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Optimal Fiscal and Monetary Policy in Economies With Capital*

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We reexamine the optimal fiscal and monetary policy in combined shopping-time monetary models with capital accumulation. Four models are constructed to investigate how the production cost of money and the utility derived from physical capital affect the toolkit of fiscal and monetary policy. It is shown that the optimality of the Friedman rule hinges on the production cost of money, while capital-in-utility overturns the Chamley-Judd zero capital income taxation theorem. When the production cost of money approaches zero, the Friedman rule is optimal; however, when consumers derive utility from capital, the limiting capital income tax is generally not zero.

Key Words: Inflation tax; Capital income tax; Friedman rule; Capital in utility.
JEL Classification Numbers: E40, E52, H20, H21.

1. INTRODUCTION

The problem of optimal fiscal and monetary policy has been analyzed in numerous studies. Most of these studies examine fiscal and monetary policies separately. Dynamic taxation theorists focus on how to tax factor incomes in dynamic models without money.[1] Some researchers in monetary theory conduct their analyses in monetary models without capital accumulation[2]. other researchers investigate optimal monetary policies in monetary models with capital but without consideration of dynamic fiscal policies[3]. Additionally, a few researchers examine optimal fiscal and mon-

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[1]See Judd (1985), Chamley (1986), Zhu (1992), Jones et al. (1993, 1997), Aiyagari (1995), Correia (1996), Golosov et al. (2003).

[2]See Calvo (1983), Lucas and Stokey (1983), Correia and Teles (1996, 1999), Chari et al. (1996).

[3]See Sidrauski (1967), Fischer (1979), Stockman (1981), Chamley (1985).

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etary policy within monetary economies without capital.[4] The purpose of this paper is to reexamine optimal fiscal and monetary policy within a combined monetary model that includes capital accumulation. We construct four models with different combinations of the production cost of capital and the utility generated by physical capital. Some of these models reproduce a zero norminal interest rate or zero limiting capital income tax, while others generate more complex trade-offs, from which we develop interesting new insights.

Dynamic tax theory follows the standard Ramsey-Cass-Koopman (RCK) framework. The most importang result in this research agenda is the famous Chamley-Judd zero capital income taxation theorem, developed by Chamley (1986) and Judd (1985). This theroem raises an important question in the theory of public finance: Is physical capital special as a stock? In a generalized model with human capital and effective labor, Jones et al. (1997) establish that under certain conditions[5] both capital and labor income taxes can be set to zero in the steady state. Moreover, if preferences satisfy an additional condition, all taxes can be chosen to be asymptotically zero. Thus, there is nothing special about physical capital as a stock variable, and the taxation rules on factor income depend on model specifications. A large literature in this research area yields very different conclusions based on various channels, as noted by Lucas (1990), Zhu (1992), Jones et al. (1993, 1997), Aiyagari (1995), Correia (1996), Golosov et al. (2003), among others. On the other hand, a substantial body of literature on optimal monetary policy is motivated by Friedman's (1969) seminal contribution, in which he proposed a monetary policy rule that could generate zero nominal interest rates on assets with a riskless nominal return. There are many supporters and opponents of the Friedman rule, and most base their arguments on the uniform commodity taxation theorem developed by Atkinson and Stiglitz (1972) or the optimal taxations for intermediate good proposed by Diamond and Mirrlees (1971). Economists have come to realize that both theorems cannot be applied directly and require additional preconditions, as suggested by Sidrauski (1967), Fischer (1979), Chamley (1985), Kimbrough (1986), Faig (1988), Guidotti and Vegh (1993), Chari et al. (1996), Correia and Teles (1996), and Woodford (1990), among others. In fact, a simple argument for the Friedman rule is that a good that is costless to produce should be priced at zero. Correia and Teles (1996) argue that this simple rule regarding the production cost of money plays a key role in determining the optimality of the Friedman rule. They show that if the production cost of real money approaches zero, the Friedman rule is

[4]See Lucas and Stokey (1983), Chari et al. (1991), Correia et al. (2008).

[5]Jone et al. (1997) provide these three conditions: (1) there are no profits from accumulating either capital stock, (2) the tax code is sufficiently rich, and (3) there is no role for relative prices to reduce the value of fixed sources of income.

optimal; if not, the Friedman rule is not optimal, and the optimal inflation tax depends on the degree of homogeneity of the transaction function.

In the paper, we utilize the shopping-time monetary model examined by Kimbrough (1986), Faig (1988), Guidotti and Vegh (1993), Chari et al. (1996), Correia and Teles (1996), and Woodford (1990), focusing on how two important channels-the production cost of money and the utility derived from physical capital-affect optimal fiscal and monetary policies across four models with different combinations. While many classical results are recovered in these generalized models, numerous new insights are also developed.

In the costless-money model without capital in utility (CIU) (Model 1), we recover the well-known results of zero limiting capital income taxation and a zero nominal interest rate, as developed by Chamley (1986) and Correia and Teles (1996), respectively. In Model 2, when money is costly to produce, the Friedman rule is not optimal, and the optimal inflation rate depends on the tax rate on the labor force employed in the money sector. Consequently, the tax structure for capital income is altered accordingly. When consumers derive utility from the capital stock, as in Model 3, the Chamley-Judd zero capital income taxation theorem does not hold; that is, the limiting capital income tax is generally not zero, even though the Friedman rule remains optimal. Incorporating capital in utility generates a non-pecuniary return in the asset pricing equaiton (i.e., consumption Euler equation or the no-arbitrage condition), which contradicts the negative effect of capital taxes on the pecuniary return, making the sign of the limiting capital tax ambiguous. In Model 4, with costly money and CIU, neither the Friedman rule nor the Chamley-Judd zero capital taxation theorem holds in general. The interaction between the production cost of money and CIU plays a crucial role in determining optimal fiscal and monetary policy.

The remainder of the paper is organized as follows. Section 2 examines a baseline costless-money model without capital-in-utility, in which both the inflation tax and the limiting capital income tax are zero. Section 3 investigates a costly-money model without capital-in-utility and demonstrates that the Friedman rule does not hold. In section 4, we introduce physical capital into the utility function of the models discussed in Sections 2 and 3, exploring how capital-in-utility alters the results regarding optimal monetary policy. Section 5 presents the concluding remarks.

2. THE MODEL WITH COSTLESS MONEY (MODEL 1)

2.1. Model setup

In this section, we consider a monetary economy with capital accumuation and costless money. An infinitely lived representative household derives utility from consumption and leisure streams $\{c_t, l_t\}_{t=0}^{\infty}$, which yield higher values of utility:

$$\sum_{t=0}^{\infty} \beta^t u(c_t, l_t), \tag{1}$$

where $\beta \in (0,1)$, $c_t \geq 0$, and $l_t \geq 0$ represent consumption and leisure at time t, respectively. Additionally, u_c and u_l are both greater than 0, u_{cc} and u_{ll} are less than 0, and u_{cl} is greater than or equal to 0.[6] The household is endowed with one unit of time per period, which can be allocated to leisure l_t, labor n_t, and shopping s_t. Thus, the time allocation equation is given by:

$$l_t + n_t + s_t = 1. \tag{2}$$

To acquire the consumption good, the household allocates time to shopping. The amount of shopping time, s_t, is positively related to the consumption level c_t and negatively related to the household's holdings of real money balances, denoted by $\frac{m_{t+1}}{p_t} \equiv \widehat{m}_{t+1}$. Specifically, the shopping/transaction technology is:

$$s_t = H\left(c_t, \frac{m_{t+1}}{p_t}\right), \tag{3}$$

where $H, H_c, H_{cc}, H_{m/p,m/p} \geq 0$ and $H_{m/p}, H_{c,m/p} \leq 0$.[7] The shopping technology is assumed to be homogeneous of degree $v \geq 0$ in consumption c_t and real money balances $\frac{m_{t+1}}{p_t}$:

$$s_t = H(c_t, \widehat{m}_{t+1}) = c_t^v H\left(1, \frac{\widehat{m}_{t+1}}{c_t}\right), \text{ for } c_t > 0.$$

By Euler's theorem, we have:

$$H_c(c_t, \widehat{m}_{t+1})\, c_t + H_{\widehat{m}}(c_t, \widehat{m}_{t+1})\, \widehat{m}_{t+1} = vH(c_t, \widehat{m}_{t+1}). \tag{4}$$

For any consumption level c_t, we assume that there exists a point of satiation in real money balances, ψc, such that:

$$H(c_t, \widehat{m}_{t+1}) = H_{\widehat{m}}(c_t, \widehat{m}_{t+1}) = 0, \text{ for } \widehat{m}_{t+1} \geq \psi c.$$

[6] $u_{ii} < 0$ indicates that the marginal utility of any commodity decreases with its own consumption, while $u_{ij} > 0$ implies that the marginal utility of one commodity increases with the consumption of another commodity.

[7] $H_{m/p} < 0$ and $H_{m/p,m/p} \geq 0$ indicate that an increase in the real quantity of money decreases the time spent on transactions, albeit at a decreasing rate. The restriction on the second derivative of the transaction function ensures that the isoquants of the transaction production function are convex, and that the demand for money depends negatively on the nominal interest rate.

It is not worthwhile to increase real balance holdings beyond this point, as doing so does not save any additional resources.

