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Uncertain Paternity, Power Utility, and Fractional Moments: The Case of Binomially Distributed Reproductive Success*

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In this paper, Newton's Theorem is used to derive a formula for the fractional moment of the binomial distribution. The formula is general enough to handle a continuous number of draws and thereby facilitates the analysis of representative agent models where discrete quantities are typically reflected by continuous variables. An application of the formula illustrates that it is easily implemented and can be quickly calculated using standard mathematical software.

Key Words: Uncertain Paternity; Binomial Distribution; Expected Power Utility; Fractional Moment; Newton's Theorem.
JEL Classification Numbers: D10, J11, J13.

1. INTRODUCTION

Economists seem to have a liking for continuous variables. Even though real world decisions are often discrete by their very nature, economic analyses thereof usually start with writing down the continuous versions of the original decision problems. One stark justification for this approach can be found in the concept of the representative agent because representative agent models aim to reflect and explain average behavior in an economy. Beckerian theories of fertility provide prominent examples where continuous choice frameworks are used to explain discrete real world decisions such as a family's number of kids (Becker, 1991).

Difficulties arise, however, when the variable of interest is of a stochastic nature which typically means that it follows a specific (discrete) probability distribution. As an example, consider the notion of paternal uncertainty which is attracting more and more attention within the economic literature

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on fertility (see, for example, the papers by Anke Becker (2019), Lena Edlund (2013), Brishti Guha (2012), among others). Uncertain paternity occurs when promiscuous behavior lowers the probability of fathering a child below one. As a consequence, the representative man's number of own offspring follows a binomial distribution with two parameters: the first is the number of his female mating partners and the second is the aforementioned probability of fatherhood. While the number of mating partners is obviously discrete in nature, the representative agent concept suggests a continuous choice.

In this paper, we will derive an easy to implement formula for the fractional moment of the binomial distribution that is general enough to handle continuous choices of the number of female mating partners.

2. PREFERENCES, REPRODUCTIVE TECHNOLOGIES, AND FRACTIONAL MOMENTS

This section briefly sketches a few ingredients of a Beckerian fertility model with paternal uncertainty. Two fundamental asymmetries between women and men are taken into account: asymmetry in offspring recognition and asymmetry in reproductive capacity.[1] The model considers a society that is populated by a large number of single-period lived fertile women and men with equal preferences. Individuals of the same sex are assumed to be homogeneous.

2.1. Female Agents and Limited Reproductive Capacity

The representative woman derives welfare $W_♀$ from reproductive success, which is a function of the (discrete) number $K_♀$ of own biological offspring:

$$W_♀ = V(K_♀) = K_♀^{\mu}, \qquad 0 < \mu < 1. \tag{1}$$

Note that the power utility specification $V(K) = K^{\mu}$ possesses derivatives of all orders. In particular, the function is increasing at a diminishing rate:

$$\frac{\partial^2 V(K)}{\partial K^2} < V(0) = 0 < \frac{\partial V(K)}{\partial K}. \tag{2}$$

Moreover, the Arrow-Pratt measure of relative risk aversion is constant and equal to $1 - \mu$. To account for physiological constraints in female fertility, the reproductive capacity of women is assumed to be bounded by one (see also Willis, 1999):

$$K_♀ \in \{0, 1\}. \tag{3}$$

[1]For a detailed description of this model please refer to Bethmann and Kvasnicka (2011).

Note that function V measures reproductive success solely in terms of the number of own offspring. This is a simplification compared to the original model in Bethmann and Kvasnicka (2011) where parental investments in child quality play an important role as well. For the representative woman, for example, securing additional male child quality contributions provides an incentive to engage in extra-pair mating.

