive a sharp characterization of a firm's investment problem. This characterization in turn simplifies the derivation of a stationary equilibrium.

I first solve a firm's optimal investment problem in partial equilibrium by fixing prices. I characterize the optimal investment policy by a trigger value and a target value such that the firm adjusts its capital stock to the target value whenever the capital stock falls to the trigger value. Given the trigger and target values, I characterize the long-run stationary distribution of capital, and hence the long-run average capital stock. I then solve

---

[1]Early partial equilibrium models of aggregate investment with micro-level lumpiness include Caballero and Engel (1999), Caballero et al. (1995), and Cooper et al. (1999).

analytically the approximate trigger and target values by the Taylor expansion for a small adjustment cost. This analytical approximate solution allows me to derive sharp comparative statics results regarding the role of fixed costs and irreversibility and the effects of an unexpected permanent corporate tax change.

After analyzing a single firm's decision problem, I turn to aggregate behavior in general equilibrium. I show that although the presence of investment frictions (irreversibility or fixed costs) lowers consumer welfare, it may raise or reduce the long-run average capital stock. In particular, the long-run average capital stock in the presence of both fixed costs and irreversibility is smaller than that in the presence of irreversibility only. This result holds true both in general and partial equilibrium. However, the long-run average capital stock without any investment frictions may be smaller or higher than that with investment frictions depending on the level of the volatility of capital shocks. In addition, this relation may be opposite in a general equilibrium analysis to that in a partial equilibrium analysis. I also show that an increase in the volatility of capital shocks may raise equilibrium aggregate capital, but reduce welfare in models with investment frictions.

I now consider the effects of an unexpected permanent corporate income tax change on aggregate capital accumulation. I show that a change in the corporate income tax rate changes the investment trigger and target values. As a result, it changes the size and rate of capital adjustment. That is, changes in corporate tax rate have effects from both extensive and intensive margins. I also show that the impact of a corporate tax change is larger when fixed adjustment costs are larger. Specifically, the percentage changes in economic variables such as capital and output are larger when fixed costs are larger. Finally, I show that the partial equilibrium impact of tax changes is much larger than the general equilibrium impact, revealing that the general equilibrium price feedback is important for a long-run tax policy analysis.

My paper is related to Abel and Eberly (1998, 1999). Abel and Eberly (1998) analyze a single-firm model with investment irreversibility and fixed adjustment costs. They show that the quasi-fixity of capital eliminates the dichotomy between factor mix and scale, and can give rise to labor hoarding. They do not consider long-run capital accumulation and tax policy. Abel and Eberly (1999) study the effects of irreversibility on long-run capital accumulation. They do not consider fixed adjustment costs and tax policy. Importantly, unlike my heterogeneous-firm general-equilibrium framework, the preceding two studies analyze a single-firm partial-equilibrium framework.

Although the effects of corporate tax policy on capital accumulation with convex adjustment costs have been extensively studied in the literature

(e.g., Abel (1982) and Auerbach (1989)), to the best of my knowledge, my paper provides the first analysis of the effects of corporate tax policy in the presence of nonconvex adjustment costs.

The remainder of the paper proceeds as follows. Section 2 presents the model. Section 3 presents the partial equilibrium analysis, while Section 4 turns to the general equilibrium analysis. Section 5 concludes. Technical details and proofs are relegated in appendices.

2. THE MODEL

I introduce corporate income tax into a single-firm investment model with fixed adjustment costs and irreversibility (e.g., Abel and Eberly (1998)), and embed this model in a general equilibrium framework. The model economy consists of a continuum of production units, a representative household, and a government. I refer to a production unit as a firm. Time is continuous and the horizon is infinite. Assume that there is no aggregate uncertainty and that firms face idiosyncratic shocks to capital. By a law of large numbers, all aggregate quantities and prices are deterministic over time, although at the firm level each firm still faces idiosyncratic uncertainty. I will focus on steady-state stationary equilibrium in which all aggregate variables are constant over time.[2]

2.1. Firms

There are two types of firms in the economy, with one type of firms incurring irreversibility and fixed adjustment costs and the other type not facing any investment frictions. All firms pay corporate income tax at the flat rate θ. All firms are ex ante identical in terms of production function and are subject to idiosyncratic shocks specific to capital only. They differ ex post in that they may experience different histories of shocks. Formally, if a firm does not make any investment, its capital stock evolves according to the following process:

$$dk_t = -\delta k_t dt + k_t \sigma dW_t, \tag{1}$$

where $\sigma > 0$, δ is the depreciation rate, and W is a standard Brownian motion specific to the firm. Different firms experience independent Brownian motion shocks. These shocks are at the firm level and there is no aggregate shock. Note that my modeling of uncertainty is different from most papers in the literature where uncertainty is typically modeled as productivity

---

[2]Hopenhayn and Rogerson (1993) show that this stationary equilibrium framework is useful for the study of long-run tax policy. Unlike their model, I do not consider entry and exit in this paper. Other studies also consider a similar framework (see, e.g., Gomes (2001) and Miao (2005)).

shocks or demand shocks (e.g., Abel and Eberly (1998)).[3] As will be seen below, this modeling simplifies analysis significantly because it makes the firm's decision problem a one-dimensional one.

Each firm combines labor and capital to produce output according to a decreasing-returns-to-scale production function given by $F(k, l)$, where k and l denote capital and labor factors, respectively. Assume that $F(\cdot)$ takes the following Cobb-Douglas form:

$$F(k, l) = k^{\alpha_k} l^{\alpha_l}, \ \alpha_k, \alpha_l \in (0, 1) \ \text{and} \ \alpha_k + \alpha_l < 1. \tag{2}$$

I can then derive the operating profit function $\pi(k; w)$ by solving the following static labor choice problem:

$$\pi(k; w) = \max_{l \geq 0} \{F(k, l) - wl\}, \tag{3}$$

where w denotes the steady-state constant wage rate. This problem gives the labor demand:

$$l^d(k; w) = \left(\frac{\alpha_l}{w}\right)^{\frac{1}{1-\alpha_l}} k^\alpha, \tag{4}$$

the output supply:

$$y(k; w) = F\left(k, l^d(k; w); w\right) = \left(\frac{\alpha_l}{w}\right)^{\frac{\alpha_l}{1-\alpha_l}} k^\alpha, \tag{5}$$

and the operating profit function:

$$\pi(k; w) = H(w) k^\alpha \tag{6}$$

where

$$H(w) = \left(\frac{\alpha_l}{w}\right)^{\frac{\alpha_l}{1-\alpha_l}} (1 - \alpha_l), \tag{7}$$

and

$$\alpha = \frac{\alpha_k}{1 - \alpha_l}. \tag{8}$$

Firms may make investments to adjust their capital stock. A fraction $1 - \lambda$ of firms do not incur any investment frictions. These firms' decision problems are standard and described in Appendix A. In Appendix A, I show that the optimal frictionless capital stock is such that the marginal revenue product of capital is equal to the tax-adjusted user cost of capital, as summarized in the following:

---

[3]My modelling of uncertainty is similar to that in Caballero (1993). Unlike my paper, Caballero (1993) also considers aggregate shocks.

LEMMA 1. *When a firm does not face any investment frictions, it adjusts capital stock to the level k^* satisfying*

$$\pi'(k^*; w) = \frac{r}{1 - \theta} + \delta. \tag{9}$$

For the remaining firms, investment is irreversible and subject to fixed adjustment costs. We follow Khan and Thomas (2003, 2008) and measure fixed adjustment costs as real wage costs.[4] In particular, whenever a firm makes investments, it must hire labor $\xi > 0$ and pay fixed costs ξw. Unlike Khan and Thomas (2003, 2008), I assume ξ is a constant parameter. As is well known in the investment literature (e.g., Abel and Eberly (1998)), when facing fixed costs, the firm makes discrete investments Δk_{τ_i} at distinct points in time τ_i, $i = 1, 2, ...$ At the investment time, the capital stock jumps such that:

$$k_{\tau_i} = k_{\tau_i-} + \Delta k_{\tau_i}. \tag{10}$$

Because investment is irreversible, $\Delta k_{\tau_i} > 0$.

Firms maximize the present value of profits. We can formulate a firm's decision problem in the presence of fixed adjustment costs as follows:

$$V(k) = \max E\left[\int_0^\infty e^{-rt}\left((1 - \theta)\pi(k_t; w) + \theta\delta k_t\right)dt - \sum_{i=1}^\infty e^{-r\tau_i}\left(\Delta k_{\tau_i} + \xi w\right)\right], \tag{11}$$

where V is the value function, and $r > 0$ is the steady-state constant interest rate. Note that depreciation has tax allowance. In this problem, the capital sock is the only state variable. I will show in the next section that whenever a firm makes capital adjustment, the adjustment amount is constant, i.e., $\Delta k_{\tau_i} = \Delta k$ for all τ_i. In addition, the rate of adjustment is also equal to a constant value Γ. By a law of large numbers, we can interpret this rate as the number of adjusting firms in the steady state.

2.2. Aggregation

Because there is a continuum of firms that are subject to idiosyncratic shocks, there is a cross sectional distribution of firms over the state k. We focus on the steady state in which the cross sectional distribution does not change over time. In this case, we refer to it as a stationary distribution.

---

[4]There are several ways to model fixed adjustment costs. Fixed costs may be proportional to the demand shock (Abel and Eberly (1998)), profits (Caballero and Engel (1999) and Cooper and Haltiwanger (2006)), or the capital stock (Cooper and Haltiwanger (2006)). Caballero and Engel (1999) introduce randomness to the fixed cost and study a generalized (S,s) approach.

As we show in Lemma 1, firms will adjust capital to the level k^* if they do not face investment frictions. Thus, we only need to characterize the stationary distribution of firms that are subject to irreversibility and fixed adjustment costs, denoted by $\mu^* (\cdot; w)$. Given this distribution, we can compute the following aggregate quantities:

- aggregate capital stock

$$K (\mu^*; w) = \lambda \int k \mu^* (dk; w) + (1 - \lambda) k^*, \qquad (12)$$

- aggregate output supply

$$Y (\mu^*; w) = \lambda \int y (k; w) \mu^* (dk; w) + (1 - \lambda) y (k^*; w), \qquad (13)$$

- aggregate labor demand

$$L^d (\mu^*; w) = \lambda \left[\int l^d (k; w) \mu^* (dk; w) + \Gamma \xi \right] + (1 - \lambda) l^d (k^*; w), \qquad (14)$$

- aggregate investment

$$I (\mu^*; w) = \lambda \Gamma \Delta k + (1 - \lambda) \delta k^*. \qquad (15)$$

2.3. Household

There is a representative agent in the economy. The agent trades shares of the firms and a riskfree bond in zero supply. He consumes and supplies labor to the firms. Because there is no aggregate uncertainty, his preferences are represented by the deterministic utility function:

$$\int_0^\infty e^{-\beta t} U (C_t, L_t) \, dt, \qquad (16)$$

where β denotes the subjective discount rate, C_t denotes the consumption rate, and L_t denotes the labor rate. Assume that U is strictly increasing, strictly concave, continuously differentiable, and satisfies the usual Inada condition.