The single good is produced using labor n_t and capital k_t. Output can be consumed by households, used by the government, or used to increase the capital stock. The resource constraint equation is:

$$c_t + k_{t+1} - (1-\delta)\,k_t + g_t = F\left(k_t, n_t\right), \tag{5}$$

where $\delta \in (0,1)$ represents the capital depreciation rate, and $\{g_t\}_{t=0}^{\infty}$ is an exogenous sequence of government purchases. We assume a standard increasing and concave production function that exhibits constant returns to scale. By Euler's theorem on homogeneous functions, the assumption of linear homogeneity of F implies that: $F\left(k_t, n_t\right) = F_k\left(k_t, n_t\right) k_t + F_n\left(k_t, n_t\right) n_t$, where F_k and F_n represent the marginal products of capital and labor, respectively.

Government. The government finances its stream of purchases $\{g_t\}_{t=0}^{\infty}$ by levying proportional factor taxes on capital and labor income, issuing new debts, and printing new currency. In this case with costless money, the production of money requires no real resources. The government's budget constraint is:

$$g_t = \tau_t^k r_t k_t + \tau_t^n w_t n_t + \frac{B_{t+1}}{R_t} - B_t + \frac{M_{t+1} - M_t}{p_t}, \tag{6}$$

where r_t and w_t are the market-determined rental rate of capital and wage rate for labor, τ_t^k and τ_t^n are flat-rate, time-varying taxes on capital and labor earnings, and R_t is the gross rate of return on one-period bonds held from t to $t+1$.[8] B_t is government indebtedness to the private sector, denominated in goods at time t, and M_t is the stock of currency issued by the government as of the beginning of period t. Interest earnings on bonds are assumed to be tax-exempt, which is neutral for bond exchanges between the government and the private sector. We assume that the government can fully and credibly commit to future tax rates, thereby evading the issue of time consistency, as raised in Kydland and Prescott (1977).

Households. A representative household chooses $\{c_t, l_t, k_{t+1}, b_{t+1}, m_{t+1}\}_{t=0}^{\infty}$ to maximize expression (1) subject to the transaction technology (3), the time allocation constraint (2) and the sequence of budget constraints:

$$c_t + k_{t+1} + \frac{b_{t+1}}{R_t} + \frac{m_{t+1}}{p_t} = \left(1 - \tau_t^k\right) r_t k_t + \left(1 - \tau_t^n\right) w_t n_t + (1-\delta)\,k_t + b_t + \frac{m_t}{p_t}, \tag{7}$$

[8]One-period government bonds cannot be accumulated like private capital. Hence, we do not introduce government bonds into the utility function of the representative consumer in Models 3 and 4 with capital-in-utility.

for $t \geq 0$, given k_0,b_0,m_0. Here, $m_{t+1} \geq 0$,[9] where $m_t + 1$ represents nominal money balances held between times t and $t+1$; p_t is the price level; and b_t is the real value of one-period government bond holdings, denominated in units of consumption at time t. By substituting the shopping technology (3) and the time allocation equation (2) into (7), introducing the Lagrange multiplier λ_t to represent the marginal utility of wealth at time t, and constructing the Lagrangian, we solve for the following the first-order conditions:

$$c_t : u_c\left(c_t, l_t\right) = \lambda_t\left[\left(1-\tau_t^n\right) w_t H\left(c_t, \widehat{m}_{t+1}\right)+1\right], \tag{8}$$

$$k_{t+1} : \lambda_t = \beta\lambda_{t+1}\left[\left(1-\tau_{t+1}^k\right) r_{t+1}+1-\delta\right], \tag{9}$$

$$l_t : u_l\left(c_t, l_t\right) = \lambda_t\left(1-\tau_t^n\right) w_t, \tag{10}$$

$$b_{t+1} : \frac{\lambda_t}{R_t} = \beta\lambda_{t+1}, \tag{11}$$

$$m_{t+1} : \left[\left(1-\tau_t^n\right) w_t H_{m/p}\left(c_t, \widehat{m}_{t+1}\right)+1\right] \frac{\lambda_t}{p_t} = \beta\frac{\lambda_{t+1}}{p_{t+1}}. \tag{12}$$

From the equations (8) and (10), we have the following relationship:

$$\frac{u_l\left(c_t, l_t\right)}{u_c\left(c_t, l_t\right)-u_l\left(c_t, l_t\right) H_c\left(c_t, \widehat{m}_{t+1}\right)} = \left(1-\tau_t^n\right) w_t, \tag{13}$$

which shows that the marginal rate of substitution (MRS) between consumption and leisure equals the after-tax price ratio of leisure to consumption. By substituting equation (8) and the after-tax wage $\left(1-\tau_t^n\right) w_t$ into (9), we derive the consumption Euler equation

$$\begin{aligned} & \left(\begin{array}{c} u_c\left(c_t, l_t\right)- \\ u_l\left(c_t, l_t\right) H_c\left(c_t, \widehat{m}_{t+1}\right) \end{array}\right) \\ = \; & \beta\left(\begin{array}{c} u_c\left(c_{t+1}, l_{t+1}\right)- \\ u_l\left(c_{t+1}, l_{t+1}\right) H_c\left(c_{t+1}, \widehat{m}_{t+2}\right) \end{array}\right)\left[\left(1-\tau_{t+1}^k\right) r_{t+1}+1-\delta\right]. \end{aligned} \tag{14}$$

Equations (9) and (11) imply the no-arbitrage condition for trades in capital and bonds, guaranteeing that these two assets offer the same rate of return:

$$R_t = \left(1-\tau_{t+1}^k\right) r_{t+1}+1-\delta. \tag{15}$$

[9]Maximization of expression (1) is subject to $m_{t+1} \geq 0$ for all $t \geq 1$, since households cannot issue money; however, no restrictions apply to the sign of b_{t+1}, meaning that b_{t+1} can be negative for $t \geq 1$.

By substituting this equation (15) into the consumption Euler equation (14), we obtain the following expression for the real interest rate:

$$R_t = \frac{[u_c(c_t, l_t) - u_l(c_t, l_t) H_c(t)]}{\beta [u_c(c_{t+1}, l_{t+1}) - u_l(c_{t+1}, l_{t+1}) H_c(t+1)]}. \tag{16}$$

The combination of equations (11) and (12) gives us:

$$\frac{R_t - R_{mt}}{R_t} = -(1 - \tau_t^n) w_t H_{m/p}(t) \left(= \frac{i_t}{1 + i_t} \equiv I_t \right), \tag{17}$$

which equates the cost to the benefit of holding the marginal unit of real money balances from t to time $t+1$, expressed in time t consumption goods. Here, $R_m t \equiv \frac{p_t}{p_t + 1}$ is the real gross return on money held between time t and time $t+1$, i.e., the inverse of the inflation rate, and $1 + i_t \equiv \frac{R_t}{R_m t}$ is the gross nominal interest rate. The real return on money $R_m t$ must not exceed the return on bonds R_t, otherwise agents could exploit this discrepancy by issuing bonds and holding arbitrarily large quantities of money. Therefore, the net nominal interest rate i_t must be non-negative, i.e., $i_t \geq 0$.

Firms. In each period, the representative firm takes the rental rate of capital r_t and the wage rate w_t as given, rents capital and labor from households, and maximizes profits: $F(k_t, n_t) - r_t k_t - w_t n_t$, where $F(k_t, n_t)$ is the production function. The firm's goal is to maximize profits by choosing optimal levels of capital k_t and labor n_t. The first-order conditions for this optimization problem are:

$$F_k(k_t, n_t) = r_t, F_n(k_t, n_t) = w_t, \tag{18}$$

which state that the firm should employ inputs until the marginal product of the last unit is equal to its rental price. Under the assumption of constant returns to scale, we obtain the standard result that pure profits are zero.

2.2. Primal approach to the Ramsey problem

We examine the second-best fiscal and monetary policy by utilizing the Primal approach developed by Atkinson and Stiglitz (1980) and Lucas and Stokey (1983). For this purpose, we present the following useful definitions.

Definition 2.1. A competive equilibrium is an allocation $\{c_t, l_t, n_t, s_t, k_{t+1}, b_{t+1}, m_{t+1}\}_{t=0}^{\infty}$, a price system $\{p_t, w_t, r_t, R_t\}_{t=0}^{\infty}$, and a government policy $\left\{g_t, \tau_t^k, \tau_t^n, B_{t+1}, M_{t+1}\right\}_{t=0}^{\infty}$ such that: (a) Given the price system and the government poicy, the allocation solves both the firm's problem and the household's problem with $b_t = B_t$ (private bonds equal government bonds) and $m_t = M_t$ (private money balances equal government money balances) for all t. (b) Given the allocation and the price system, the government policy satisfies the sequence of government budget constraint (6) for all t. (3) The time allocation constraint (2) and the resource constraint (5) are safisfied for all t.

There are many competitive equilibria, indexed by different government policies. This multiplicity of equillibria motivates the Ramsey problem, which seeks to determine the optimal set of government policies.

DEFINITION 2.2. Given k_0, b_0, and m_0, the Ramsey problem is to choose a competitive equilibrium that maximizes expression (1).