A necessary condition for having a child is that the woman does not abstain from mating. In other words, it requires at least one male mating partner to reproduce:

$$K_{♀} = \min\{1, N_{♀}\}, \tag{4}$$

where $N_{♀} \in \mathbb{N}_0$ denotes the number of male mating partners the representative female has chosen. As can be seen from (4), additional male partners do not increase the number of offspring because of limited reproductive capacity. While extra-pair mating might help women to secure (additional) child quality contributions in the original version of the model, we will bypass all child quality decisions as well as mating market considerations in this paper. Here, we simply assume that the representative woman's chosen number of male mating partners ($N_{♀}$) is taken as given by male participants in the mating market.

2.2. Male Agents and Paternal Uncertainty

Like women, men derive utility from reproductive success. Unlike women, however, men are restricted in their reproductive capacity only by access to fertile partners of the opposite sex. They may therefore father more than one child by mating several women. Moreover, unlike women, men do not recognize their offspring. However, men can infer the total number of matings $N_{♀}$ each of their female partners has in equilibrium. For a child born by one of his female partners, a man's probability of fatherhood δ is therefore inversely related to the number of male partners $N_{♀}$ the female has mated, i.e.:

$$\delta = N_{♀}^{-1} \qquad \Leftrightarrow \qquad 1 - \delta = 1 - N_{♀}^{-1}, \tag{5}$$

where $1 - \delta$ measures the degree of paternal uncertainty. If women are monogamous, men can be absolutely certain about biological parenthood. If women are promiscuous, however, paternity is uncertain. In this case, the actual but unknown number of own offspring $K_{♂}$ of a man that mates $N_{♂} \in \mathbb{N}_0$ females can be any integer up to and including $N_{♂}$:

$$K_{♂} \in \{0, .., N_{♂}\}\,. \tag{6}$$

Given $N_♂$ and δ, the probability of a man to father exactly $K_♂$ children therefore follows the binomial distribution with parameters $N_♂$ and δ:

$$\mathbb{P}\left(K_♂; N_♂, \delta\right) = \binom{N_♂}{K_♂} (1-\delta)^{N_♂ - K_♂}\, \delta^{K_♂}. \tag{7}$$

Denoting by $\mathbb{E}$ the expectations operator under the above binomial distribution, the expected number of offspring of a man with $N_♂$ female partners is hence given by:

$$\mathbb{E}\left[K_♂\right] = \delta N_♂. \tag{8}$$

2.3. Power Utility and Fractional Moments

Because paternity is uncertain, men can only maximize expected reproductive success. Together with the power utility specification this implies that the representative man's welfare $W_♂$ is determined by the (fractional) μ^{th} moment of the binomial distribution with parameters $N_♂$ (number of female partners) and δ (probability of fatherhood):

$$W_♂ = \mathbb{E}\left[V(K_♂)\right] = \mathbb{E}[K_♂^{\mu}]. \tag{9}$$

According to Hoffmann-Jørgensen (1994, p. 303), the fractional moment of a non-negative random variable can be calculated using the following formula:

$$\mathbb{E}[K_♂^{\mu}] = \frac{\mu}{\Gamma(1-\mu)} \int_0^{\infty} \frac{1 - M(-z)}{z^{\mu+1}}\, \mathrm{d}z, \tag{10}$$

where Γ is Euler's Gamma function[2] and $M(\tau)$ denotes the respective moment generating function with the property that the n^{th} moment about the origin is given by the n^{th} derivative evaluated at zero. Formula (10) can be obtained by using techniques of fractional calculus and taking the derivative of the μ^{th} order of $M(\tau)$ (for details see Wolfe, 1975). For our purposes, it is important to note that $K_♂$ is indeed a non-negative random variable and that the moment generating function of the underlying binomial distribution is given by:

$$M(\tau) = \left(1 - \delta + \delta \mathrm{e}^{\tau}\right)^{N_♂}. \tag{11}$$

[2]The Gamma function $\Gamma(\nu) = \int_0^{\infty} t^{\nu-1}\mathrm{e}^{-t}\,\mathrm{d}t$ extends the factorial function for arbitrary real numbers, except zero and negative integers: $\Gamma(n+1) = n!$ for all $n \in \mathbb{N}_0$. For $\nu \in \mathbb{R}\backslash\{0, -1, -2, ..\}$, it satisfies the functional equation $\Gamma(\nu+1) = \nu\Gamma(\nu)$. The incomplete gamma function $\Gamma(\nu, z) = \int_z^{\infty} t^{\nu-1}\mathrm{e}^{-t}\,\mathrm{d}t$ coincides with the Gamma function when $z = 0$.