As is well known, in order for consumption to be constant over time in a steady state, the interest rate r must be equal to the discount rate β. In addition, in the steady state, the agent's problem reduces to the following static problem (e.g., Gomes (2001) and Hopenhayn and Rogerson (1993)):

$$\max_{C,L} U (C, L) \qquad (17)$$

subject to the static budget constraint:

$$C = wL + D + T, \tag{18}$$

where D denotes aggregate dividends and T denotes government transfers. Aggregate dividends are equal to total after-tax profits minus total investment spending and fixed adjustment costs:

$$
\begin{aligned}
D \ = \ & \lambda \int \left[(1 - \theta) \pi (k; w) + \theta \delta k \right] d\mu^* (k) - \lambda \Gamma (\Delta k + w\xi) \\
& + (1 - \lambda) (1 - \theta) (\pi (k^*; w) - \delta k^*).
\end{aligned} \tag{19}
$$

The consumption-labor choice implies the first-order condition: $-U_2 (C, L) = wU_1 (C, L)$. For simplicity, I use the following utility specification commonly used in macroeconomics (e.g., Gomes (2001) and Hopenhayn and Rogerson (1993)):

$$U (C, L) = \log C + h (1 - L). \tag{20}$$

In this case, the first-order condition becomes

$$hC = w. \tag{21}$$

2.4.   Government

Because the focus of the paper is on the distortionary effect of corporate income tax on investment, we assume that the tax revenue collected by the government is rebated to the household in a lump-sum manner. Thus, we abstract from the wealth effect and the other distortionary effect associated with using distortionary taxation to finance government spending on goods and services.[5] Because the government collects corporate income taxes, and transfers these tax revenues to the household, the government budget constraint is given by

$$T = \lambda\theta \int (\pi (k; w) - \delta k) \, \mu^* (dk; w) + (1 - \lambda) \theta (\pi (k^*; w) - \delta k^*).$$

2.5.   Stationary Equilibrium

A stationary competitive equilibrium consists of a constant wage rate w, and a constant interest rate $r = \beta$, capital adjustment level Δk, the

---

[5]Incorporating government spending would complicate our analysis since a tax cut must eventually be financed with some combination of other tax increases or spending cuts. The analysis of how the dividend and capital gains tax cut is financed is beyond the scope of the present paper and is left for future research.

rate of adjustment Γ, a stationary distribution μ^* of firms that are subject to investment irreversibility and fixed costs, and aggregate quantities, C, $I(\mu^*; w)$, $Y(\mu^*; w)$, $L^d(\mu^*; w)$, and L such that (i) each firm solves problem (11); (ii) the agent chooses C and L to maximize utility (17) subject to budget constraint (18); (iii) μ^* is a stationary distribution and aggregate quantities satisfy equations (13)-(15); and (iv) markets clear:

$$L^d(\mu^*; w) = L, \qquad (22)$$

$$C + I(\mu^*; w) = Y(\mu^*; w). \qquad (23)$$

3. PARTIAL EQUILIBRIUM ANALYSIS

In order to analyze the general equilibrium effects of corporate income tax, I first analyze a single firm's decision problem in partial equilibrium. I thus fix the wage rate w and suppress it without risk of confusion throughout this section.

3.1. Optimal Investment Policy

To solve the firm's problem (11), we note that the value function V satisfies the following Hamilton-Jacobi-Bellman (HJB) equation when the firm does not make capital adjustment:

$$rV(k) = (1-\theta)\pi(k; w) + \theta\delta k - \delta k V'(k) + \frac{\sigma^2 k^2}{2} V''(k). \qquad (24)$$

This equation has an intuitive interpretation. The expression on the left-hand side is the require return. This return is equal to the sum of three components given in the right-hand side of equation (24). The first component is after-tax profits (including depreciation allowance) $(1-\theta)\pi(k; w) + \theta\delta k$. The second component is the loss due to capital depreciation, $\delta k V'(k)$. The third component is the expected capital gain $\sigma^2 k^2 V''(k)/2$.

The general solution to equation (24) is given by

$$V(k) = G_1 k^{\beta_N} + G_2 k^{\beta_P} + \Pi(k), \qquad (25)$$

where G_1 and G_2 are coefficients to be determined, $\Pi(k)$ is the present value of after-tax profits given by:

$$\Pi(k) = \frac{(1-\theta)H(w)k^\alpha}{r + \alpha\delta + \alpha(1-\alpha)\sigma^2/2} + \frac{\delta\theta k}{r+\delta}, \qquad (26)$$

and β_N and β_P are the negative and positive roots of the equation:

$$v(\beta) \equiv r + \delta\beta + \frac{\sigma^2}{2}\beta(1-\beta) = 0. \qquad (27)$$

I next derive boundary conditions. Because investment is irreversible and subject to fixed adjustment costs, there are two values k_b and k_c satisfying the following properties: When its capital stock falls below to the trigger value k_b, the firm immediately makes investment to bring its capital stock to the target level k_c. When its capital stock is higher than k_b, the firm does not adjust capital, and simply lets capital depreciate. At the trigger k_b and the target k_c, the value function satisfies the following value-matching condition:

$$V(k_b) = V(k_c) - (\xi w + (k_c - k_b)). \tag{28}$$

This equation says that the increase in firm value after the purchase of capital must be equal to the sum of the fixed costs and the purchase spending. In addition, the following smooth-pasting conditions must also be satisfied:

$$V'(k_b) = 1, \tag{29}$$
$$V'(k_c) = 1. \tag{30}$$

These equations say that the marginal valuation of capital must be equal to the purchase price of capital immediately before and immediately after investing.

The final boundary condition is a no-bubble condition. Capital can be arbitrarily large, but firm value cannot explode relative to its present value of profits. This implies that $G_2 = 0$. We now have three equations (28)-(30) for three unknowns G_1, k_b, and k_c. Because these equations do not permit a closed-form solution, I first derive an analytical approximate solution in the next subsection, and then consider a numerical solution in Section 3.2.

3.2. Approximate Solution for the Investment Trigger and Target

To derive an approximate solution, I use the general solution (25) to rewrite the three boundary conditions as follows:

$$G_1 \beta_N k_b^{\beta_N - 1} + \Pi'(k_b) = 1,$$
$$G_1 \beta_N k_c^{\beta_N - 1} + \Pi'(k_c) = 1,$$

$$G_1 k_b^{\beta_N} + \Pi(k_b) = G_1 k_c^{\beta_N} + \Pi(k_c) - (\xi w + (k_c - k_b)).$$

Eliminating G_1, I obtain a system of two equations:

$$J^1(k_b, k_c) = 0, \tag{31}$$
$$J^2(k_b, k_c) = 0, \tag{32}$$

where I define:

$$J^1\left(k_b, k_c\right) \equiv \left(1 - \Pi'\left(k_b\right)\right) k_c^{\beta_N - 1} - \left(1 - \Pi'\left(k_c\right)\right) k_b^{\beta_N - 1}, \qquad (33)$$

$$J^2\left(k_b, k_c\right) \equiv \frac{1 - \Pi'\left(k_c\right)}{\beta_N} k_c + \Pi\left(k_c\right) - \frac{1 - \Pi'\left(k_b\right)}{\beta_N} k_b - \Pi\left(k_b\right) - \left(\xi w + \left(k_c - k_b\right)\right).$$
$$(34)$$

In Appendix C, I follow Abel and Eberly (1998) and derive the Taylor series expansions of $J^1\left(k_b, k_c\right)$ and $J^2\left(k_b, k_c\right)$. I then set these expansions equal to zero to solve for the approximate solutions for k_b and k_c. As in Abel and Eberly (1998), I choose the irreversible investment trigger as the point around which to expand, and choose the third order of the expansion.

When the fixed adjustment cost $\xi = 0$, investment is purely irreversible. The following lemma characterizes the optimal irreversible investment policy (see Appendix B for a proof):

LEMMA 2. *The optimal irreversible investment policy is characterized by an investment trigger k_i given by:*

$$k_i = \left[\frac{\left(1 - \beta_N\right)}{\left(\alpha - \beta_N\right)\left(r + \delta\right)} \frac{v\left(\alpha\right)}{\alpha H\left(w\right)} \left(\frac{r}{1 - \theta} + \delta\right)\right]^{\frac{1}{\alpha - 1}}, \qquad (35)$$

such that the firm buys capital to prevent its capital stock from falling below the trigger k_i. When its capital stock is higher than k_i, the firm does not adjust its capital stock. The trigger value k_i is lower than the optimal frictionless capital stock k^*. The trigger value k_i decreases with θ.

To interpret equation (35), I rewrite it as

$$\pi'\left(k_i; w\right) = \frac{\left(1 - \beta_N\right) v\left(\alpha\right)}{\left(\alpha - \beta_N\right)\left(r + \delta\right)} \left(\frac{r}{1 - \theta} + \delta\right). \qquad (36)$$

This equation says that the marginal revenue product of capital at k_i is equal to the tax-adjusted user cost of capital. The tax-adjusted user cost of capital consists of the tax-adjusted interest rate and the depreciation rate. Compared with case of frictionless investment characterized in Lemma 1, the tax-adjusted user cost of capital in the presence of irreversibility contains a multiplicative factor $\left(1 - \beta_N\right) v\left(\alpha\right) / \left[\left(\alpha - \beta_N\right)\left(r + \delta\right)\right]$. In Appendix B, I show that this factor is greater than 1, which reflects the option value of waiting. Equation (36) also reveals that an increase in the corporate income tax rate raises the tax-adjusted user cost of capital, and thus reduces the trigger value k_i.