To construct the Ramsey problem, we firstly iteratively substitute the flow budget constraint (7) to derive the household's present-value budget constraint:

$$\begin{aligned}\sum_{t=0}^{\infty} q_t^0 \left(c_t + \frac{i_t}{1+i_t}\widehat{m}_{t+1}\right) &= \sum_{t=0}^{\infty} q_t^0 \left(1-\tau_t^n\right) w_t n_t \\ &+ \left[\left(1-\tau_0^k\right) r_0 + 1 - \delta\right] k_0 + b_0 + \frac{m_0}{p_0}, \quad (19)\end{aligned}$$

where $q_t^0 = \prod_{i=0}^{t-1} R_i^{-1}$ is the Arrow-Debreu price, with the numeriare $q_0^0 = 1$. We have also imposed the transversality conditions to prevent arbitrage or infinite wealth accumulation: $\lim_{T\to\infty} q_T^0 \frac{b_{T+1}}{R_T} = 0$ and $\lim_{T\to\infty} q_T^0 \widehat{m}_{T+1} = 0$. Substituting equation (16) into the definition of the Arrow-Debreu price leads to:

$$q_t^0 = \beta^t \frac{u_c\left(c_t, l_t\right) - u_l\left(c_t, l_t\right) H_c\left(t\right)}{u_c\left(c_0, l_0\right) - u_l\left(c_0, l_0\right) H_c\left(0\right)}. \quad (20)$$

Substituting (13), (17), (20), and (4) into the present-value budget constraint (19) and rearranging it, we obtain the implementability condition:

$$\sum_{t=0}^{\infty} \beta^t \left[u_c\left(c_t, l_t\right) c_t - u_l\left(c_t, l_t\right)\left(1 - l_t - (1-v) H\left(c_t, \widehat{m}_{t+1}\right)\right)\right] = A_1, \quad (21)$$

where A_1 is given by

$$\begin{aligned} A_1 &= A\left(c_0, l_0, k_0, b_0, m_0, \tau_0^k\right) \\ &= \left[u_c\left(c_0, l_0\right) - u_l\left(c_0, l_0\right) H_c\left(0\right)\right]\left[\left(\left(1-\tau_0^k\right) r_0 + 1 - \delta\right) k_0 + b_0 + \frac{m_0}{p_0}\right].\end{aligned}$$

The Ramsey problem is to maximize expression (1) subject to the implementability condition (21) and the feasibility constraint (5). Let ϕ be the Lagrange multiplier associated with equation (21), and define:

$$\begin{aligned}&U\left(c_t, l_t, \widehat{m}_{t+1}, \phi\right)\\=&u\left(c_t, l_t\right)+\phi\left[u_c\left(c_t, l_t\right) c_t-u_l\left(c_t, l_t\right)\left(1-l_t-(1-v) H\left(c_t, \widehat{m}_{t+1}\right)\right)\right].\end{aligned}$$

Next, we construct the Lagrangian:

$$\begin{aligned}\mathcal{L}=&\sum_{t=0}^{\infty} \beta^t\left\{U\left(c_t, l_t, \widehat{m}_{t+1}, \phi\right)\right.\\&\left.+\theta_t\left[F\left(k_t, 1-l_t-H\left(c_t, \widehat{m}_{t+1}\right)\right)+(1-\delta) k_t-c_t-g_t-k_{t+1}\right]\right\}-\phi A_1,\end{aligned}$$

where $\{\theta_t\}_{t=0}^{\infty}$ is a sequence of Lagrange multipliers associated with the resource constraint. After deriving the first-order conditions with respect to c_t, l_t, k_{t+1}, and $\widehat{m}_{t+1}$, for all $t \geq 0$, we combine them and obtain the following optimality conditions:

$$\frac{U_l\left(c_t, l_t, \widehat{m}_{t+1}, \phi\right)}{U_c\left(c_t, l_t, \widehat{m}_{t+1}, \phi\right)}=\frac{F_n\left(k_t, n_t\right)}{F_n\left(k_t, n_t\right) H_c\left(c_t, \widehat{m}_{t+1}\right)+1}, \qquad (22)$$

$$\begin{aligned}&\frac{U_c\left(c_t, l_t, \widehat{m}_{t+1}, \phi\right)}{\left(F_n\left(k_t, n_t\right) H_c\left(c_t, \widehat{m}_{t+1}\right)+1\right)}\\=&\frac{\beta U_c\left(c_{t+1}, l_{t+1}, \widehat{m}_{t+2}, \phi\right)}{\left(F_n\left(k_{t+1}, n_{t+1}\right) H_c\left(c_{t+1}, \widehat{m}_{t+2}\right)+1\right)}\left[F_k\left(k_{t+1}, n_{t+1}\right)+1-\delta\right], \qquad (23)\end{aligned}$$

$$\left[(v \phi+1) u_l\left(c_t, l_t\right)+\phi\left(u_{c l}\left(c_t, l_t\right) c_t-u_{l l}\left(c_t, l_t\right) n_t\right)\right] H_{\widehat{m}}\left(c_t, \widehat{m}_{t+1}\right)=0, t \geq 0. \qquad (24)$$

2.3. Optimal policy and its intuition

PROPOSITION 1. *In a monetary model with capital accumulation and costless money, the optimal monetary policy follows the Friedman rule. In other words, the optimal inflation tax is zero, $I_t = 0$, which implies that the nominal interest rate is also zero, $i_t = 0$. Moreover, in the long run, the optimal capital income tax is zero, $\tau^k = 0$.*

Proof. The first-order condition for real balances (24) is satisfied when either $H_{\widehat{m}}(t) = 0$ or

$$(v \phi+1) u_l\left(c_t, l_t\right)+\phi\left(u_{c l}\left(c_t, l_t\right) c_t-u_{l l}\left(c_t, l_t\right) n_t\right)=0. \qquad (25)$$

The Lagrange multiplier ϕ of the implementability condition, which measures the utility costs of distorting taxes, is nonnegative. Since the left-hand side of equation (25) is strictly positive, this equation cannot hold, and the solution must be $H_{\widehat{m}}(t) = 0$. From equation (17), we conclude that the optimal inflation tax is zero, $I_t = 0$, which implies that the net nominal interest rate is zero, i.e., $i_t = 0$. Thus, the social planner adheres to the Friedman rule, ensuring the economy is satiated with real money balances. To examine the limiting capital income tax, consider the special case where $g_t = g$ for all $t \geq T$ and assume that the Ramsey problem converges to a stationary solution. If the allocation beome tme-invariant, with constant values of c, l, n, $\widehat{m}$, and k, then since $U_c(t)$ and $[F_n(t) H_c(t) + 1]$ converge to constants, the stationary version of equation (23) implies

$$1 = \beta \left[F_k(k, n) + 1 - \delta \right].$$

Because c, l, and $\widehat{m}$ are constant in the limit, equations (14) and (15) imply that $R_t \left(= \frac{q_t^0}{q_{t+1}^0} \right) \to \beta^{-1}$ and

$$1 = \beta \left[\left(1 - \tau^k\right) F_k(k, n) + 1 - \delta \right].$$

Combining these two equalities implies that $\tau^k = 0$, meaning that the long-run optimal capital income tax is zero. ▮

As shown above, the baseline model can be viewed as an extension of Correia and Teles (1996), incorporating capital accumulation, or as an extension of Chamley (1986), introducing money through a transaction technology. Proposition 1 shows that in a combined monetary model with capital accumulation, we simultaneously recover the optimality of the Friedman rule with a zero norminal interest rate and the Chamley-Judd zero capital taxation theorem.

Mathematically, the optimality of the Friedman rule in this section generalizes results from other shopping-time monetary models, such as Kimbrough (1986), Faig (1988), Guidotti and Vegh (1993), Chari et al. (1996), Correia and Teles (1996), and Woodford (1990). However, the intuition behind the zero norminal interest rate is closely related to Correia and Teles (1996), who offer a simple argument: a good that is costless to produce should be priced at zero. Since the marginal cost of holding real money balances is zero, the marginal revenues from holding money should also be zero. This implies that the net norminal interest rate is zero ($i_t = 0$) and, therefore, the optimal inflation tax is zero ($I_t = 0$). This conclusion will be verified in the following costly-money models. In a related line of research, Sidrauski (1967) and Chamley (1985) develop money-in-utility (MIU) models that also establish the optimality of the Friedman rule.

Proposition 1 shows that the limiting capital income tax rate remains zero in shopping-time monetary economies. In other words, introducing money through transaction technologies has no effect on the consumer's savings behavior and, therefore, does not alter the zero capital income tax result from the standard RCK model. However, shopping-time models affect the optimal allocation of the consumer's time endowment, distorting the determination of the limiting labor income tax rate. As shown in Appendix A, the term $u_l H_c$ in the formula of the limiting labor income tax may be positive, zero or negative.