After some calculus, the substitution of the moment generating function (11) into formula (10) leads us to the following representation of the μ^{th} fractional moment:[3]

$$\mathbb{E}[K_{♂}^{\mu}] = \frac{\delta\nu}{\Gamma(1-\mu)} \int_0^{\infty} \frac{\mathrm{e}^{-t}(1-\delta+\delta\mathrm{e}^{-t})^{\nu-1}}{t^{\mu}}\, \mathrm{d}t, \tag{12}$$

where $N_{♂} \in \mathbb{N}_0$ was replaced with $\nu \geq 1$ in order to stress that we are considering a continuous number of mating partners from now on.[4] Moreover, note that the integral on the right hand side of (12) evaluates to $\Gamma(1-\mu)$ when $\nu = 1$ holds such that $\mathbb{E}[K_{♂}^{\mu}] = \delta\nu$ in this case. Hence, we focus on $\nu > 1$ from now on.

Interestingly, function $M(\tau)$ also plays a role when we replace the utility function $V(K)$ with the negative exponential, i.e $\tilde{V}(K) = \frac{1-\exp(-\alpha K)}{\alpha}$, where $\alpha > 0$ is the Arrow-Pratt measure of constant absolute risk aversion. For in this case, it can be shown that expected reproductive success is determined by a monotonous transformation of function M evaluated at $\tau = -\alpha$:

$$\mathbb{E}\big[\tilde{V}(K_{♂})\big] = \tfrac{1-M(-\alpha)}{\alpha}. \tag{13}$$

Consequently, expected utility can be calculated easily for various degrees of reproductive success δ and continuous numbers of draws ν in this case.[5]

3. FRACTIONAL MOMENTS OF THE BINOMIAL DISTRIBUTION AND NEWTON'S THEOREM

In this section we will use Newton's generalization of the binomial theorem to rewrite the integrand in (12) in terms of an infinite series. As a quick reminder, let us briefly state how the binomial theorem for a natural number $a \in \mathbb{N}_0$ expands the polynomial $(x+y)^a$ with $x, y \in \mathbb{R}$ into a finite sum as follows:

$$\begin{aligned}(x+y)^a &= \tbinom{a}{0}x^0y^a + \tbinom{a}{1}x^1y^{a-1} + \ldots + \tbinom{a}{a-1}x^{a-1}y^1 + \tbinom{a}{a}x^ay^0 \\ &= \sum_{k=0}^{a} \tbinom{a}{k}x^ky^{a-k}. \end{aligned} \tag{14}$$

[3]The result in equation (12) can be obtained by applying the rule of integration by parts together with an application of L'Hôpital's rule.

[4]The case $N_{♂} = 0$ is trivial and of no interest in the current discussion. As argued above, paternity is certain in a monogamous society ($\nu = 1$) and turns uncertain when agents become polygamous ($\nu > 1$).

[5]See the appendix for more details about the case of constant absolute risk aversion. I am indebted to an anonymous referee for the valuable suggestion to investigate this case.