I now set $\widetilde{k}_b = k_b - k_i$ and $\widetilde{k}_c = k_c - k_i$. The following proposition characterizes the approximate solution.

PROPOSITION 1. *When ξ is sufficiently small,*

$$\tilde{k}_c \approx \left[\frac{3}{2} \left(1 - \frac{\delta\theta}{r+\delta} \right)^{-1} \frac{\xi w k_i^2}{(1-\alpha)(1-\beta_N)} \right]^{\frac{1}{3}} > 0. \qquad (37)$$

and

$$\tilde{k}_b \approx -\tilde{k}_c < 0. \qquad (38)$$

The value $\tilde{k}_b$ is the wedge between the trigger value k_b in the presence of fixed costs and the trigger value k_i in the absence of fixed costs. Proposition 1 shows that the relationship between this wedge and fixed costs is cubic, a result similar to Abel and Eberly (1998). It is easy to show that the derivative of the wedge $\tilde{k}_b$ with respect to the fixed cost parameter ξ is minus infinity when evaluated at $\xi = 0$. This means that even a very small value of fixed cost will lower the trigger value substantially below k_i. A similar result holds for the wedge $\tilde{k}_c$ between the target k_c and the trigger k_i.

3.3. The Size and Rate of Capital Adjustment

In the presence of irreversibility and fixed adjustment costs, optimal investment is lumpy and is characterized by a trigger value k_b and a target value k_c. I now use the approximate solution to the trigger and target values derived in the previous subsection to analyze properties of the size and rate of capital adjustment.

I start with the size of capital adjustment $k_c - k_b$. The following proposition shows the comparative statics result.

PROPOSITION 2. *Suppose ξ is sufficiently small. Then the size of capital adjustment $k_c - k_b \approx 2\tilde{k}_c$ increases with fixed cost ξ. It decreases with the corporate income tax rate θ if and only if*

$$2r > \delta (1-\theta)(1-\alpha). \qquad (39)$$

When the fixed adjustment cost is higher, the firm must make a larger capital adjustment in order to raise capital stock and hence profits to a higher level so that the increase in profits can cover the larger fixed costs. The effect of the corporate income tax rate θ on the size of capital adjustment is ambiguous. The intuition is that an increase in θ lowers the after-tax profits, but also lowers the tax allowance on depreciation. If the depreciation rate is small enough the first effect dominates so that the firm

prefers to reduce the size of capital adjustment in response to an increase in θ.

I next turn to the rate of capital adjustment. I first derive the duration between successive capital adjustments. In Section 3.1, I show that the capital stock is equal to k_c immediately after investing, and the next time investment occurs when the capital stock first falls to the trigger value k_b. Thus, the duration between successive adjustments is the first passage time of the capital stock process (k_t) from the target k_c to the trigger k_b. I denote this first passage time by $T(k_c, k_b)$. By a standard argument (e.g., Abel and Eberly (1998)), I can compute this expected first passage time:

$$E[T(k_c, k_b)] = \frac{\ln\left(\frac{k_c}{k_b}\right)}{\delta + \frac{\sigma^2}{2}}. \tag{40}$$

Using Proposition 1, I derive the following result:

LEMMA 3. *For small fixed cost ξ,*

$$\ln\left(\frac{k_c}{k_b}\right) = \frac{2\tilde{k}_c}{k_i} + \frac{2}{3}\left(\frac{\tilde{k}_c}{k_i}\right)^3. \tag{41}$$

In addition, it increases with θ, σ^2, and ξ.

An immediate corollary of this lemma is that the duration between successive adjustments increases with the corporate income tax rate θ and the fixed adjustment cost parameter ξ. The intuition is that an increase in corporate income tax rate θ or the fixed cost parameter ξ raises the geometric distance between the trigger and the target values. Thus, the firm delays the occurrence of the next time that it must pay the fixed cost.

The effect of volatility σ on the duration time is ambiguous because there are two opposing forces: First, for fixed trigger and target values k_b and k_c, an increase in volatility σ reduces the expected first passage time from k_b to k_c. Second, an increase in volatility σ changes the trigger and target values and raises the distance between these two values as measured by $\ln(k_c/k_b)$. This effect increases the duration time.

Once the duration between successive adjustments is known, I can compute the rate of capital adjustment per unit of time as $\Gamma = 1/E[T(k_c, k_b)]$. The following result follows immediately from the previous analysis.

PROPOSITION 3. *The rate of capital adjustment is given by*

$$\Gamma = \frac{\delta + \frac{\sigma^2}{2}}{\ln\left(\frac{k_c}{k_b}\right)}. \tag{42}$$

For small fixed cost ξ, Γ decreases with θ and ξ.

In response to an increase in θ or ξ, the firm delays the occurrence of the next time that it must pay the fixed cost. Thus, the firm reduces the frequency of making successive adjustments.

3.4. Stationary Distribution and Long-Run Average Capital Stock

It is interesting to study the role of irreversibility and fixed adjustment costs in long-run capital accumulation. Several studies (e.g., Abel and Eberly (1999) and Bentolila and Bertola (1990)) have examined the effect of irreversibility on long-run capital stock in various ways. The key issue is how to measure long-run capital stock. In the literature on irreversible investment under uncertainty, uncertainty is typically modeled as demand shocks driven by a geometric Brownian motion process. In this case, one needs to know the joint stationary distribution of shocks and capital stock (or the marginal revenue product of capital) in order to compute the long-run capital stock, even though the firm's problem can be reduced to one dimensional by exploiting the homogeneity property. Because the shock is nonstationary, the joint stationary distribution does not exist. This creates conceptual difficulty in computing the long-run capital stock. To deal with this issue, Bentolila and Bertola (1990) compute the long-run capital stock using the stationary distribution of the marginal revenue product of capital for a given value of shocks. Abel and Eberly (1999) compute the long-run capital stock as the expected value of capital with respect to the (nonstationary) distribution of shocks.

In my model, there is a single state variable, which allows me to derive the stationary distribution of the capital stock (k_t) and its expected long-run value. To do so, I make the transformation by defining:

$$z_t = \ln k_t, \ \ z_b = \ln k_b, \ \ z_c = \ln k_c.$$

Then (z_t) follows the following arithmetic Brownian motion process:

$$dz_t = \left(-\delta - \sigma^2/2\right) dt + \sigma dW_t. \tag{43}$$

Define:

$$\mu \ = \ -\delta - \sigma^2/2, \tag{44}$$

$$\gamma \ = \ 2\mu/\sigma^2 = -2\delta/\sigma^2 - 1 < -1. \tag{45}$$

In Appendix D, I derive the stationary density of (z_t).

PROPOSITION 4. *The stationary density of the process (z_t) is given by*

$$f(z) = \begin{cases} A_1 e^{\gamma z} + B_1 & \text{for } z \in (z_b, z_c) \\ A_2 e^{\gamma z} & \text{for } z \in (z_c, \infty) \end{cases}, \tag{46}$$

where

$$A_1 = \frac{e^{-\gamma z_b}}{z_b - z_c},$$
$$A_2 = \frac{e^{-\gamma z_b} - e^{-\gamma z_c}}{z_b - z_c},$$
$$B_1 = \frac{1}{z_c - z_b}.$$

Using the this stationary density, I can compute the long-run average capital stock K_f in the presence of irreversibility and fixed costs.

PROPOSITION 5. *The long-run average capital stock K_f is given by*

$$K_f = \frac{(k_c - k_b)}{\ln(k_c/k_b)} \left(1 + \frac{\sigma^2}{2\delta}\right). \tag{47}$$

Equations (47) and (42) imply that

$$\delta K_f = (k_c - k_b)\Gamma. \tag{48}$$

This equation gives two ways to represent long-run average investment. Since the long-run average capital stock K_f is constant, the long-run average investment must be equal to depreciation, δK_f. On the other hand, in the presence of fixed costs, investment is lumpy. Thus, the long-run average investment is equal to the size of capital adjustment $(k_c - k_b)$ multiplied by the rate of capital adjustment Γ.

It is interesting to compare with the case with frictionless investment and the case with purely irreversible investment. Let K^* and K_i denote the corresponding long-run average capital stock, respectively. Lemma 1 shows that all firms adjust capital stock to the level k^* when there is no investment friction. Thus, we have $K^* = k^*$.

PROPOSITION 6. *The long-run average capital stock with investment irreversibility is given by*

$$K_i = \left(1 + \frac{\sigma^2}{2\delta}\right) k_i. \tag{49}$$

When ξ is sufficiently small,

$$K_f \approx \frac{1}{1 + \left(\tilde{k}_c/k_i\right)^2/3} \left(1 + \frac{\sigma^2}{2\delta}\right) k_i < K_i. \tag{50}$$

In addition, K_f decreases with ξ.

This proposition shows that when the fixed adjustment cost is sufficiently small, its presence makes the firm accumulates less capital $K_f < K_i$, compared with the case of irreversibility only. The intuition is that the presence of adjustment costs makes the firm more reluctant to adjust capital. We note that there is no unambiguous relation between K_i and the frictionless capital stock K^*. Although Lemma 2 implies that $k_i < K^*$. The multiplier factor in equation (49) may cause $K_i > K^*$. The intuition is similar to that described in Abel and Eberly (1999). There are two opposing effects. First, irreversibility increases the user cost of capital as shown in equation (36). This user cost of capital effect causes the firm to accumulate less capital. Working in the opposite direction, there is a hangover effect. Because of irreversibility, the firm cannot sell capital when accumulating too much capital due to positive capital shocks. It may end up with more capital than it would choose without irreversibility. Similarly, there is no unambiguous relation between K^* and K_f.

I now consider the effect of corporate income taxes on long-run capital accumulation.

PROPOSITION 7. *When ξ is sufficiently small,*

$$\frac{\partial \ln K_f}{\partial \theta} < \frac{\partial \ln K_i}{\partial \theta} = \frac{\partial \ln K^*}{\partial \theta} < 0. \tag{51}$$

In addition, $\left|\frac{\partial \ln K_f}{\partial \theta}\right|$ increases with ξ.