3. THE MODEL WITH COSTLY MONEY (MODEL 2)

3.1. Setup

In this section, we derive the optimal monetary policy and the limiting capital tax results for the case where money requires resources for its production. We assume that the government (the central bank)[10] employs labor (n_{2t}) and capital (k_{2t}) to produce real money balances with a constant-return-to-scale (CRS) production technology. The CRS property implies that the government earns no profits from producing real money balances. For the government, producing money provides an additional financing method for its expenditures[11]. For individuals, holding money saves (time) resources, which can be allocated to either leisure or additional labor supply. For analytical convenience, we assume the production function for real balances is Cobb-Douglas, namely:

$$\frac{m_{t+1}}{p_t} = k_{2t}^{\alpha_2} n_{2t}^{1-\alpha_2}, \alpha_2 \in (0,1). \tag{26}$$

For ease of exposition, we assume that the production technology for the consumption good is also Cobb-Douglas, but with different factor income shares than those in the production of money. Specifically, the production function is given by $F(k_{1t}, n_{1t}) = k_{1t}^{\alpha_1} n_{1t}^{1-\alpha_1}$, where $\alpha_1 \in (0,1)$ and $\alpha_1 \neq \alpha_2$. We allow for different tax rates on capital and labor used in the production of both the consumption good and money. Labor used in the production of the consumption good (n_{1t}) and money (n_{2t}) is taxed at rates τ_{1t}^n and τ_{2t}^n, respectively. Similarly, capital used in both sectors (k_{1t} and k_{2t}) is taxed at rate τ_{1t}^k and τ_{2t}^k, respectively. The transaction technology is also given by (3). The flow budget constraint and time allocation equation for

[10]In most countries, the central bank is the sole producer of fait money.

[11]The government taxes the factors used money production. The net benefits from producing money equal the total revenues $\left(\frac{m_{t+1}}{p_t} + \tau_{2t}^k r_{2t} k_{2t} + \tau_{2t}^n w_{2t} n_{2t}\right)$ minus the production costs $(r_{2t}k_{2t} + w_{2t}n_{2t})$. Due to the CRS property of the production function, the net value is $\left(\tau_{2t}^k r_{2t} k_{2t} + \tau_{2t}^n w_{2t} n_{2t}\right)$. Hence, producing money provides an additional financing method for government expenditures.

households are defined, for $t \geq 0$, by

$$c_t + k_{1t+1} + k_{2t+1} + \frac{b_{t+1}}{R_t} + \frac{m_{t+1}}{p_t} = \sum_{i=1,2} \left[\left(1 - \tau_{it}^k\right) r_{it} + (1 - \delta_i) \right] k_{it} + \sum_{i=1,2} \left(1 - \tau_{it}^n\right) w_{it} n_{it} + b_t + \frac{m_t}{p_t},$$

and

$$l_t + s_t + n_{1t} + n_{2t} = 1, \tag{27}$$

respectively. The resource constraint[12] is

$$c_t + k_{1t+1} + k_{2t+1} - (1 - \delta_1) k_{1t} - (1 - \delta_2) k_{2t} + g_t = F(k_{1t}, n_{1t}) = k_{1t}^{\alpha_1} n_{1t}^{1-\alpha_1}. \tag{28}$$

No arbitrage implies that the after-tax net rental rates of capital and the after-tax wage rates must be equalized across sectors:

$$\left(1 - \tau_{1t}^k\right) r_{1t} + (1 - \delta_1) = \left(1 - \tau_{2t}^k\right) r_{2t} + (1 - \delta_2), \left(1 - \tau_{1t}^n\right) w_{1t} = \left(1 - \tau_{2t}^n\right) w_{2t}. \tag{29}$$

Let $k_t = k_{1t} + k_{2t}$ and $n_t = n_{1t} + n_{2t}$. Thus, the household's flow budget constraint (FBC) can be rewritten as

$$c_t + k_{t+1} + \frac{b_{t+1}}{R_t} + \frac{m_{t+1}}{p_t} = \left(1 - \tau_{1t}^k\right) r_{1t} k_t + \left(1 - \tau_{1t}^n\right) w_{1t} n_t + (1 - \delta_1) k_t + b_t + \frac{m_t}{p_t}, \tag{30}$$

which is the same condition as in (7), with $\left(\tau_t^k, \tau_t^n, r_t, w_t, \delta\right)$ replaced by $\left(\tau_{1t}^k, \tau_{1t}^n, r_{1t}, w_{1t}, \delta_1\right)$. The constraints for the private problem are the budget constraint (30) and the transaction technology (3) for all $t \geq 0$. The first-order conditions of the private problem are identical to those in Model 1, Section 2, but with the aforementioned parameter replacements. Therefore, we have:

$$\frac{u_l(c_t, l_t)}{u_c(c_t, l_t) - u_l(c_t, l_t) H_c(t)} = \left(1 - \tau_{1t}^n\right) w_{1t}, \tag{31}$$

$$\begin{aligned} &[u_c(t) - u_l(t) H_c(t)] \\ =&\beta \left[u_c(t+1) - u_l(t+1) H_c(t+1)\right] \left[\left(1 - \tau_{1t+1}^k\right) r_{1t+1} + 1 - \delta_1\right]. \end{aligned} \tag{32}$$

[12]Note that combining the household's budget constraint (30) and the government's budget constraint (35), we can recover the resource constraint of the economy (28).

$$R_t = \left(1-\tau^k_{1t+1}\right) r_{1t+1} + 1 - \delta_1$$
$$= \frac{\left[u_c\left(c_t,l_t\right) - u_l\left(c_t,l_t\right) H_c\left(c_t,\widehat{m}_{t+1}\right)\right]}{\beta\left[u_c\left(c_{t+1},l_{t+1}\right) - u_l\left(c_{t+1},l_{t+1}\right) H_c\left(c_{t+1},\widehat{m}_{t+2}\right)\right]}. \quad (33)$$

$$\frac{R_t - R_{mt}}{R_t} = -\left(1-\tau^n_{1t}\right) w_{1t} H_{m/p}\left(c_t,\widehat{m}_{t+1}\right) = I_t. \quad (34)$$

Since the production cost of money is borne by the government, the government's budget constraint (GBC) is modified as follows:

$$g_t + r_{2t}k_{2t} + w_{2t}n_{2t} + B_t = \sum_{i=1,2}\left(\tau^k_{it}r_{it}k_{it} + \tau^n_{it}w_{it}n_{it}\right) + \frac{B_{t+1}}{R_t} - B_t + \frac{M_{t+1}-M_t}{p_t}. \quad (35)$$

The optimal production of both consumption goods and real balances gives rise to the marginal productivity conditions:

$$r_{it} = \alpha_i k_{it}^{\alpha_i - 1} n_{it}^{1-\alpha_i}, w_{it} = \left(1-\alpha_i\right) k_{it}^{\alpha_i} n_{it}^{-\alpha_i}, i = 1,2. \quad (36)$$

3.2. The Ramsey problem

The Ramsey problem is to choose $\{c_t, l_t, k_{t+1}, k_{2t}, m_{t+1}\}_{t=0}^{\infty}$ to maximize welfare, (1), subject to the implementability condition (21) with $\left(\tau^k_0, r_0, \delta\right)$ replaced by $\left(\tau^k_{10}, r_{10}, \delta_1\right)$, and the resource constraints[13], for $t \geq 0$,

$$c_t + k_{t+1} - \left(1-\delta_1\right) k_t - \left(\delta_1 - \delta_2\right) k_{2t} + g_t$$
$$= \left(k_t - k_{2t}\right)^{\alpha_1}\left(1 - l_t - H\left(c_t,\widehat{m}_{t+1}\right) - \widehat{m}_{t+1}^{\frac{1}{1-\alpha_2}} k_{2t}^{-\frac{\alpha_2}{1-\alpha_2}}\right)^{1-\alpha_2}. \quad (37)$$

An interior solution of the Ramsey problem requires the following optimality conditions,

$$c_t : U_c\left(c_t, l_t, \widehat{m}_{t+1}, \phi\right) = \theta_t\left[\left(1-\alpha_1\right)\left(\frac{k_t - k_{2t}}{n_{1t}}\right)^{\alpha_1} H_c(t) + 1\right], t \geq 1 \quad (38)$$

$$l_t : U_l\left(c_t, l_t, \widehat{m}_{t+1}, \phi\right) = \theta_t\left(1-\alpha_1\right)\left(\frac{k_t - k_{2t}}{n_{1t}}\right)^{\alpha_1}, t \geq 1 \quad (39)$$

$$k_{t+1} : \theta_t = \beta\theta_{t+1}\left(\alpha_1\left(\frac{k_{t+1} - k_{2t+1}}{n_{1t+1}}\right)^{\alpha_1 - 1} + 1 - \delta_1\right), t \geq 0, \quad (40)$$

[13]Notice that by substituting (27) and (26) into (28), we recover the resource constraint (37).

$$k_{2t} : \alpha_1 \left(\frac{n_{1t}}{k_t - k_{2t}} \right)^{1-\alpha_1}$$
$$= \frac{(1-\alpha_1)\,\alpha_2}{(1-\alpha_2)} \left(\frac{k_t - k_{2t}}{n_{1t}} \right)^{\alpha_1} \left(\frac{\widehat{m}_{t+1}}{k_{2t}} \right)^{\frac{1}{1-\alpha_2}} + (\delta_1 - \delta_2)\,, t \geq 0, \qquad (41)$$

$$\widehat{m}_{t+1} : \phi\,(1-v)\,u_l\,(c_t, l_t)\,H_{\widehat{m}}\,(t)$$
$$= \theta_t\,(1-\alpha_1) \left(\frac{k_t - k_{2t}}{n_{1t}} \right)^{\alpha_1} \left(H_{\widehat{m}}\,(t) + \frac{1}{1-\alpha_2} \left(\frac{\widehat{m}_{t+1}}{k_{2t}} \right)^{\frac{\alpha_2}{1-\alpha_2}} \right), t \geq 0. \qquad (42)$$

Here, ϕ and $\theta_t, t \geq 0$, are the multipliers associated with the implementability condition (21) and the resource constraints (37), respectively. Condition (41) is used to determine k_{2t}. Condition (42) differs from condition (24) (for the problem without costs of producing money) by the inclusion of the extra term $\frac{(\widehat{m}_{t+1}/k_{2t})^{\alpha_2/(1-\alpha_2)}}{(1-\alpha_2)}$.