As we will see, Newton's generalization by and large resembles the discrete original in (14). However, two small differences have to be discussed. First and foremost, in their most basic version, binomial coefficients $\binom{a}{k} = \frac{a!}{k!\,(a-k)!}$ are defined only for natural numbers $a, k \in \mathbb{N}_0$ with $0 \leq k \leq a$ which is not helpful when considering an arbitrary real number α. Instead, we define the binomial coefficient for arbitrary real numbers $\alpha, \kappa \in \mathbb{R}$ as follows:

$$\binom{\alpha}{\kappa} = \lim_{v \to \alpha} \lim_{w \to \kappa} \frac{\Gamma(v+1)}{\Gamma(w+1)\Gamma(v-w+1)}, \tag{15}$$

where the limits occur to handle the singularities of the Gamma function at non-positive integers.[6,7] The second difference refers to the summation in (14). Instead of the finite sum, the generalized binomial formula in the following theorem contains an infinite series.

Theorem 1 (Newton). *Let $x, y \in \mathbb{R}$ where $0 \leq \|x\| < \|y\|$ and let $\alpha \in \mathbb{R}$. Then the expansion of the binomial $(x+y)^\alpha$ is given by an infinite series as follows:*

$$(x+y)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k y^{\alpha-k}, \tag{16}$$

where the infinite series in fact converges.

A proof of Newton's Binomial Theorem can be found in many textbooks on real analysis (see, e.g. Protter, 1998). In the following, we will apply the theorem to the expression $(1 - \delta + \delta e^{-t})^{\nu-1}$ within the integrand of (12). For this purpose, we must assign the roles of x and y with $\|x\| < \|y\|$. A first attempt could be to set $x = \delta e^{-t}$ and $y = 1 - \delta$. Clearly, we can ignore the absolute value operators in this case because both δe^{-t} and $1-\delta$ cannot be negative. However, the t in the exponential function requires our attention. Obviously, the value $t_0 = \log(\frac{\delta}{1-\delta})$ is important in this context because there $1 - \delta = \delta e^{-t_0}$ holds. In fact we can distinguish the following cases:

$$t_0 < 0 \quad \Leftrightarrow \quad \delta \in (0, \tfrac{1}{2}), \tag{17a}$$

$$t_0 = 0 \quad \Leftrightarrow \quad \delta = \tfrac{1}{2}, \tag{17b}$$

$$t_0 > 0 \quad \Leftrightarrow \quad \delta \in (\tfrac{1}{2}, 1). \tag{17c}$$

[6] In the rest of the paper these limits will play no role so that $\binom{\alpha}{\kappa} = \frac{\Gamma(\alpha+1)}{\Gamma(\kappa+1)\Gamma(\alpha-\kappa+1)}$ always holds.

[7] The appendix to this paper lists a few useful properties of generalized binomial coefficients.

First, we consider the case in (17a) where $\delta \in (0, \frac{1}{2})$. Obviously, $\|x\| = \delta e^{-t} < \frac{1}{2} < 1 - \delta = \|y\|$ holds for all $t \in [0, \infty)$ and we apply the binomial theorem (16) with $x = \delta e^{-t}$ and $y = 1 - \delta$:

$$\mathbb{E}[K^{\mu}_{♂}] = \tfrac{\nu(1-\delta)^{\nu}}{\Gamma(1-\mu)} \int_0^{\infty} \sum_{k=0}^{\infty} \tbinom{\nu-1}{k} \left(\tfrac{\delta}{1-\delta}\right)^{k+1} \tfrac{e^{-(k+1)t}}{t^{\mu}}\, dt, \qquad \delta \in (0, \tfrac{1}{2}). \quad (18)$$

Because the sequence of functions $f_k(t) = \binom{\nu-1}{k}\left(\frac{\delta}{1-\delta}\right)^{k} \frac{e^{-(k+1)t}}{t^{\mu}}$ converges, we may switch integration and summation in (18) by the Fubini-Tonelli Theorem.[8] The subsequent evaluation of the integrals then yields:

$$\mathbb{E}[K^{\mu}_{♂}] = \nu(1-\delta)^{\nu} \sum_{k=0}^{\infty} \tbinom{\nu-1}{k} \left(\tfrac{\delta}{1-\delta}\right)^{k+1} \tfrac{1}{(k+1)^{1-\mu}}. \qquad (19a)$$