As expected, Proposition 7 shows that an increase in the corporate income tax rate reduces the long-run average capital stock. It also shows that the presence of irreversibility alone does not affect the magnitude of the impact of a corporate income tax change on the long-run average capital stock. That is, in response to a small change in the corporate income tax rate, the percentage change in the long-run average capital stock is the same for both the case of frictionless investment and the case of irreversible investment. Importantly, Proposition 7 establishes that the presence of fixed adjustment costs causes the long-run capital stock to be more sensitive to changes in the corporate income tax rate. In addition, the magnitude of

the impact following a change in the corporate income tax rate increases with fixed adjustment costs. The intuition is that a change in the corporate income tax rate changes both the trigger and target values. Thus, it changes both the size and the rate of capital adjustment. Because the size of the fixed adjustment cost affects the trigger and target values, it affects the impact of tax changes. In particular, an increase in the fixed adjustment costs widens the gap between the trigger and target values, thereby amplifying the impact of tax changes.

4. GENERAL EQUILIBRIUM ANALYSIS

So far, I have focused on a partial equilibrium analysis by fixing the wage rate. In a general equilibrium, the wage rate is endogenously determined. Thus, a change in corporate income tax rate has a price feedback effect in the sense that it also changes the equilibrium wage rate, which in turn affects a firm's investment decision. Thus, a general equilibrium analysis may provide a very different quantitative result than a partial equilibrium analysis. It may even deliver an opposite qualitative result. In addition, because a general equilibrium model incorporates household behavior, it also has implications for welfare.

In this section, I turn to a general equilibrium analysis by endogenizing the wage rate. To do so, I use the resource constraint (23) to derive $C = Y(\mu^*; w) - I(\mu^*; w)$. I then substitute this expression into equation (21). I finally obtain one equation for one unknown, w. This nonlinear equation does not admit a closed-form solution. I thus resort to numerical solutions.

4.1. Baseline Parameter Values

I select baseline parameter values in order to solve the model numerically. Note that the model is stylized and cannot be calibrated to confront with data. The purpose of the numerical solutions is to illustrate the working of the model mechanism, but not to match data.

First, I set the corporate income tax rate $\theta = 0.34$. I then consider parameters for preferences. I set the subjective discount rate $\beta = 0.04$ so that it is equal to the 4 percent interest rate. I choose the parameter h so that the steady-state fraction of time spent on working is equal to about 1/3.

I next consider parameters for technology. I set $\alpha_l = 0.64$ so that the labor share is equal to 64 percent. I then choose the curvature parameter α in the profit function given in equation (6). Cooper and Haltiwanger (2006) and Hennessy and Whited (2005, 2007) estimate this parameter to be between 0.5 to 0.7. Here I set $\alpha = 0.6$. Using equation (8), I can then deduce that $\alpha_k = 0.216$. I set the depreciate rate $\delta = 0.1$, as standard in macroeconomics literature. As in Khan and Thomas (2003), I choose

$\xi = 0.003$ so that fixed adjustment costs account for 1 percent of working time.

I need to assign value for the remaining two parameters: λ and σ. Since there is no direct estimate for these values in the literature, I simply set $\sigma = 0.2$, as a baseline value. I also set $\lambda = 0.5$ so that half of the firms in the economy are subject to investment frictions.

In summary, the parameter values are listed in Table 1.

TABLE 1.

Baseline parameter values.

α_k	α_l	δ	σ	ξ	λ	β	h	θ
0.216	0.64	0.1	0.2	0.003	0.5	0.04	2.25	0.34

4.2.   Welfare and Long-Run Average Capital

In Section 3.4, I have analyzed the role of irreversibility and fixed adjustment costs in long-run capital accumulation in a partial equilibrium model by fixing the wage rate. I now consider the case of general equilibrium. Table 2 presents values of long-run average capital obtained, respectively, in the models with irreversible investment and fixed costs, with irreversible investment but without fixed costs, and with frictionless investment for two different volatility values $\sigma = 0.2$ and $\sigma = 0.4$. The numbers in brackets are the partial equilibrium values when the wage rate is fixed at the level obtained in the model with the baseline parameter values given in Table 1.

TABLE 2.

Comparison of the long-run aggregate capital stock.

	Irreversibility and fixed costs	Irreversibility	Frictionless
$\sigma = 0.2$	0.5880	0.5896 (0.5924)	0.5884 (0.5921)
$\sigma = 0.4$	0.5950 (0.5813)	0.5968 (0.5863)	0.5884 (0.5921)

Notes: Other parameters are listed in Table 1. The numbers in the brackets are values in the partial equilibrium framework by fixing the wage rate at the equilibrium value with baseline parameter values given in Table 1.

Table 2 reveals several interesting results. First, the frictionless long-run average capital stock does not depend on the uncertainty parameter σ. This result follows immediately from Lemma 1 given my modeling of iid uncertainty. Second, introducing fixed adjustment costs in the model with irreversible investment reduces the long-run average capital stock in partial equilibrium as shown in Proposition 5. This result also holds true in general equilibrium. Third, as we discuss in Section 3.4, there is no unambiguous relation between the long-run average capital stock with frictionless investment and that with investment irreversibility in a partial equilibrium

model. Table 2 shows numerically that this relation depends on the level of volatility. In particular, compared with the frictionless investment model, when $\sigma = 0.2$, introducing investment irreversibility raises the long-run average capital stock, but reduces it if fixed costs are present additionally. This result holds true in both general and partial equilibria.

By contrast, when $\sigma = 0.4$, introducing irreversibility reduces the long-run average capital stock in partial equilibrium, and introducing fixed costs reduces it further. This result, however, does not hold true in general equilibrium. In general equilibrium, irreversibility raises the long-run average capital stock. In addition, the long-run average capital stock in the frictionless model is even smaller than that in the model with both irreversibility and fixed costs. This result demonstrates that the price feedback is important. Specifically, in partial equilibrium, the high level of the frictionless capital stock raises labor demand and hence causes the wage rate to rise. The high wage rate reduces marginal revenue product of capital, thereby reducing a firm's incentive to accumulate capital. This effect may dominate so that the frictionless long-run average capital stock is actually lower than that with investment frictions. Finally, an increase in uncertainty lowers long-run average capital stock with investment frictions in partial equilibrium. This result seems intuitive because an increase in uncertainty raises the option value to wait. By contrast, because of the price feedback effect, an increase in uncertainty raises the long-run average capital stock in general equilibrium.

An important question is how idiosyncratic uncertainty and investment frictions influence welfare. For the frictionless model, idiosyncratic uncertainty does not affect welfare because it does not affect factor allocations and household utility. With investment frictions, capital is inefficiently allocated among firms, leading to welfare losses. I use the parameter values given in Table 1 to illustrate this point numerically. Compared to the frictionless investment model, irreversibility alone reduces the household utility from 0.5394 to 0.5387. Irreversibility and fixed costs together reduce the household utility further to 0.5371.

Although an increase in uncertainty raises equilibrium aggregate capital in models with investment frictions as illustrated in Table 2, it reduces welfare. In particular, for the model with fixed costs and irreversibility, when σ is raised from 0.2 to 0.4 and other parameter values are given in Table 1, the household consumption decreases from 0.3781 to 0.3761 and working hours increases from 0.3290 to 0.3299. For the model with irreversibility and without fixed costs $\xi = 0$, when σ is raised from 0.2 to 0.4 and other parameter values are given in Table 1, the household consumption decreases from 0.3785 to 0.3766 and working hours increases from 0.3288 to 0.3295. These numerical examples illustrate that capital is more inefficiently allocated as idiosyncratic uncertainty increases.

4.3.   Long-Run Effects of Corporate Tax Policy

I now analyze the effects of an unexpected permanent corporate tax change on the economy. I consider two experiments by permanently lowering and increasing the corporate tax rate by 10 percentage points, respectively. Table 3 presents the results for the model with irreversibility and fixed costs. In particular, Columns 2-4 present the general equilibrium results, while Columns 5-6 present the partial equilibrium results by fixing the wage rate at the value obtained in the model with baseline parameter values given in Table 1.

TABLE 3.

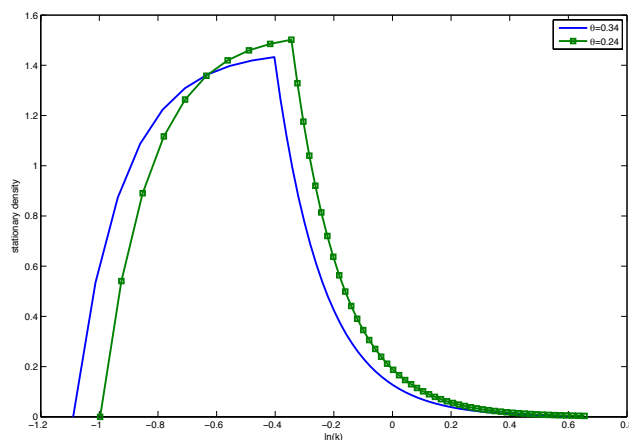Results for the baseline model with irreversibility and fixed costs

		GE		PE	
	$\theta = 0.34$	$\theta = 0.24$	$\theta = 0.44$	$\theta = 0.24$	$\theta = 0.44$
Wage w	0.8507	1.27	-1.62	0	0
Output Y	0.4369	2.10	-2.58	8.03	-9.42
Capital K	0.5880	7.45	-8.74	13.69	-15.17
Employment L	0.3290	0.83	-0.98	8.03	-9.42
Consumption C	0.3781	1.00	-1.30	4.35	-5.65
Average productivity Y/L	1.3279	1.27	-1.61	0.00	0.00
Duration time	4.5221	-4.92	6.07	-7.11	9.32
Adjustment size $k_c - k_b$	0.2640	2.24	-3.30	5.72	-7.39
Adjustment rate Γ	0.2211	5.17	-5.72	7.65	-8.53

Notes: The parameter values are given in Table 1. Except for the numbers in Column 2, other numbers are in percentage. Columns 2-4 present results for the general equilibrium model. Columns 5-6 present results for the partial equilibrium model by fixing the wage rate at the equilibrium value (0.8507) in Column 2.

Table 3 reveals several interesting results. First, a decrease in corporate income tax rate raises the equilibrium wage rate, output, capital stock, employment, welfare and average labor productivity. Second, a decrease in the corporate income tax rate changes the investment trigger and target values. As a result, it raises the capital adjustment size and the adjustment rate. That is, changes in corporate tax have effects from both extensive and intensive margins. Figure 1 illustrates the extensive margin effect by plotting the stationary density of firms. When the corporate income tax rate is reduced, the stationary density shifts to the right. Third, the partial equilibrium impact of tax changes is much larger than the general equilibrium impact. The intuition follows from the price feedback effect as described earlier.