3.3. Optimal policy and its intuition

PROPOSITION 2. *In a shopping-time monetary model with costly money, the optimal monetary policy adheres the following rules:*

$$I_t \begin{pmatrix} > \\ = \\ < \end{pmatrix} (1 - \tau_{2t}^n)\,,\ \textit{if}\ v \begin{pmatrix} < \\ = \\ > \end{pmatrix} 1. \qquad (43)$$

In the steady state, the optimal tax rate on physical capital employed in the consumption sector is zero, i.e.,

$$\tau_1^k = 0;$$

and the optimal tax rule on physical capital employed in the money sector follows:

$$\tau_2^k \begin{pmatrix} > \\ = \\ < \end{pmatrix} 0,\ \textit{if}\ (r_2 - \delta_2) \begin{pmatrix} > \\ = \\ < \end{pmatrix} (r_1 - \delta_1)\,.$$

Proof. The proof is provided in Appendix B. ▌

Proposition 2 states that if producing money uses resources of the market economy, then the Friedman rule does not generally hold. This means that the nomianl interest rate is not zero, which implies that the optimal inflation rate is also not zero. The optimal inflation tax I_t (or the net nominal interest rate $i_t = \frac{I_t}{(1-I_t)}$) depends not only on the optimal tax rate on the labor force employed in the money sector, τ_{2t}^n, but also on the degree of homogeneity of the transection technology, v. If $v < 1$, then the optimal inflation rate is larger than the after-tax return. This case is similar to, and also generalizes, the Correia and Teles (1996) model with capital.

It is shown that the limiting tax rate on capital employed in the consumption sector is also zero, i.e., $\tau_1^k = 0$, while the limiting tax on capital employed in the money sector varies. If the net (after depreciation) return rate of capital in the money sector is greater than that in the concumption sector, then the government should tax the capital employed in the money sector to eliminate arbitrage opportunities. Conversely, if the net return rate of capital in the money sector is less than that in the consumption sector, then the government should subsidize the capital employed in the money sector because the optimal capital tax rate in the consumption sector is always zero. However, if the physical capital has the same net rate of return in both sectors, then the limiting tax rate on the capital employed in the money sector is also zero.

4. MODELS WITH CAPITAL-IN-UTILITY (CIU) (MODELS 3 AND 4)

In this section, we introduce physical capital (k_t) into the household's utility function and investigate its implications for optimal fiscal and monetary policy. Kurz (1968) pioneered this kind of capital-in-utility (CIU) model within the standard Ramsey-Cass-Koopman (RCK) framework, examining its implications for growth performance. A large body of literature has since explored the theoretical and empirical implications of CIU for savings and growth (Kurz, 1968; Cole et al., 1992; Zou, 1994, 1995; He et al., 2023; Shi et al., 2024), business cycles (Boileau and Rebecca, 2007; Karnizova, 2010; Michallat and Saez, 2015), asset pricing (Bakshi and Chen, 1995; Smith, 2002; Boileau and Rebecca, 2007), wealth distribution (Luo and Young, 2009), occupational choice in labor markets (Doepke and Zilibotti, 2008), and rational bubbles (Zhou, 2016). In this section, we will examine how CIU affects optimal fiscal and monetary policy in models with costless and costely money, as discussed in Sections 2 and 3.

Keeping all other elements of Models 1 and 2 unchanged, we introduce physical capital k_t in the households' utility function in both models. Consequently, the objective function of the representative household is modified as follows:

$$\sum_{t=0}^{\infty} \beta^t u\left(c_t, l_t, k_t\right), \tag{44}$$

where $k_t \geq 0$ represents the physical capital stock at time t, and the utility function satisfies $u_k > 0$, $u_{kk} < 0$, and $u_{ik} \geq 0$ for $i \in \{c, l\}$. The dependence of the utility function on physical capital stock (capital-in-utility) with $u_k > 0$ and $u_{kk} < 0$ reflects Weber's idea that capital accumulation in a capitalist economy is driven not only by the maximization of long-run consumption but also by the utility derived from wealth itself.[14] Next, we will examine the capital-in-utility models with costless and costly money.

4.1. Costless-money model with capital-in-utility (Model 3)

In this subsection, we re-examine the costless-money model presented in Section 2, but with a different objective function (44). The household's problem is to maximize (44), subject to the budget constraint (7), time allocation equation (2), and the shopping technology constraint (3). The first-order necessary conditions with respect to c_t, l_t, b_{t+1}, and m_{t+1} remain the same as in (8), (10), (11), and (12), excepts that the arguments (c_t, l_t) of the utility function are replaced by (c_t, l_t, k_t). However, the first-order necessary condition with respect to k_{t+1} changes to:

$$k_{t+1} : \lambda_t = \beta \left\{ u_k\left(c_{t+1}, l_{t+1}, k_{t+1}\right) + \lambda_{t+1}\left[\left(1 - \tau_{t+1}^k\right) r_{t+1} + 1 - \delta\right] \right\}, \tag{45}$$

where the positive term $u_k\left(c_{t+1}, l_{t+1}, k_{t+1}\right) > 0$ introduces a new channel to savings via CIU.[15] By combining these first-order conditions and compressing the arguments of (c_t, l_t, k_t) and $(c_t, \widehat{m}_{t+1})$ as (t), we obtain:

$$\frac{u_l(t)}{u_c(t) - u_l(t) H_c(t)} = \left(1 - \tau_t^n\right) w_t, \tag{46}$$

$$\begin{aligned} &\left[u_c(t) - u_l(t) H_c(t)\right] \\ =&\beta \left\{ u_k(t+1) + \left[u_c(t+1) - u_l(t+1) H_c(t+1)\right]\left[\left(1 - \tau_{t+1}^k\right) r_{t+1} + 1 - \delta\right] \right\}, \end{aligned} \tag{47}$$

[14]Zou (1994) refers this capital-in-utility concept as the "the spirit of capitalism" approach, which has inspired much discussion in the literature. For further economic interpretations of the "spirit of capitalism" approach, see Zou (1994).

[15]This new savings motive can be more clearly understood from the steady-state version of equation (47) without taxes, $F_k = \frac{1}{\beta} - 1 + \delta - \frac{u_k}{(u_c - u_l H_c)}$. The marginal product of capital F_k is lower than in the standard model without Capital-in-Utility, due to the presence of the new positive term $\frac{u_k}{(u_c - u_l H_c)}$ (> 0).

$$R_t = \frac{\left(1-\tau_{t+1}^k\right) r_{t+1} + 1 - \delta}{\left(1 - \frac{\beta u_k(t+1)}{u_c(t) - u_l(t) H_c(t)}\right)} = \frac{\left[u_c(t) - u_l(t) H_c(t)\right]}{\beta \left[u_c(t+1) - u_l(t+1) H_c(t+1)\right]}, \tag{48}$$

$$\frac{R_t - R_{mt}}{R_t} = -\left(1-\tau_t^n\right) w_t H_{m/p}\left(c_t, \widehat{m}_{t+1}\right) = I_t, \tag{49}$$

Equation (46) expresses that the marginal rate of substitution between consumption (adjusted for its utility loss due to reduced leisure) and leisure equals their (after-tax) price ratios. In the consumption Euler equation (47), the presence of capital-in-utility (with $u_k > 0$) introduces a non-pecuniary return for physical capital, represented by $\frac{u_k(t+1)}{u_c(t+1) - u_l(t+1) H_c(t+1)}$. This is in addition to the pecuniary after-tax return of $\left[\left(1-\tau_{t+1}^k\right) r_{t+1} + 1 - \delta\right]$. Capital-in-Utility (CIT) creates a positive savings incentive, counteracting the dissavings effect caused by capital taxation. This interplay renders the signs of the limiting capital taxes ambiguous, which will be examined in the next subsection. The modified no-arbitrage condition for trades between capital and bonds (48) also includes a new positive term, $\frac{\beta u_k(t+1)}{u_c(t) - u_l(t) H_c(t)}$. Finally, equation (49) matches equation (17) from the costless-money model, but without the capital-in-utility factor.