Second, we turn to the case in (17b) where δ equals one half such that we can factor out $(\frac{1}{2})^{\nu-1}$ before we set $x = e^{-t}$ and $y = 1$. Obviously, $\|x\| < \|y\|$ holds for all positive real t:[9]

$$\mathbb{E}[K^{\mu}_{♂}] = \tfrac{\nu}{\Gamma(1-\mu)} \left(\tfrac{1}{2}\right)^{\nu} \int_0^{\infty} \sum_{k=0}^{\infty} \tbinom{\nu-1}{k} \tfrac{e^{-(k+1)t}}{t^{\mu}}\, dt, \qquad \delta = \tfrac{1}{2}. \qquad (20)$$

Again, the Fubini-Tonelli Theorem allows us to switch integration and summation in (20) and we obtain the following result:

$$\mathbb{E}[K^{\mu}_{♂}] = \tfrac{\nu}{2^{\nu}} \sum_{k=0}^{\infty} \tbinom{\nu-1}{k} \tfrac{1}{(k+1)^{1-\mu}}. \qquad (19b)$$

Third, let us consider the case in (17c) where t_0 is positive. Here, the variable of integration t is decisive when assigning the roles of x and y. Consequently, we split the area of integration and obtain the following equation with two integrals:

$$\mathbb{E}[K^{\mu}_{♂}] = \tfrac{\delta\nu}{\Gamma(1-\mu)} \left[\int_0^{t_0} \tfrac{e^{-t}(1-\delta+\delta e^{-t})^{\nu-1}}{t^{\mu}}\, dt + \int_{t_0}^{\infty} \tfrac{e^{-t}(1-\delta+\delta e^{-t})^{\nu-1}}{t^{\mu}}\, dt\right], \ \delta \in (\tfrac{1}{2}, 1). \qquad (21)$$

When $t \in [0, t_o)$, the inequality $\|\delta e^{-t}\| > \|1-\delta\|$ holds and we set $x = 1-\delta$ and $y = \delta e^{-t}$. Yet, when $t \in (t_0, \infty)$, the reversed inequality $\|\delta e^{-t}\| <$

[8] For example, consider the series $\sum_{k=0}^{\infty} f_k(t) \le \sum_{k=0}^{\infty} \|f_k(t)\| < \binom{\nu-1}{\frac{\nu-1}{2}} \frac{1-\delta}{1-2\delta} \frac{e^{-t}}{t^{\mu}} < \infty$.

[9] Note that the series in (16) converges even when $t = 0$, because we consider a positive exponent $\nu - 1 > 0$.

$\|1-\delta\|$ is true and we set $x = \delta e^{-t}$ and $y = 1-\delta$. In the specific case when $t = t_0$ holds, convergence is also ensured (see also footnote 9) such that a careful application of the binomial theorem yields the following result:

$$(1-\delta+\delta \mathrm{e}^{-t})^{\nu-1} = \begin{cases} \delta^{\nu-1} \sum\limits_{k=0}^{\infty} \binom{\nu-1}{k} \left(\frac{1-\delta}{\delta}\right)^k e^{(k+1-\nu)t}, & t \in [0, t_0], \\ (1-\delta)^{\nu-1} \sum\limits_{k=0}^{\infty} \binom{\nu-1}{k} \left(\frac{\delta}{1-\delta}\right)^k e^{-kt}, & t \in [t_0, \infty). \end{cases} \tag{22}$$

Using this in (21), we can switch the order of summation and integration once again. After evaluating the integrals, we obtain the following result:

$$\mathbb{E}[K_\sigma^\mu] = \frac{\nu}{\Gamma(1-\mu)} \sum_{k=0}^{\infty} \binom{\nu-1}{k} \left[\delta^\nu \left(\frac{1-\delta}{\delta}\right)^k \frac{\Gamma(1-\mu)-\Gamma(1-\mu,(\nu-k)t_0)}{(\nu-k)^{1-\mu}} + (1-\delta)^\nu \left(\frac{\delta}{1-\delta}\right)^{k+1} \frac{\Gamma(1-\mu,(k+1)t_0)}{(k+1)^{1-\mu}} \right]. \tag{19c}$$