I next consider the effects of fixed costs by doubling the value of the fixed costs parameter ξ. Table 4 presents the results. This table reveals that an increase in fixed costs reduces equilibrium output, capital, consumption and

FIG. 1. Stationary densities of $\ln(k)$ for $\theta = 0.34$ and $\theta = 0.24$. Parameter values are given in Table 1.

labor productivity. In particular, an increase in fixed costs raises the capital adjustment size, but reduces the capital adjustment rate. The extensive margin effect dominates so that it reduces aggregate capital.

TABLE 4.

Results for the model with irreversibility and fixed costs ($\xi = 0.006$)

	GE			PE	
	$\theta = 0.34$	$\theta = 0.24$	$\theta = 0.44$	$\theta = 0.24$	$\theta = 0.44$
Wage w	0.8502	1.28	−1.62	0	0
Output Y	0.4366	2.11	−2.58	8.08	−9.48
Capital K	0.5870	7.47	−8.76	13.76	−15.24
Employment L	0.3292	0.83	−0.98	8.08	−9.48
Consumption C	0.3779	1.00	−1.31	4.38	−5.69
Average productivity, Y/L	1.3263	1.27	−1.62	0.00	0.00
Duration time	5.7220	−4.96	6.13	−7.18	9.44
Investment size $k_c - k_b$	0.3320	2.26	−3.32	5.78	−7.45
Adjustment rate Γ	0.1748	5.22	−5.77	7.73	−8.62

Notes: Except for $\xi = 0.006$, other parameter values are given in Table 1. Except for the numbers in Column 2, other numbers are in percentage. Columns 2-4 present results for the general equilibrium model. Columns 5-6 present results for the partial equilibrium model by fixing the wage rate at the equilibrium value (0.8502) in Column 2.

Table 4 also reveals that the impact of tax changes increases with the fixed costs. But the quantitative effects are small. For example, following a 10 percentage point decrease in the corporate tax rate, the increase in

output is 2.11 percent when $\xi = 0.006$, raising only by 0.01 percentage point of output increase in the model with $\xi = 0.003$. In addition, the welfare increase is extremely small, which is almost the same as in the model with $\xi = 0.003$.

TABLE 5.

Results for the model with frictionless investment

	GE			PE		
	$\theta = 0.34$	$\theta = 0.24$	$\theta = 0.44$	$\theta = 0.34$	$\theta = 0.24$	$\theta = 0.44$
Wage w	0.8519	1.26	−1.60	0.8507	0	0
Output Y	0.4375	2.09	−2.56	0.4403	7.94	−9.32
Capital K	0.5884	7.42	−8.71	0.5921	13.58	−15.04
Employment L	0.3286	0.82	−0.97	0.3312	7.94	−9.32
Consumption C	0.3786	0.99	−1.29	0.3811	4.28	−5.56
Average productivity, Y/L	1.3311	1.26	−1.60	1.3292	0	0

Notes: The parameter values are given in Table 1. Except for the numbers in Columns 2 and 5, other numbers are in percentage. Columns 2-4 present results for the general equilibrium model. Columns 5-7 present results for the partial equilibrium model by fixing the wage rate at the equilibrium value (0.8507) in the baseline model.

TABLE 6.

Results for the model with irreversible investment

	GE			PE		
	$\theta = 0.34$	$\theta = 0.24$	$\theta = 0.44$	$\theta = 0.34$	$\theta = 0.24$	$\theta = 0.44$
Wage w	0.8516	1.26	−1.60	0.8507	0	0
Output Y	0.4375	2.09	−2.56	0.4395	7.94	−9.32
Capital K	0.5896	7.43	−8.71	0.5924	13.58	−15.04
Employment L	0.3288	0.82	−0.97	0.3307	7.94	−9.32
Consumption C	0.3785	0.99	−1.29	0.3803	4.28	−5.56
Average productivity, Y/L	1.3306	1.26	−1.60	1.3292	0	0

Notes: The parameter values are given in Table 1. Except for the numbers in Columns 2 and 5, other numbers are in percentage. Columns 2-4 present results for the general equilibrium model. Columns 5-7 present results for the partial equilibrium model by fixing the wage rate at the equilibrium value (0.8507) in the model with baseline parameter values given in Table 1.

I finally compare with the models with frictionless investment and with irreversible investment, respectively. Tables 5 and 6 present the results. These tables show that irreversibility does not change the impact of tax changes on the economy in the sense that it does not affect the percentage change in equilibrium quantities. However, irreversibility is costly to the household because it reduces equilibrium consumption and labor productivity. Comparing with Table 3, I find that the impact of tax changes is larger with irreversibility and fixed costs, for both the partial and gen-

eral equilibrium models (the partial equilibrium result is consistent with Proposition 7). In addition, the presence of fixed costs further reduces welfare.

5. CONCLUSION

This paper presents an analytically tractable continuous-time general equilibrium model with investment irreversibility and fixed adjustment costs. In the model, there is a continuum of firms that are subject to idiosyncratic shocks to capital. I derive analytical approximate solutions that permit me to characterize the long-run stationary equilibrium explicitly up to a simple algebraic equation. I find the following main results: First, although the presence of investment frictions lowers consumer welfare, it may raise or reduce the long-run average capital stock, depending on the degree of idiosyncratic uncertainty. In addition, an increase in idiosyncratic uncertainty may raise equilibrium aggregate capital, but reduce welfare in models with investment frictions. Second, an unexpected permanent change in the corporate income tax rate affects the investment trigger and target values, and hence the size and rate of capital adjustment. Thus, tax policy has both intensive and extensive margin effects. Following this tax policy, the percentage changes in equilibrium quantities are larger when fixed adjustment costs are larger. Finally, these changes are significantly smaller in a general equilibrium model than in a partial equilibrium model.

For tractability, I consider the role of irreversibility and fixed adjustment costs only. It would be interesting to consider other types of capital adjustment costs as studied in Abel and Eberly (1994) or Cooper and Haltiwanger (2006). I shall also emphasize that my model is highly stylized and cannot be used to match the data. As future research, it would be interesting to enrich the model structure and assumptions. In this case, tractability is lost and numerical methods are often needed.

APPENDIX A

Frictionless Investment

I write the capital accumulation equation without frictions as:

$$dk_t = -\delta k_t dt + \sigma k_t dW_t + dI_t, \tag{A.1}$$

where I_t denotes investment. The firm's problem is to solve

$$\max E\left[\int_0^\infty e^{-rt}\left\{((1-\theta)\,\pi\,(k_t; w) + \theta\delta k_t)\,dt - dI_t\right\}\right], \tag{A.2}$$

subject to (A.1). Substituting (A.1) into (A.2) yields:

$$\max E\left[\int_0^\infty e^{-rt}\left\{((1-\theta)\pi(k_t;w)+\theta\delta k_t)\,dt-dk_t-\delta k_t dt+\sigma k_t dW_t\right\}\right]$$

$$=\max E\left[\int_0^\infty e^{-rt}\left\{((1-\theta)\pi(k_t;w)+\theta\delta k_t)\,dt-dk_t-\delta k_t dt\right\}\right]$$

$$=\max E\left[\int_0^\infty e^{-rt}\left\{(1-\theta)(\pi(k_t;w)-\delta k_t)\,dt-dk_t\right\}\right]$$

$$=\max E\left[\int_0^\infty e^{-rt}(1-\theta)(\pi(k_t;w)-\delta k_t)\,dt-e^{-rt}k_t|_0^\infty+k_t de^{-rt}\right]$$

$$=k_0+\max E\left[\int_0^\infty e^{-rt}(1-\theta)(\pi(k_t;w)-\delta k_t)\,dt-re^{-rt}k_t dt\right],$$

where I have used the transversality condition: $\lim_{t\to\infty}E\left[e^{-rt}k_t\right]=0$. The last equation implies that I only need to solve the first-order condition:

$$(1-\theta)(\pi'(k^*;w)-\delta)=r.$$

From this equation, I obtain (9).

APPENDIX B

Irreversible Investment

When investment is irreversible, the value function V satisfies the Bellman equation (24) when the firm does not make investments. In addition, investment policy is characterized by a trigger value k_i such that when k hits k_i the firm makes investment to prevent k from falling below k_i. The value function V satisfies the smooth-pasting and high-contact conditions:

$$V'(k_i) = 1,$$
$$V''(k_i) = 0.$$

In addition, V satisfies the growth condition:

$$\lim_{k\to\infty}V(k)/\Pi(k)<\infty,$$

where $\Pi(k)$ is given in (26). The general solution is given by:

$$V(k)=G_1 k^{\beta_N}+\Pi(k),$$

where G_1 is a constant to be determined. Then we have

$$G_1\beta_N k_i^{\beta_N-1}+\Pi'(k) = 1,$$
$$G_1\beta_N(\beta_N-1)k_i^{\beta_N-2}+\Pi''(k_i) = 0.$$

Eliminating G_1 yields

$$(1 - \beta_N)(1 - \Pi'(k_i)) = k_i \Pi''(k_i).$$

Solving yields (35). Thus, we have

$$
\begin{aligned}
\pi'(k_i; w) &= \frac{(1 - \beta_N)\left(1 - \frac{\delta\theta}{r+\delta}\right)}{(\alpha - \beta_N)(1 - \theta)} v(\alpha) \\
&= \frac{1 - \beta_N}{\alpha - \beta_N} \frac{v(\alpha)}{r + \delta} \left(\frac{r}{1 - \theta} + \delta\right).
\end{aligned}
$$

I next derive the stationary distribution of capital. Making the transformation $z = \ln k$, I only need to derive the stationary distribution for (z_t). Let $z_i = \ln k_i$. The stationary density f on (z_i, ∞) satisfies the differential equation:

$$\mu f'(z) = \frac{\sigma^2}{2} f''(z),$$

where μ is given in equation (44). The general solution is given by

$$f(z) = A_1 e^{\gamma z} + B_1 \text{ for } z \in (z_i, \infty),$$

where A_1, B_1 are constants to be determined and γ is given in equation (45). The boundary conditions are

$$\lim_{z \to \infty} f(z) = 0,$$

$$\int_{z_i}^{\infty} f(z)\, dz = 1.$$

Using these boundary conditions, I can show that $B_1 = 0$ and $A_1 = -\gamma e^{-\gamma z_i}$.