The government's budget constraint and the resource constraint remain the same as those presented in Section 2, specifically (6) and (5). Next, we derive the household's present-value budget constraint:

$$\begin{aligned} &\sum_{t=0}^{\infty} \begin{pmatrix} q_t^0 \left(c_t + \frac{i_t}{1+i_t}\widehat{m}_{t+1} - \left(1-\tau_t^n\right) w_t n_t\right) \\ + q_{t+1}^0 \frac{u_k(t+1) k_{t+1}}{u_c(t+1) - u_l(t+1) H_c(t+1)} \end{pmatrix} \\ &= \left[\left(1-\tau_0^k\right) r_0 + 1 - \delta\right] k_0 + b_0 + \frac{m_0}{p_0}, \end{aligned} \tag{50}$$

and the implementability condition

$$\begin{aligned} &\sum_{t=0}^{\infty} \beta^t \{u_c\left(c_t, l_t, k_t\right) c_t - u_l\left(c_t, l_t, k_t\right)\left[1 - l_t - (1-v) H\left(c_t, \widehat{m}_{t+1}\right)\right] \\ &+ u_k\left(c_t, l_t, k_t\right) k_t\} = A_3, \end{aligned} \tag{51}$$

where

$$A_3 = \left[u_c(0) - u_l(0) H_c(0)\right]\left\{\left[\left(1-\tau_0^k\right) r_0 + 1 - \delta\right] k_0 + b_0 + \frac{m_0}{p_0}\right\} + u_k\left(c_0, l_0, k_0\right) k_0.$$

The Ramsey problem is to maximize expression (44), subject to the implementability condition (51) and the feasibility constraint (5). Solving this problem results in the following optimality conditions:

$$\frac{U_l(t)}{U_c(t)} = \frac{F_n(t)}{F_n(t)H_c(t)+1}, t \geq 1 \tag{52}$$

$$\begin{aligned}&\frac{U_c(t)}{[F_n(t)H_c(t)+1]}\\ =&\beta\left(U_k(t+1) + \frac{U_c(t+1)}{[F_n(t+1)H_c(t+1)+1]}[F_k(t+1)+1-\delta]\right), t \geq 1\end{aligned} \tag{53}$$

$$\{(1+\upsilon\phi)u_l(t) + \phi[u_{cl}(t)c_t - u_{ll}(t)n_t + u_{kl}(t)k_t]\}H_{\widehat{m}}(t) = 0, t \geq 0 \tag{54}$$

$$U_c(0) - \phi A_{3c} = \beta U_c(1)\frac{[F_k(1)+1-\delta]}{[F_n(1)H_c(1)+1]}, t = 0$$

$$U_l(0) - \phi A_{3l} = \beta U_c(1)[F_k(1)+1-\delta]\frac{F_n(0)}{F_n(1)}, t = 0.$$

Compared to equations (22)-(24) in Section 1, the only difference in equality (53) is the addition of a new term $U_k(t+1)$, and the arguments of the utility function are now (c, l, k). From this, we derive the following:

Proposition 3. *In a costless monetary model with capital-in-utility, the optimal inflation tax is always zero, i.e., $I_t = 0$, which implies that the (net) nominal interest rate is also zero, i.e., $i = 0$. Suppose that the economy converges to an interior steady state.*[16] *The optimal capital income tax rate in the steady state is given by:*

$$\tau^k = \frac{1}{F_k(u_c - u_l H_c)}\frac{u_c F_n - u_l(F_n H_c + 1)}{u_c\eta_3 - u_l\eta_1}[u_k(\eta_1 - \eta_3 H_c) - \eta_2(u_c - u_l H_c)], \tag{55}$$

which shows that the optimal capital income tax is positive, zero, or negative, depending on $[u_k(\eta_1 - \eta_3 H_c) - \eta_2(u_c - u_l H_c)]$ is greater than, equal

[16]Unlike the standard Ramsey model, we cannot prove the existence and uniqueness of a (non-degenerate) steady state. In this model, the steady-state version of the consumption Euler equation is: $\frac{1}{\beta} = \frac{u_k}{(u_c - u_l H_c)} + \left[(1-\tau^k)F_k + 1 - \delta\right]$. The new term $\frac{u_k}{(u_c - u_l H_c)}$ complicates solving for the steady state, potentially leading to multiple equilibria, as discussed by Kurz (1968). Therefore, this paper assumes the existence of a steady state and focuses on the optimal taxation problem.

to, or less than zero. Namely,

$$\tau^k \begin{pmatrix} > \\ = \\ < \end{pmatrix} 0 \Leftrightarrow \left[u_k \left(\eta_1 - H_c\eta_3\right) - \eta_2 \left(u_c - u_l H_c\right)\right] \begin{pmatrix} > \\ = \\ < \end{pmatrix} 0.$$

Meanwhile, the formula for the optimal labor income tax rate in the steady state is as follows:

$$\tau^n = \frac{\phi}{1+\phi} \frac{1}{\left(u_c - u_l H_c\right) F_n} \left[\left(F_n H_c + 1\right)\eta_3 - F_n \eta_1\right], \tag{56}$$

which shows that the optimal labor income tax is positive, zero, or negative if and only if $\left[\left(F_n H_c + 1\right)\eta_3 - F_n \eta_1\right]$ *is greater than, equal to, or less than zero. Specifically,,*

$$\tau^n \begin{pmatrix} > \\ = \\ < \end{pmatrix} 0 \Leftrightarrow \left[\left(F_n H_c + 1\right)\eta_3 - F_n \eta_1\right] \begin{pmatrix} > \\ = \\ < \end{pmatrix} 0,$$

where

$$\begin{aligned} \eta_1 &= u_{cc}c - u_{lc}n + u_l\left(1-v\right) H_c + u_{kc}k, \\ \eta_2 &= u_{ck}c - u_{lk}n + u_{kk}k, \\ \eta_3 &= u_{cl}c - u_{ll}n + u_{kl}k. \end{aligned}$$

Proof. The proof is placed in Appendix C. ▮

Proposition 3 states that, in the monetary growth model with capital in the utility function, the Friedman rule remains optimal, while the Chamley-Judd zero capital taxation theorem does not hold. The optimality of the Friedman rule in this case suggests that the optimal inflationt tax hinges on the production cost of real money balances, independent of capital accumulation and capital in utility. As the production cost of money approaches zero, the net norminal interest rate will also converges to zero. However, in this scenario, the limiting capital income tax is generally not zero, since the key term $u_k\left(\eta_1 - \eta_3 H_c\right) - \eta_2\left(u_c - u_l H_c\right)$ in equation (55) does not generally equal zero. Therefore, if the representative consumer values utility derived from the physical capital stock, the Chamley-Judd zero capital income taxation theorem will be overturned. Furthermore, the sign of the optimal capital tax rate depends solely on the specification of the utility function and transaction technology, rather than the production technology, as indicated by the term $u_k\left(\eta_1 - \eta_3 H_c\right) - \eta_2\left(u_c - u_l H_c\right)$ in equation

(55). The sign of the limiting capital tax rate can be positive, negative or zero, implying that capital should be taxed, subsidized or left untaxed in the long run. Similarly, the sign of the optimal labor income tax depends on the sign of the term $[(F_n H_c + 1)\eta_3 - F_n \eta_1]$.[17]

The ambiguous effects on optimal taxation from capital in utility arise from the non-pecuniary return on capital, represented by $\frac{u_k}{(u_c - u_l H_c)}$, in the following asset-pricing equation (a rearranged version of the consumption Euler equation (47)):

$$1 = \underbrace{\beta \frac{u_c(t+1) - u_l(t+1) H_c(t+1)}{u_c(t) - u_l(t) H_c(t)}}_{\text{SDF}} \left\{ \underbrace{\frac{u_k(t+1)}{u_c(t+1) - u_l(t+1) H_c(t+1)}}_{\text{non-pecuniary return}} + \underbrace{\left[\left(1 - \tau^k_{t+1}\right) r_{t+1} + 1 - \delta\right]}_{\text{pecuniary return}} \right\}. \tag{57}$$

Taxing capital discourages MPK-driven capital accumulation, which-under the standard Ramsey settings-leads to lower steady-state capital. However, lower steady-state capital increases the numerator of the non-pecuniary comonent due to $u_{kk} < 0$, thereby encouraging capital-in-utility-driven capital accumulation. These two effects work in opposite directions, making it difficult to determine which one dominates. As a result, the sign of the limiting capital income tax cannot be determined in general. In fact, if the implied change in the steady state is relatively small, the entire non-pecuniary term may increase, further encouraging capital–in-utility-driven capital accumulation. Consequently, taxing capital may be relatively more or less distortionary in the CIU specification than in the standard neoclassical model, meaning that capital taxation has an ambiguous effect on steady-state capital accumulation in a model with capital in utility. Hence, the limiting capital tax rate can take any sign. In particular, if capital is not part of the utility function (i.e., $u_k = 0$, which implies $u_k(\eta_1 - \eta_3 H_c) - \eta_2(u_c - u_l H_c) = 0$), then the limiting capital income tax is zero (i.e., $\tau^k = 0$). Meanwhile, the formula for the limiting labor income tax reduces to the one used in Model 1. The degenerate case without CIU is essentially the same as Model 1, discussed in Section 2. In the absence of CIU, the asset-pricing equation simplifies to the standard form:

$$1 = \underbrace{\beta \frac{u_c(t+1) - u_l(t+1) H_c(t+1)}{u_c(t) - u_l(t) H_c(t)}}_{\text{SDF}} \underbrace{\left[\left(1 - \tau^k_{t+1}\right) r_{t+1} + 1 - \delta\right]}_{\text{pecuniary return}}.$$

This shows that taxing capital leads to lower levels of physical capital, which harms long-run economic growth. Hence, physical capital should

[17]Li et al. (2020) derived similar results regarding the indeterminacy of limiting factor income taxation in a non-monetary model with capital in the utility function.

remain untaxed. These results align the Chamley-Judd zero capital income taxation theorem in a neoclassical growth model, without with or without money.