4. APPLICATION: COMPUTING EXPECTED UTILITY WHEN PATERNITY IS UNCERTAIN

In the previous section, we have derived simple formulas for the fractional μ^{th} moment of the binomial distribution with probability of success δ and numbers of trials ν. Restrictions on the parameter values were given by: $\mu \in (0,1)$, $\delta \in (0,1)$, and $\nu \in [1,\infty)$. The following equation summarizes our results:

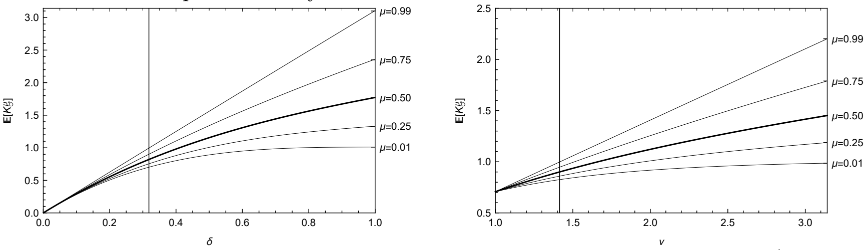
$$\mathbb{E}[K_\sigma^\mu] = \begin{cases} \nu(1-\delta)^\nu \sum\limits_{k=0}^{\infty} \binom{\nu-1}{k} \left(\frac{\delta}{1-\delta}\right)^{k+1} \frac{1}{(k+1)^{1-\mu}}, & \delta \in (0, \frac{1}{2}], \\ \nu(1-\delta)^\nu \sum\limits_{k=0}^{\infty} \binom{\nu-1}{k} \left[\left(\frac{\delta}{1-\delta}\right)^{\nu-k} \frac{\Gamma(1-\mu)-\Gamma(1-\mu,(\nu-k)t_0)}{\Gamma(1-\mu)(\nu-k)^{1-\mu}} + \left(\frac{\delta}{1-\delta}\right)^{k+1} \frac{\Gamma(1-\mu,(k+1)t_0)}{\Gamma(1-\mu)(k+1)^{1-\mu}} \right], & \delta \in (\frac{1}{2}, 1). \end{cases} \tag{19}$$

The formula in (19) could be useful in many areas of economic research and related fields. For instance, the binomial distribution plays a prominent role in the field of finance where risk neutrality of agents is still a widely used assumption. Hence, the formula can help model the behavior of risk-averse agents in financial markets. In the paper at hand, we stick to the Beckerian theory of fertility where paternity is uncertain. We illustrate the usefulness of the formula in (19) by demonstrating how the fractional moment (and hence expected utility) depends on the various parameters of the model.

According to equation (19), the calculation of the fractional moment can be performed by evaluating an infinite sum where each summand is essentially determined by a binomial coefficient and an expression containing

Gamma functions. From a practical point of view the numerical evaluation is not very challenging. Modern computer algebra software packages can efficiently calculate the Gamma function (and its incomplete version) with very high precision. The same is true for generalized binomial coefficients.

FIG. 1. Expected Utility for Various Coefficients of Relative Risk Aversion



Notes: The thick black lines in both diagrams show the half moment ($\mu = \frac{1}{2}$) of the binomial distribution with parameters ν and δ. The thin lines display variations of the constant degree of relative risk aversion μ. In the left diagram, different probabilities to father a child $\delta \in (0, 1)$ are considered while setting the number of female mating partners equal to the circular constant $\nu = \pi$. In the right diagram, the number of female mating partners ν varies between one and π while keeping the probability to father a child constant at $\delta = \frac{1}{\sqrt{2}}$. Intersections with the vertical lines indicate mating market clearing when sex ratios are balanced (i.e. $\delta = \frac{1}{\nu}$).