APPENDIX C
Approximate Solution for the Trigger and Target

I use an approximation method similar to that in Abel and Eberly (1998). I approximate $J^1(k_b, k_c)$ and $J^2(k_b, k_c)$ around $k_b = k_c = k_i$ and $\xi = 0$. Define

$$\tilde{k}_b = k_b - k_i \text{ and } \tilde{k}_c = k_c - k_i.$$

Ignoring ξ, I apply the Taylor expansion to derive

$$
\begin{aligned}
J^j\left(k_b, k_c\right) = & J^j\left(k_i, k_i\right) + J_1^j\left(k_i, k_i\right)\tilde{k}_b + J_2^j\left(k_i, k_i\right)\tilde{k}_c \\
& + \frac{1}{2}J_{11}^j\left(k_i, k_i\right)\tilde{k}_b^2 + J_{12}^j\left(k_i, k_i\right)\tilde{k}_b\tilde{k}_c + \frac{1}{2}J_{22}^j\left(k_i, k_i\right)\tilde{k}_b^2 \\
& + \frac{1}{6}J_{111}^j\left(k_i, k_i\right)\tilde{k}_b^3 + \frac{1}{2}J_{112}^j\left(k_i, k_i\right)\tilde{k}_b^2\tilde{k}_c + \frac{1}{2}J_{122}^j\left(k_i, k_i\right)\tilde{k}_b\tilde{k}_c^2 + \frac{1}{6}J_{222}^j\left(k_i, k_i\right)\tilde{k}_c^3
\end{aligned}
$$

I first derive partial derivatives for $J^1\left(k_b, k_c\right)$. Differentiating equation (33) yields:

$$
J_1^1\left(k_b, k_c\right) = -k_c^{\beta_N-1}\Pi''\left(k_b\right) - \left(\beta_N - 1\right)\left(1 - \Pi'\left(k_c\right)\right)k_b^{\beta_N-2},
$$

and

$$
J_2^1\left(k_b, k_c\right) = \left(1 - \Pi'\left(k_b\right)\right)\left(\beta_N - 1\right)k_c^{\beta_N-2} + k_b^{\beta_N-1}\Pi''\left(k_c\right).
$$

From these equations, one can easily verify that

$$
J_1^1\left(k_i, k_i\right) = 0, \ \ J_2^1\left(k_i, k_i\right) = 0.
$$

Taking second-order differentiation of equation (33) yields:

$$
J_{11}^1\left(k_b, k_b\right) = -k_c^{\beta_N-1}\Pi'''\left(k_b\right) - \left(\beta_N - 1\right)\left(\beta_N - 2\right)\left(1 - \Pi'\left(k_c\right)\right)k_b^{\beta_N-3},
$$

and

$$
J_{12}^1\left(k_b, k_c\right) = -\left(\beta_N - 1\right)k_c^{\beta_N-2}\Pi''\left(k_b\right) + \left(\beta_N - 1\right)k_b^{\beta_N-2}\Pi''\left(k_c\right).
$$

Evaluating at $\left(k_i, k_i\right)$ yields:

$$
\begin{aligned}
J_{11}^1\left(k_i, k_i\right) &= -k_i^{\beta_N-1}\Pi'''\left(k_i\right) - \left(\beta_N - 1\right)\left(\beta_N - 2\right)\left(1 - \Pi'\left(k_i\right)\right)k_i^{\beta_N-3} \\
&= k_i^{\beta_N-1}\left(-\Pi'''\left(k_i\right) + \left(\beta_N - 2\right)k_i^{-1}\Pi''\left(k_i\right)\right) \\
&= \alpha\left(\alpha - 1\right)H\left(1 - \theta\right)\left(\beta_N - \alpha\right)k_i^{\alpha+\beta_N-4}/v\left(\alpha\right),
\end{aligned}
$$

and

$$
J_{12}^1\left(k_i, k_i\right) = 0.
$$

Similarly, we can derive that

$$
J_{22}^1\left(k_b, k_c\right) = \left(1 - \Pi'\left(k_b\right)\right)\left(\beta_N - 1\right)\left(\beta_N - 2\right)k_c^{\beta_N-3} + k_b^{\beta_N-1}\Pi'''\left(k_c\right),
$$

$$J_{22}^1(k_i, k_i) = -(\beta_N - 2) k_i^{\beta_N - 2} \Pi''(k_i) + k_i^{\beta_N - 1} \Pi'''(k_i)$$

$$= k_i^{\beta_N - 1} \left[\frac{(1 - \theta) H}{v(\alpha)} \alpha (\alpha - 1)(\alpha - 2) k_i^{\alpha - 3} - (\beta_N - 2) \frac{(1 - \theta) H}{v(\alpha)} \alpha (\alpha - 1) k_i^{\alpha - 2} \right]$$

$$= \alpha (\alpha - 1)(\alpha - \beta_N) \frac{(1 - \theta) H}{v(\alpha)} k_i^{\beta_N + \alpha - 4}.$$

Taking third-order differentiation of equation (33) yields:

$$J_{111}^1(k_b, k_c) = -k_c^{\beta_N - 1} \Pi''''(k_b) - (\beta_N - 1)(\beta_N - 2)(\beta_N - 3)(1 - \Pi'(k_c)) k_b^{\beta_N - 4},$$

$$J_{111}^1(k_i, k_i) = k_i^{\beta_N - 1} \left[\frac{-\alpha (\alpha - 1)(\alpha - 2)(\alpha - 3)(1 - \theta) H}{v(\alpha)} k_i^{\alpha - 4} + (\beta_N - 2)(\beta_N - 3) k_i^{-4} \Pi''(k_i) \right]$$

$$= k_i^{\beta_N + \alpha - 5} \frac{(1 - \theta) H \alpha (\alpha - 1)}{v(\alpha)} ((\beta_N - 2)(\beta_N - 3) - (\alpha - 2)(\alpha - 3))$$

$$= k_i^{\beta_N + \alpha - 5} \frac{(1 - \theta) H \alpha (\alpha - 1)}{v(\alpha)} (\beta_N - \alpha)(\alpha + \beta_N - 5),$$

$$J_{112}^1(k_b, k_c) = -(\beta_N - 1) k_c^{\beta_N - 2} \Pi'''(k_b) + (\beta_N - 1)(\beta_N - 2) \Pi''(k_c) k_b^{\beta_N - 3},$$

$$J_{112}^1(k_i, k_i) = (\beta_N - 1) k_i^{\beta_N - 2} \left[-\Pi'''(k_i) + (\beta_N - 2) \Pi''(k_i) k_i^{-1} \right]$$

$$= (\beta_N - 1) k_i^{\beta_N - 2} \frac{(1 - \theta) H}{v(\alpha)} \left[-\alpha (\alpha - 1)(\alpha - 2) k_i^{\alpha - 3} + (\beta_N - 2) k_i^{\alpha - 3} \alpha (\alpha - 1) \right]$$

$$= k_i^{\alpha + \beta_N - 5} (\beta_N - 1) \alpha (\alpha - 1) \frac{(1 - \theta) H}{v(\alpha)} (\beta_N - \alpha),$$

$$J_{122}^1(k_b, k_c) = -(\beta_N - 1)(\beta_N - 2) k_c^{\beta_N - 3} \Pi''(k_b) + (\beta_N - 1) k_b^{\beta_N - 2} \Pi'''(k_c),$$

$$J_{122}^1(k_i, k_i) = (\beta_N - 1) k_i^{\beta_N - 2} \frac{(1 - \theta) H}{v(\alpha)} \left[\alpha (\alpha - 1)(\alpha - 2) k_i^{\alpha - 3} - (\beta_N - 2) \alpha (\alpha - 1) k_i^{\alpha - 3} \right]$$

$$= (\beta_N - 1) k_i^{\alpha + \beta_N - 5} \alpha (\alpha - 1)(\alpha - \beta_N) \frac{(1 - \theta) H}{v(\alpha)},$$

$$J_{222}^1 (k_b, k_c) = (1 - \Pi' (k_b) - b) (\beta_N - 1) (\beta_N - 2) (\beta_N - 3) k_c^{\beta_N - 4} + k_b^{\beta_N - 1} \Pi'''' (k_c),$$

$$J_{222}^1 (k_i, k_i) = -J_{111}^1 (k_i, k_i).$$

I now derive partial derivatives for $J^2 (k_b, k_c)$. We begin with simplifying $J^2 (k_b, k_c)$ using equation (34):

$$
\begin{aligned}
J^2 (k_b, k_c) &= \left(\frac{1}{\beta_N} - 1 \right) (k_c - k_b) + \Pi (k_c) - \Pi (k_b) \\
&\quad - \frac{1}{\beta_N} (\Pi' (k_c) k_c - \Pi' (k_b) k_b) - \xi w \\
&= \left(\frac{1}{\beta_N} - 1 \right) (k_c - k_b) + \frac{(1 - \theta) H}{v (\alpha)} (k_c^\alpha - k_b^\alpha) + \frac{\delta \theta}{r + \delta} (k_c - k_b) \\
&\quad - \frac{1}{\beta_N} \left[\frac{(1 - \theta) H \alpha}{v (\alpha)} (k_c^\alpha - k_b^\alpha) + \frac{\delta \theta}{r + \delta} (k_c - k_b) \right] - \xi w \\
&= \left(\frac{1}{\beta_N} - 1 \right) \left(1 - \frac{\delta \theta}{r + \delta} \right) (k_c - k_b) + \frac{(1 - \theta) H}{v (\alpha)} \left(1 - \frac{\alpha}{\beta_N} \right) (k_c^\alpha - k_b^\alpha) - \xi w \\
&= \left(\frac{1}{\beta_N} - 1 \right) \left(1 - \frac{\delta \theta}{r + \delta} \right) \left[k_c - k_b - \frac{1}{\alpha} k_i^{1 - \alpha} (k_c^\alpha - k_b^\alpha) \right] - \xi w.
\end{aligned}
$$