Comparing our model to one without capital in utility, we find that zero capital tax result does not hold in all cases. As argued by Jones et al. (1997), there is nothing inherently special about physical capital as a stock variable. Similarly, the limiting tax on labor income (as a flow variable) is also ambiguous, and its sign depends on the specifications of both the utility function and the production technology.

To develop further intuition on optimal capital taxation, we assume that there is no money in the economy (i.e., $s_t = H(c_t, \widehat{m}_{t+1}) = 0$). This instantaneous utility function of the representative consumer is additively separable with respect to its three arguments, given by:

$$u(c,l,k) = \gamma_c u(c) + \gamma_l v(l) + \gamma_k w(k), \gamma_i > 0, i \in \{c,l,k\}. \qquad (58)$$

Thus, we know that $u' > 0$, $u'' < 0$, $v' > 0$, $v'' < 0$, $w' > 0$, and $w'' < 0$, due to the assumed properties of $u(c,l,k)$. Consequently we have the following:

Corollary 1. *Assume that there is no money, and the utility function is as defined in (58). The limiting capital income tax is positive, zero, or negative, if and only if, the capital elasticity of marginal utility of capital is less than, equal to, or greater than the consumption elasticity of marginal utility of consumption. Specifically,,*

$$\tau^k \begin{pmatrix} > \\ = \\ < \end{pmatrix} 0 \Longleftrightarrow \frac{w''(k)k}{w'(k)} \begin{pmatrix} < \\ = \\ > \end{pmatrix} \frac{u''(c)c}{u'(c)}.$$

Meanwhile, the optimal labor income tax is nonnegative, namely,

$$\tau^n = \frac{1}{u_c F_n} \frac{\Phi}{1+\Phi} (-u_{ll} n - u_{cc} c F_n) \geq 0.$$

Furthermore, if the utility function exhibits constant relative risk aversion (CRRA), i.e.,

$$u(c,l,k) = \frac{\gamma_c \left(c^{1-1/\theta_c} - 1\right)}{(1-1/\theta_c)} + \frac{\gamma_l \left(l^{1-1/\theta_l} - 1\right)}{(1-1/\theta_l)} + \frac{\gamma_k \left(k^{1-1/\theta_k} - 1\right)}{(1-1/\theta_k)}, \qquad (59)$$

where $\theta_i, i \in \{c,l,k\}$, are the constant elasticities of intertemporal substitution (EIS) for three types of utility goods, then we know that:

$$\tau^k \begin{pmatrix} > \\ = \\ < \end{pmatrix} 0 \Longleftrightarrow \theta_k \begin{pmatrix} < \\ = \\ > \end{pmatrix} \theta_c.$$

Proof. Corollary 1 can be easily proven by substitution. ∎

Corollary 1 explores a special case with additively separable utility functions by assuming away money and shopping technologies. It shows that optimal capital taxes depend on the relative magnitudes of the marginal utility elasticities for different goods (consumption and capital goods). Specifically, if the marginal utility of capital responds more sensitively to one percent change in the capital stock, compared to the marginal utility of consumption responding to a one percent change in consumption, then the optimal capital tax will be positive. Conversely, if the sensitivity of consumption is greater, the optimal capital tax will be negative. If both goods exhibit the same sensitivity, the optimal capital tax will be zero. Simple calculations yields the following elasticities:

$$\epsilon_c = -\frac{u'(c)}{u''(c)\,c}, \epsilon_n = -\frac{v'(1-n)}{v''(1-n)\,n}, \epsilon_k = -\frac{w'(k)}{w''(k)\,k}.$$

In particular, we examine the case of constant relative risk aversion (CRRA). If the elasticity of intertemporal substitution (EIS) for consumption goods, denoted θ_c, is larger than (equal to, or less than) that of capital goods, θ_k, then the limiting capital income tax is positive (zero, or negative).

4.2. The costly-money model with capital-in-utility (Model 4)

In this section, we examine a costly-money model with capital-in-utility. This case is formulated by either introducing physical capital into the utility function (Model 2) or by incorporating the production technology of real money balances (Model 3).

The household's optimization problem is to maximize the objective function, (44), subject to the budget constraint, (30), the time allocation equation, (27), and the shopping technology, (3). The first-order necessary conditions for this optimization problem are:

$$\frac{u_l(c_t, l_t, k_t)}{u_c(c_t, l_t, k_t) - u_l(c_t, l_t, k_t)\, H_c(c_t, \widehat{m}_{t+1})} = (1-\tau_{1t}^n)\, w_{1t}, \tag{60}$$

$$\begin{aligned} & u_c(t) - u_l(t)\, H_c(t) \\ =& \beta \left\{ u_k(t+1) + \left[\left(1-\tau_{1t+1}^k\right) r_{1t+1} + 1 - \delta_1 \right] \left[u_c(t+1) - u_l(t+1)\, H_c(t+1) \right] \right\}, \end{aligned} \tag{61}$$

$$R_t = \frac{\left(1-\tau_{1t+1}^k\right) r_{1t+1} + 1 - \delta}{\left(1 - \frac{\beta u_k(t+1)}{u_c(t) - u_l(t_t) H_c(c_t, \widehat{m}_{t+1})}\right)}$$
$$= \frac{u_c(c_t, l_t, k_t) - u_l(c_t, l_t, k_t) H_c(c_t, \widehat{m}_{t+1})}{\beta \left[u_c(c_{t+1}, l_{t+1}, k_{t+1}) - u_l(c_{t+1}, l_{t+1}, k_{t+1}) H_c(c_{t+1}, \widehat{m}_{t+2})\right]}, \quad (62)$$

$$\frac{R_t - R_{mt}}{R_t} = -\left(1-\tau_{1t}^n\right) w_{1t} H_{m/p}(c_t, \widehat{m}_{t+1}) = I_t. \quad (63)$$

Compared to the first-order conditions (31)-(34) in Model 2, there is an additional term involving u_k in the equations (61) and (62), where the arguments in the utility function are now (c, l, k).

The household's present-value budget constraint and the implementability condition are given by (50) and (51), respectively, with $\left(\tau_0^k, r_0, \delta, k_0\right)$ replaced by $\left(\tau_{10}^k, r_{10}, \delta_1, k_{10}\right)$. The resource constraint remains the same as in Model 2, (37).

The Ramsey problem is to maximize the objective function, (44), subject to the implementability condition, (51), and the resource constraint, (37). The corresponding optimality conditions are:

$$U_c(t) = \theta_t \left[(k_t - k_{2t})^{\alpha_1} (1-\alpha_1) n_{1t}^{-\alpha_1} H_c(c_t, \widehat{m}_{t+1}) + 1\right], t \geq 1$$

$$l_t : U_l(t) = \theta_t (k_t - k_{2t})^{\alpha_1} (1-\alpha_1) n_{1t}^{-\alpha_1}, t \geq 1$$

$$\theta_t = \beta \left\{U_k(t+1) + \theta_{t+1}\left[\alpha_1 (k_{t+1} - k_{2t+1})^{\alpha_1 - 1} n_{1t+1}^{1-\alpha_1} + 1 - \delta_1\right]\right\}, t \geq 0$$

$$\alpha_1 \left(\frac{k_t - k_{2t}}{n_{1t}}\right)^{\alpha_1 - 1} = \frac{(1-\alpha_1)\alpha_2}{(1-\alpha_2)} \left(\frac{k_t - k_{2t}}{n_{1t}}\right)^{\alpha_1} \left(\frac{\widehat{m}_{t+1}}{k_{2t}}\right)^{\frac{1}{1-\alpha_2}} + (\delta_1 - \delta_2),$$

$$\phi(1-v) u_l(c_t, l_t, k_t) H_{\widehat{m}}(c_t, \widehat{m}_{t+1})$$
$$= \theta_t (1-\alpha_1) \left(\frac{k_t - k_{2t}}{n_{1t}}\right)^{\alpha_1} \left(H_{\widehat{m}}(c_t, \widehat{m}_{t+1}) + \frac{1}{1-\alpha_2}\left(\frac{\widehat{m}_{t+1}}{k_{2t}}\right)^{\frac{\alpha_2}{1-\alpha_2}}\right), t \geq 0$$

where

$$U(t) = u(c_t, l_t, k_t)$$
$$+ \phi \left[u_c(c_t, l_t, k_t) c_t - u_l(c_t, l_t, k_t)\left[1 - l_t - (1-v) H(c_t, \widehat{m}_{t+1})\right] + u_k(c_t, l_t, k_t) k_t\right],$$

$$U_c(t) = u_c(t) + \phi\left[u_{cc}(t) c_t + u_c(t) - u_{lc}(t) n_t + u_l(t)(1-v) H_c(t) + u_{kc}(t) k_t\right],$$

$$U_l(t) = u_l(t) + \phi\left[u_{cl}(t)c_t - u_{ll}(t)n_t + u_l(t) + u_{kl}(t)k_t\right],$$

$$U_k(t+1) = u_k(t+1)$$
$$+\phi\left[u_{ck}(t+1)c_{t+1} - u_{lk}(t+1)n_{t+1} + u_{kk}(t+1)k_{t+1} + u_k(t+1)\right].$$