As can be seen in Figure 1, the fractional moment $\mathbb{E}[K_{\sigma}^{\mu}]$ becomes a linear function of both the probability of fatherhood as well as the number of female mating partners when the degree of relative risk aversion approaches zero (i.e. μ is close to one). For very low values of μ, utility from reproductive success merely reflects the chance of having own children or not whereas the number of own children is less important (see the left panel of Figure 1). A similar observation can be made when the number of mating partners is increased while keeping the degree of paternal uncertainty constant: a low degree of relative risk aversion renders the utility function into an indicator function for reproductive success that is insensitive to the number of own children.

5. CONCLUSION

We have derived an easy to implement formula to calculate fractional moments of the binomial distribution. The formula is general enough to allow for non-integer numbers of draws, a fact that makes our result particularly interesting for representative agent frameworks where discrete real-world quantities are typically modeled by real numbers. Areas where our result might prove useful include the analysis of financial markets with risk-averse agents or macroeconomic models with indivisible labor, to name just a few examples from economics.

To illustrate our result, we have determined the expected utility of a representative male participant in the mating market given his chosen number of mating partners (ν) and given the probability of success (δ). From an individual perspective the number of mating partners is clearly an integer. However, the analysis of average behavior as it is done in a representative agent setting obviously requires a non-integer correspondence. Our result which was summarized in (19) satisfies this requirement. Furthermore, our finding is easily implemented and does not require vast computing power. On an average consumer notebook, it only takes a few seconds to calculate the roughly one thousand fractional moments needed to generate the plots in Figure 1.

APPENDIX A

A.1. CONSTANT ABSOLUTE RISK AVERSION

1. Expected Reproductive Success: Let us consider the negative exponential utility function: $\tilde{V}(K) = \frac{1-\exp(-\alpha K)}{\alpha}$ where parameter $\alpha > 0$ is the Arrow-Pratt measure of constant absolute risk aversion. Given the probability of success δ and the number of draws $N_♂$, expected utility from reproduction is calculated as follows:

$$\mathbb{E}\big[V(K_♂)\big] = \sum_{k=0}^{N_♂} \binom{N_♂}{k} (1-\delta)^{N_♂-k} \delta^k \left(\frac{1-\mathrm{e}^{-\alpha k}}{\alpha}\right) \tag{A.1}$$

$$\Leftrightarrow \qquad \mathbb{E}\big[V(K_♂)\big] = \frac{1}{\alpha} - \frac{1}{\alpha} \sum_{k=0}^{N_♂} \binom{N_♂}{k} (1-\delta)^{N_♂-k} \left(\delta \mathrm{e}^{-\alpha}\right)^k \tag{A.2}$$

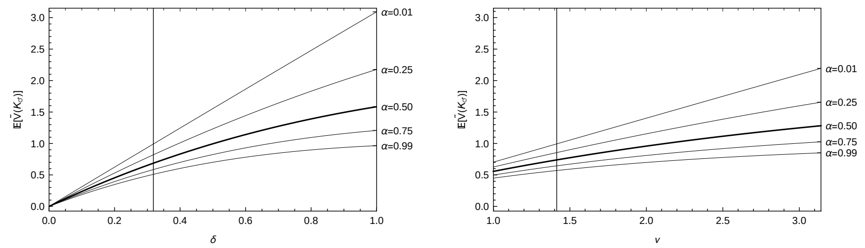
$$\Leftrightarrow \qquad \mathbb{E}\big[V(K_♂)\big] = \frac{1}{\alpha} - \frac{1}{\alpha} \left(1-\delta+\delta\mathrm{e}^{-\alpha}\right)^{N_♂} \quad \left[= \frac{1-M(-\alpha)}{\alpha}\right]. \tag{A.3}$$