We then derive partial derivatives:

$$J_1^2 (k_b, k_c) = \left(\frac{1}{\beta_N} - 1 \right) \frac{r + \delta (1 - \theta)}{r + \delta} \left(k_i^{1 - \alpha} k_b^{\alpha - 1} - 1 \right),$$

$$J_1^2 (k_i, k_i) = 0,$$

$$J_2^2 (k_b, k_c) = \left(\frac{1}{\beta_N} - 1 \right) \left(1 - \frac{\delta \theta}{r + \delta} \right) \left(1 - k_i^{1 - \alpha} k_c^{\alpha - 1} \right),$$

$$J_2^2 (k_i, k_i) = 0,$$

$$J_{11}^2 (k_b, k_c) = \left(\frac{1}{\beta_N} - 1 \right) \left(1 - \frac{\delta \theta}{r + \delta} \right) (\alpha - 1) k_i^{1 - \alpha} k_b^{\alpha - 2},$$

$$J_{11}^2 (k_i, k_i) = \left(\frac{1}{\beta_N} - 1 \right) \left(1 - \frac{\delta \theta}{r + \delta} \right) (\alpha - 1) k_i^{-1},$$

$$J_{12}^2 \left(k_b, k_c \right) = 0,$$

$$J_{22}^2 \left(k_b, k_c \right) = - \left(\frac{1}{\beta_N} - 1 \right) \left(1 - \frac{\delta\theta}{r+\delta} \right) (\alpha - 1) \, k_i^{1-\alpha} k_c^{\alpha-2},$$

$$J_{22}^2 \left(k_i, k_i \right) = -J_{11}^2 \left(k_i, k_i \right),$$

$$J_{111}^2 \left(k_b, k_c \right) = \left(\frac{1}{\beta_N} - 1 \right) \left(1 - \frac{\delta\theta}{r+\delta} \right) (\alpha - 1)(\alpha - 2) \, k_i^{1-\alpha} k_b^{\alpha-3},$$

$$J_{111}^2 \left(k_i, k_i \right) = \left(\frac{1}{\beta_N} - 1 \right) \left(1 - \frac{\delta\theta}{r+\delta} \right) (\alpha - 1)(\alpha - 2) \, k_i^{-2},$$

$$J_{112}^2 \left(k_b, k_c \right) = 0, \;\; J_{122}^2 \left(k_b, k_c \right) = 0,$$

$$J_{222}^2 \left(k_b, k_c \right) = - \left(\frac{1}{\beta_N} - 1 \right) \left(1 - \frac{\delta\theta}{r+\delta} \right) (\alpha - 1)(\alpha - 2) \, k_i^{1-\alpha} k_c^{\alpha-3},$$

$$J_{222}^2 \left(k_i, k_i \right) = - \left(\frac{1}{\beta_N} - 1 \right) \left(1 - \frac{\delta\theta}{r+\delta} \right) (\alpha - 1)(\alpha - 2) \, k_i^{-2}.$$

We finally use equations (31)-(32) and the preceding Taylor expansions to derive

$$
\begin{aligned}
0 \;=\; J^1 \left(k_b, k_c \right) &= \frac{1}{2} \alpha (\alpha - 1) \frac{(1-\theta) H}{v(\alpha)} (\beta_N - \alpha) \, k_i^{\beta_N + \alpha - 4} \left(\tilde{k}_b^2 - \tilde{k}_c^2 \right) \\
&+ \frac{1}{6} \alpha (\alpha - 1) \frac{(1-\theta) H}{v(\alpha)} (\beta_N - \alpha) \, k_i^{\beta_N + \alpha - 5} (\alpha + \beta_N - 5) \left(\tilde{k}_b^3 - \tilde{k}_c^3 \right) \\
&+ \frac{1}{2} \alpha (\alpha - 1) \frac{(1-\theta) H}{v(\alpha)} (\beta_N - \alpha) \, k_i^{\beta_N + \alpha - 5} (\beta_N - 1) \left(\tilde{k}_b^2 \tilde{k}_c - \tilde{k}_c^2 \tilde{k}_b \right)
\end{aligned}
$$

and

$$
\begin{aligned}
0 = J^2(k_b, k_c) = & -\xi w \\
& + \frac{1}{2}\left(\frac{1}{\beta_N} - 1\right)\left(1 - \frac{\delta\theta}{r+\delta}\right)(\alpha - 1)k_i^{-1}\left(\tilde{k}_b^2 - \tilde{k}_c^2\right) \\
& + \frac{1}{6}\left(\frac{1}{\beta_N} - 1\right)\left(1 - \frac{\delta\theta}{r+\delta}\right)(\alpha - 1)(\alpha - 2)k_i^{-2}\left(\tilde{k}_b^3 - \tilde{k}_c^3\right) \\
= & -\xi w + \left(\frac{1}{\beta_N} - 1\right)\left(1 - \frac{\delta\theta}{r+\delta}\right)(\alpha - 1) \\
& \times k_i^{-1}\frac{1}{2}\left[\left(\tilde{k}_b^2 - \tilde{k}_c^2\right) + \frac{1}{3}(\alpha - 2)k_i^{-1}\left(\tilde{k}_b^3 - \tilde{k}_c^3\right)\right]
\end{aligned}
$$

It follows from the preceding two equations that

$$
\tilde{k}_b^2 - \tilde{k}_c^2 = -\frac{1}{3}k_i^{-1}(\alpha + \beta_N - 5)\left(\tilde{k}_b^3 - \tilde{k}_c^3\right) - k_i^{-1}(\beta_N - 1)\left(\tilde{k}_b^2\tilde{k}_c - \tilde{k}_c^2\tilde{k}_b\right) \tag{C.1}
$$

$$
\xi w = \left(\frac{1}{\beta_N} - 1\right)\left(1 - \frac{\delta\theta}{r+\delta}\right)(\alpha - 1)\frac{k_i^{-1}}{2}\left[\left(\tilde{k}_b^2 - \tilde{k}_c^2\right) + \frac{1}{3}(\alpha - 2)k_i^{-1}\left(\tilde{k}_b^3 - \tilde{k}_c^3\right)\right] \tag{C.2}
$$

I next substitute the right side of equation (C.1) for $\tilde{k}_b^2 - \tilde{k}_c^2$ in equation (C.2) and rearrange terms to derive

$$
\begin{aligned}
\xi w = & \left(\frac{1}{\beta_N} - 1\right)\left(1 - \frac{\delta\theta}{r+\delta}\right)(\alpha - 1)\frac{k_i^{-2}}{6}\left(\tilde{k}_b - \tilde{k}_c\right) \\
& \times\left[(3 - \beta_N)\left(\tilde{k}_b + \tilde{k}_c\right)^2 - 2\beta_N\tilde{k}_b\tilde{k}_c\right].
\end{aligned} \tag{C.3}
$$

Equation (C.1) also implies that

$$
k_i\left(\tilde{k}_b + \tilde{k}_c\right) + \frac{1}{3}(\alpha + \beta_N - 5)\left(\tilde{k}_b + \tilde{k}_c\right)^2 + \frac{1}{3}(2\beta_N - \alpha + 2)\tilde{k}_b\tilde{k}_c = 0 \tag{C.4}
$$

I will verify later that $\tilde{k}_b$ is of the order of $\tilde{k}_c$, $\tilde{k}_b\tilde{k}_c$ is of the order of $\tilde{k}_c^2$, and $\left(\tilde{k}_b + \tilde{k}_c\right)^2$ is of the order of $\tilde{k}_c^4$. Given this fact, I ignore all terms with order higher than $\tilde{k}_c$ in equation (C.4). I then obtain that $\tilde{k}_b = -\tilde{k}_c$. Substituting $\tilde{k}_b = -\tilde{k}_c$ into equation (C.3) and ignoring all terms with order higher than $\tilde{k}_c$, I obtain equation (37).

I next derive $\tilde{k}_c$. Ignoring the term $\left(\tilde{k}_b + \tilde{k}_c\right)^2$ in equation (C.4), I obtain:

$$
\tilde{k}_b = \frac{-3k_i}{3k_i + (2\beta_N - \alpha + 2)\tilde{k}_b}\tilde{k}_c. \tag{C.5}
$$

Given this solution, we know that $\tilde{k}_b\tilde{k}_c$ is of the order $\tilde{k}_c^2$. In addition,

$$\tilde{k}_b + \tilde{k}_c = \frac{(2\beta_N - \alpha + 2)}{3k_i + (2\beta_N - \alpha + 2)\tilde{k}_b}\tilde{k}_c^2$$

is of the order of $\tilde{k}_b^2$, and

$$\left(\tilde{k}_b + \tilde{k}_c\right)^2 = \left[\frac{(2\beta_N - \alpha + 2)}{3k_i + (2\beta_N - \alpha + 2)\tilde{k}_b}\right]^2 \tilde{k}_c^4$$

is of the order of $\tilde{k}_c^4$. These results verify my previous claim.

APPENDIX D

Proofs

Proof of Lemma 1:

See Appendix A.

Proof of Lemma 2:

The optimal investment policy and the trigger value k_i are derived in Appendix B. From equation (35), one can immediately see that k_i decreases with θ. We now show that $k_i < k^*$. Using (35) and (9) and noticing that

$$v(\alpha) = \frac{(\alpha - \beta_N)(\alpha - \beta_P)}{\beta_N \beta_P}r, \tag{D.1}$$

I only need to prove

$$\frac{(\alpha - \beta_P)(1 - \beta_N)}{\beta_N \beta_P} > 1 + \frac{\delta}{r}.$$

Since $\beta_N < 0$, the previous inequality is equivalent to

$$(\alpha - \beta_P)(1 - \beta_N) < \beta_N \beta_P + \beta_N \beta_P \frac{\delta}{r},$$

or

$$\alpha(1 - \beta_N) - \beta_P < \beta_N \beta_P \frac{\delta}{r}. \tag{D.2}$$

Using the fact that

$$\beta_P + \beta_N = \frac{2\delta}{\sigma^2} + 1 \text{ and } \beta_N \beta_P = -\frac{2r}{\sigma^2},$$

I can rewrite (D.2) as

$$\alpha\left(\beta_P - \frac{2\delta}{\sigma^2}\right) - \beta_P < -\frac{2\delta}{\sigma^2}.$$

That is

$$\beta_P > \frac{2\delta}{\sigma^2}.$$

This can be easily verified since $v\left(2\delta/\sigma^2\right) > 0$. Q.E.D.

Proof of Proposition 1:

See Appendix C.

Proof of Proposition 2:

Proposition 1 implies that the size of capital adjustment $k_b - k_c \approx 2k_c$ increases with fixed cost ξ. Let $x = (1 - \theta)$. By (37) and Lemma 2, we can show that the sign of $\partial \tilde{k}_c/\partial x$ is the same as the sign of $2r - \delta\left(1 - \theta\right)\left(1 - \alpha\right)$. Since $\partial \tilde{k}_c/\partial \theta = -\partial \tilde{k}_c/\partial x$, we obtain the desired result. Q.E.D.