2. Expected Utility for Various Coefficients of Absolute Risk Aversion: Strictly speaking equation (A.3) is restricted to discrete numbers of draws $N_♂$. Note, however, that it can easily be extended to a continuous choice. Using our notation above this means that we can replace $N_♂$ with ν:

$$\mathbb{E}\big[V(K_♂)\big] = \frac{1}{\alpha} - \frac{1}{\alpha} \left(1-\delta+\delta\mathrm{e}^{-\alpha}\right)^{\nu}. \tag{A.4}$$

In contrast to the power utility case, we obtain a closed-form solution when considering negative exponential preferences. Consequently, we can directly calculate expected utility and easily produce graphical illustrations (see Figure 1).

FIG. 1. Expected Utility for Various Coefficients of Relative Risk Aversion



NOTE: This figure repeats Figure 1 in the main text, replacing the power utility function with negative exponential preferences. Each line refers to a coefficient of absolute risk aversion α as indicated.

A.2. USEFUL PROPERTIES OF BINOMIAL COEFFICIENTS

1. General Properties: Let the generalized binomial coefficient be defined as in (15). For all $\alpha \in \mathbb{R}$:

$$\binom{\alpha+1}{k+1} = \binom{\alpha}{k} + \binom{\alpha}{k+1}, \quad k \in \mathbb{Z}; \qquad \binom{\alpha}{k} = \frac{\alpha}{k}\binom{\alpha-1}{k-1}, \quad k \in \mathbb{Z}\backslash\{0\}, \tag{A.5}$$

$$\sum_{k\in\mathbb{Z}} \binom{\alpha}{k} = 2^{\alpha}, \quad \alpha > -1; \qquad \sum_{k\in\mathbb{Z}} (-1)^k \binom{\alpha}{k} = 0, \quad \alpha > 0, \tag{A.6}$$

$$\sum_{k=0}^{\lfloor \frac{a}{2} \rfloor} \binom{a}{2k} = 2^{a-1}, \quad a \in \mathbb{N}; \qquad \sum_{k=0}^{\lfloor \frac{a-1}{2} \rfloor} \binom{a}{2k+1} = 2^{a-1}, \quad a \in \mathbb{N}. \tag{A.7}$$

See Gould (1972) for these and many other results. Proofs for the following Properties 2, 3 and 4 are available from the author upon request.

2. Greatest Coefficient: For any upper index $\alpha \in [0, \infty)$ and lower index $\kappa \in [0, \alpha]$ the generalized binomial coefficient $\binom{\alpha}{\kappa}$ is greatest when $\kappa = \frac{\alpha}{2}$.

3. Alternating Signs: For any upper index $\alpha \in [0, \infty)$ and lower index $k \in \mathbb{N}_0$ with $k \geq \lceil \alpha \rceil$ the generalized binomial coefficients are alternately

positive and negative as follows:

$$\binom{\alpha}{\alpha+n} = 0, \qquad \forall n \in \mathbb{N} \quad \text{and} \quad \alpha \in \mathbb{N}_0, \tag{A.8a}$$

$$\binom{\alpha}{\lceil\alpha\rceil+2n} > 0, \qquad \forall n \in \mathbb{N}_0 \quad \text{and} \quad \alpha \notin \mathbb{N}_0, \tag{A.8b}$$

$$\binom{\alpha}{\lceil\alpha\rceil+2n+1} < 0, \qquad \forall n \in \mathbb{N}_0 \quad \text{and} \quad \alpha \notin \mathbb{N}_0. \tag{A.8c}$$

4. Convergence: For any upper index $\alpha \in [0, \infty)$ with $\alpha \notin \mathbb{N}_0$ and lower index $k \in \mathbb{N}_0$ with $k \geq \lceil\alpha\rceil$ the absolute value of the generalized binomial coefficient is declining, that is $\|\binom{\alpha}{k}\| > \|\binom{\alpha}{k+1}\|$.

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