Proof of Lemma 3:

By Proposition 1 and the Taylor expansion, we have

$$
\begin{aligned}
\ln\left(\frac{k_c}{k_b}\right) &= \ln\left(\frac{k_i + \tilde{k}_c}{k_i + \tilde{k}_b}\right) = \ln\left(\frac{k_i + \tilde{k}_c}{k_i - \tilde{k}_c}\right) \\
&= \ln\left(1 + \frac{\tilde{k}_c}{k_i}\right) - \ln\left(1 - \frac{\tilde{k}_c}{k_i}\right) \\
&= \frac{2\tilde{k}_c}{k_i} + \frac{2}{3}\left(\frac{\tilde{k}_c}{k_i}\right)^3.
\end{aligned}
$$

It follows from Proposition 1 that $\tilde{k}_c$ increases with ξ. Thus, the preceding equation implies that $\ln\left(k_c/k_b\right)$ increases with ξ. The preceding equation also implies that to show $\ln\left(k_c/k_b\right)$ increases with θ and σ^2, we only need to show $\tilde{k}_c/k_i$ increases with θ and σ^2. By Proposition 1, it suffices to show that

$$\left(1 - \frac{\delta\theta}{r + \delta}\right)^{-\frac{1}{3}} k_i^{-\frac{1}{3}}$$

increases with θ and that $\left(1 - \beta_N\right) k_i$ decreases with σ^2. The first result immediately follows from Lemma 2. Using Lemma 2 again, I show that

$$
\begin{aligned}
\frac{\partial\left[\left(1 - \beta_N\right) k_i\right]}{\partial \sigma^2} &= -\frac{\partial \beta_N}{\partial \sigma^2} k_i + \left(1 - \beta_N\right)\frac{\partial k_i}{\partial \sigma^2} \\
&= \frac{\partial \beta_N}{\partial \sigma^2}\left[\left(1 - \beta_N\right)\frac{\partial k_i}{\partial \beta_N} - k_i\right] \\
&= k_i\frac{\partial \beta_N}{\partial \sigma^2}\left[\frac{-1}{\alpha - \beta_N} - 1\right] < 0,
\end{aligned}
$$

where the last inequality follows from the fact that $\beta_N < 0$ and $\partial \beta_N/\partial \sigma^2 > 0$. Q.E.D.

Proof of Proposition 3:

It follows from equation (40) and Lemma 3. Q.E.D.

Proof of Proposition 4:

The proof is similar to that of Proposition 1 in Caballero (1993). Here I outline the proof. I refer the reader to Caballero (1993) for details. Except on points z_b and z_c, the stationary density $f(z)$ on $(z_b, z_c) \cup (z_c, \infty)$ satisfies the differential equation:

$$\mu f'(z) = \frac{\sigma^2}{2} f''(z),$$

where μ is given in equation (44). The general solution is

$$f(z) = \begin{cases} A_1 e^{\gamma z} + B_1 & \text{for } z \in (z_b, z_c) \\ A_2 e^{\gamma z} + B_2 & \text{for } z \in (z_c, \infty) \end{cases},$$

where A_1, A_2, B_1, and B_2 are coefficients to be determined and γ is given in equation (45). The boundary conditions are

$$f(z_b) = 0,$$

$$\lim_{z \downarrow z_c} f(z) = \lim_{z \uparrow z_c} f(z),$$

$$\lim_{z \downarrow z_c} f'(z) - \lim_{z \uparrow z_c} f'(z) = \lim_{z \uparrow \infty} f'(z) - \lim_{z \downarrow z_b} f'(z),$$

$$\int_{z_b}^{\infty} f(z)\, dz = 1.$$

Using these boundary conditions, we can show that $B_2 = 0$,

$$\begin{aligned} A_1 e^{\gamma z_b} + B_1 &= 0, \\ A_1 e^{\gamma z_c} + B_1 &= A_2 e^{\gamma z_c}, \\ A_2 \gamma e^{\gamma z_c} - A_1 \gamma e^{\gamma z_c} &= -A_1 \gamma e^{\gamma z_b}, \end{aligned}$$

and

$$\int_{z_b}^{z_c} \left(A_1 e^{\gamma z} + B_1 \right) dz + \int_{z_c}^{\infty} A_2 e^{\gamma z}\, dz = 1.$$

Solving yields the desired result in Proposition 4. Q.E.D.

Proof of Proposition 5:

Note that

$$K_f = \int_{z_b}^{\infty} e^z f(z),$$

where the stationary density f is given in Proposition 4. Using Proposition 4, we can derive (47).   Q.E.D.

Proof of Proposition 6:

Using the stationary density f derived in Appendix B, we can derive the aggregate capital stock when investment is irreversible:

$$K_i = \int_{z_i}^{\infty} e^z f(z)\, dz = \frac{\gamma}{\gamma+1} e^{z_i} = \left(1 + \frac{\sigma^2}{2\delta}\right) k_i.$$

When ξ is sufficiently small, we use Lemma 3, Propositions 3 and 5 to show that

$$K_f = \frac{(k_c - k_b)}{\ln(k_c/k_b)} \left(1 + \frac{\sigma^2}{2\delta}\right) = \left(1 + \frac{\sigma^2}{2\delta}\right) \frac{2\tilde{k}_c}{\frac{2\tilde{k}_c}{k_i} + \frac{2}{3}\left(\frac{\tilde{k}_c}{k_i}\right)^3}.$$

Simplifying yields (50). By Proposition 1, $\tilde{k}_c$ increases with ξ. Thus, K_f decreases with ξ.   Q.E.D.

Proof of Proposition 7:

Using Lemmas 1-2 and Proposition 6, I can show that $\partial \ln K_f / \partial \theta = \partial \ln K^* / \partial \theta < 0$. Using Proposition 6, I show that when ξ is sufficiently small,

$$\frac{\partial \ln K_f}{\partial \theta} = \frac{\partial \ln K_i}{\partial \theta} - \frac{\partial \ln \left(1 + \left(\frac{\tilde{k}_c}{k_i}\right)^2\right)}{\partial \theta}.$$

In the proof of Lemma 3, I show that $\tilde{k}_c / k_i$ increases with θ. Thus, we have $\frac{\partial \ln K_f}{\partial \theta} < \frac{\partial \ln K_i}{\partial \theta} < 0$.   Q.E.D.

REFERENCES

Abel, A. B., 1982. Dynamic effects of permanent and temporary tax policies in a q model of investment. *Journal of Monetary Economics* **9**, 353-373.

Abel, A. B. and J. C. Eberly, 1994. A unified model of investment under uncertainty. *American Economic Review* **84**, 1369-1384.

Abel, A. B. and J. C. Eberly, 1998. The Mix and Scale of Factors with Irreversibility and Fixed Costs of Investment, in Bennett McCallum and Charles Plosser (eds.) *Carnegie-Rochester Conference Series on Public Policy* **48**, 101-135.

Abel, A. B. and J. C. Eberly, 1999. The Effects of Irreversibility and Uncertainty on Capital Accumulation. *Journal of Monetary Economics* **44**, 339-377.

Auerbach, A., J., 1989. Tax reform and adjustment costs: the impact on investment and market value. *International Economic Review* **30**, 939-942.

Bachmann, R., Caballero, R., and E. Eduardo, 2008. Aggregate implications of lumpy investment: new evidence and a DSGE model. NBER working paper No. 12336.

Bentolila, S. and G. Bertola, 1990. Firing Costs and Labor Demand: How Bad is Eurosclerosis? *Review of Economic Studies* **57**, 381-402.

Bertola, G. and R. J. Caballero, 1994. Irreversibility and Aggregate Investment. *Review of Economic Studies* **61**, 223-246.

Caballero, R. J., 1993. Durable Goods: An Explanation for their Slow Adjustment. *Journal of Political Economy* **101**, 353-384.

Caballero, R. J., and E. M. R. A. Engel, 1999. Explaining investment dynamics in U.S. manufacturing: A generalized (S,s) approach, *Econometrica* **67**, 783-826.

Caballero, R. J., E. M.R.A. Engel, and J. C. Haltiwanger, 1995. Plant Level Adjustment and Aggregate Investment Dynamics. *Brookings Papers on Economic Activity* **2**, 1-39.

Cooper, R. W. and J. C. Haltiwanger, 2006. On the Nature of Capital Adjustment Costs. *Review of Economic Studies* **73**, 611-633.

Cooper, R. W., J. C. Haltiwanger, and L. Power, 1999. Machine Replacement and Business Cycle: Lumps and Bumps. *American Economic Review* **89**, 921-946.

Domes, M. and T. Dunne, 1998. Capital Adjustment Patterns in Manufacturing Plants. *Review of Economic Dynamics* **1**, 409-429.

Gomes, J., 2001. Financing investment. *American Economic Review* **91**, 1263-1285.

Gourio, F. and A. Kashyap, 2007. Investment Spikes: New Facts and a General Equilibrium Exploration. *Journal of Monetary Economics* **54: Supplement 1**, pp. 1-22.

Hennessy, C. and T. Whited, 2005. Debt dynamics. *Journal of Finance* **60**, 1129-1165.

Hennessy, C. and T. Whited, 2007. How costly is external financing? evidence from a structural estimation. *Journal of Finance* **62**, 1705-1745.

Hopenhayn, H. and R. Rogerson, 1993. Job turnover and policy evaluation: a general equilibrium analysis. *Journal of Political Economy* **101**, 915-938.

Khan A. and J. K. Thomas, 2003. Nonconvex Factor Adjustments in Equilibrium Business Cycles Models: Does Nonlinearity Matter? *Journal of Monetary Economics* **50**, 331-360.

Khan, Aubhik and Julia Thomas, 2008. Idiosyncratic Shocks and the Role of Nonconvexities in Plant and Aggregate Investment Dynamics. *Econometrica* **76(2)**, 395-436.

Miao, J., 2005. Optimal capital structure and industry dynamics. *Journal of Finance* **6**, 2621-2659.

Ramey, V. A. and M. D. Shapiro, 2001. Displaced Capital: A Study of Aerospace Plant Closings. *Journal of Political Economy* **109**, 958-992.

Thomas, J. K., 2002. Is Lumpy Investment Relevant for Business Cycles. *Journal of Political Economy* **110**, 508-534.

Veracierto, M. L., 2002. Plant-Level Irreversible Investment and Equilibrium Business Cycles. *American Economic Review* **92**, 818-197.