Thus, we have the following

PROPOSITION 4. *In a costly-money model with capital in the utility function, the optimal monetary policy follows these rules:*

$$I_t \begin{pmatrix} > \\ = \\ < \end{pmatrix} (1-\tau_{2t}^n),\ \textit{if } v \begin{pmatrix} < \\ = \\ > \end{pmatrix} 1. \tag{64}$$

Suppose the economy converges to an interior steady state. At the steady state, the formula for the limiting tax on capital employed in the consumption sector is:

$$\tau_1^k = \frac{1}{F_{k_1}(u_c - u_l H_c)} \frac{u_c F_{n_1} - u_l(F_{n_1}H_c + 1)}{u_c\eta_3 - u_l\eta_1}\left[u_k(\eta_1 - \eta_3 H_c) - \eta_2(u_c - u_l H_c)\right].$$

It is positive, zero, or negative if and only if $[u_k(\eta_1 - \eta_3 H_c) - \eta_2(u_c - u_l H_c)]$ *is greater than, equal to, or less than zero, respectively, i.e.,*

$$\tau_1^k \begin{pmatrix} > \\ = \\ < \end{pmatrix} 0 \Leftrightarrow \left[u_k(\eta_1 - H_c\eta_3) - \eta_2(u_c - u_l H_c)\right] \begin{pmatrix} > \\ = \\ < \end{pmatrix} 0.$$

The formula for the limiting tax on capital employed in the money sector is:

$$\tau_2^k = \frac{(r_2-\delta_2)-(r_1-\delta_1)}{r_2} + \frac{r_1}{r_2}\tau_1^k. \tag{65}$$

Then, we know that:

$$\tau_2^k \begin{pmatrix} > \\ = \\ < \end{pmatrix} \frac{(r_2-\delta_2)-(r_1-\delta_1)}{r_2},\ \textit{if } \tau_1^k \begin{pmatrix} > \\ = \\ < \end{pmatrix} 0.$$

Proof. The proof of the optimal monetary policy rules presented here is very similar to the case with costly money in Model 2, and the derivations of τ_1^k and τ^n are comparable to those with costless money in Model 3. Therefore, we omit them here. The results on τ_2^k stem from the no-arbitrage condition of factor mobility, i.e. (29). ▮

Proposition 4 indicates that in the model with costly money and capital in utility (CIU), the Friedman rule is not optimal in genenal, and the optimal inflation tax depends on the optimal tax on the labor employed in the money sector, τ_{2t}^{n}, as well as the degree of homogeneity of the transaction function, v. Note that the optimal tax rates τ_{2t}^{n} in the expressions of (43) and (64) differ, as they are endogenously determined within the analytical framework of Ramsey taxation.

In this context, the limiting taxes on capital income are more complex. The sign of the limiting tax on capital employed in the consumption good is determined by the sign of the expression $u_k(\eta_1 - H_c\eta_3) - \eta_2(u_c - u_l H_c)$, which is indeterminate. The rationale for this indeterminacy is analogous to Model 3, which we have omitted here. The limiting tax rate on capital employed in the money sector depends on two factors: the relative values of the net real returns of capital employed in the two sectors, $\frac{(r_2-\delta_2)-(r_1-\delta_1)}{r_2}$ and $\frac{r_1}{r_2}$, and the limiting tax rate on capital employed in the consumption sector, τ_1^k. If the limiting tax τ_1^k is zero (i.e., $\tau_1^k = 0$), then the limiting tax τ_2^k equals the difference between the net real returns of capital in both sectors (i.e., $\tau_2^k = \frac{(r_2-\delta_2)-(r_1-\delta_1)}{r_2}$). If the limiting tax τ_1^k is positive (i.e., $\tau_1^k > 0$), then the limiting tax τ_2^k is greater than the difference in net real returns of capital in both sectors (i.e., $\tau_2^k > \frac{(r_2-\delta_2)-(r_1-\delta_1)}{r_2}$); and vice versa.

5. CONCLUSION

In this paper, we reexamine the optimal fiscal and monetary policy within a combined shopping-time monetary model that includes capital accumulation. By exploring different combinations of two important channels (i.e., the production cost of money and capital in utility), we analyze four models and derive several interesting results. In the costless-money model without capital in utility (CIU), we recover classical results in dynamic taxation theory and optimal monetary theory: both the Friedman rule and the Chamley-Judd zero capital income taxation theorem hold. However, when money production is costly, the Friedman rule is not optimal, and the optimal inflation rate depends on the tax rate applied to the labor force employed in the money sector as well as the homogeneity of the transaction technology. Consequently, the tax structure for capital income changes accordingly. When consumers value utility from the physical capital stock, the Chamley-Judd theorem does not hold, and the limiting taxes on physical capital deviate from zero due to trade-offs between the non-pecuniary and pecuniary returns of capital accumulation. In the more complx Model 4, neither the Friedman rule nor the Chamley-Judd theorem applies. As the production cost of money and capital in utility (CIU) interact to determine the optimal fiscal and monetary policy.

APPENDIX A

Proof of Proposition 1. First, we derive the implementability condition. By iterating the household's flow budget constraint from period zero, we obtain:

$$\begin{aligned} b_0 =& q_T^0 \frac{b_{T+1}}{R_T} + q_T^0 \frac{m_{T+1}}{p_T} + \sum_{t=0}^{T} q_t^0 c_t + \sum_{t=0}^{T-1} q_t^0 \frac{i_t}{1+i_t} \widehat{m}_{t+1} \\ & - \sum_{t=0}^{T} q_t^0 \left(1-\tau_t^n\right) w_t n_t + \sum_{t=0}^{T} q_t^0 k_{t+1} - \sum_{t=0}^{T-1} q_{t+1}^0 \left[\left(1-\tau_{t+1}^k\right) r_{t+1} + 1 - \delta\right] k_{t+1} \\ & - \left[\left(1-\tau_0^k\right) r_0 + 1 - \delta\right] k_0 - \frac{m_0}{p_0}. \end{aligned}$$

Using the no-arbitrage condition (15), we take limits on both sides as $T \rightarrow +\infty$ and impose the transversality conditions $\lim_{T\rightarrow+\infty} q_T^0 \frac{b_{T+1}}{R_T} = 0$ and $\lim_{T\rightarrow+\infty} q_T^0 \frac{m_{T+1}}{p_T} = 0$. This yields the present-value budget constraint:

$$\sum_{t=0}^{+\infty} q_t^0 \left[c_t - \left(1-\tau_t^n\right) w_t n_t + \frac{i_t}{1+i_t} \widehat{m}_{t+1}\right] = \left[\left(1-\tau_0^k\right) r_0 + 1 - \delta\right] k_0 + b_0 + \frac{m_0}{p_0}.$$

Substituting (13), (17), (20), and (4) into the present-value budget constraint and rearranging, we obtain the implementability condition (21):

$$\sum_{t=0}^{\infty} \beta^t \left[u_c\left(c_t, l_t\right) c_t - u_l\left(c_t, l_t\right)\left(1 - l_t - (1-v) H\left(c_t, \widehat{m}_{t+1}\right)\right)\right] = A,$$

where A is given by

$$\begin{aligned} A =& A\left(c_0, l_0, k_0, b_0, m_0, \tau_0^k\right) \\ =& \left[u_c\left(c_0, l_0\right) - u_l\left(c_0, l_0\right) H_c(0)\right] \left[\left(\left(1-\tau_0^k\right) r_0 + 1 - \delta\right) k_0 + b_0 + \frac{m_0}{p_0}\right]. \end{aligned}$$

Second, we solve the Ramsey problem using the Primal approach. The Ramsey problem is to maximize expression (1) subject to the implementability condition (21) and the feasibility constraint (5). Let ϕ be a Lagrange multiplier on equation (21) and define:

$$U\left(c_t, l_t, \widehat{m}_{t+1}, \phi\right) = u\left(c_t, l_t\right) + \phi\left[u_c\left(c_t, l_t\right) c_t - u_l\left(c_t, l_t\right)\left(1 - l_t - (1-v) H\left(c_t, \widehat{m}_{t+1}\right)\right)\right].$$

Then, we construct the Lagrangian:

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \{U(c_t, l_t, \widehat{m}_{t+1}, \phi) \\ +\theta_t [F(k_t, 1 - l_t - H(c_t, \widehat{m}_{t+1})) + (1-\delta) k_t - c_t - g_t - k_{t+1}]\} - \phi A,$$

where $\{\theta_t\}_{t=0}^{\infty}$ is a sequence of Lagrange multipliers. First-order conditions for this problem are:

$$c_t : U_c(c_t, l_t, \widehat{m}_{t+1}, \phi) = \theta_t [F_n(k_t, n_t) H_c(c_t, \widehat{m}_{t+1}) + 1], t \geq 1$$

$$l_t : U_l(c_t, l_t, \widehat{m}_{t+1}, \phi) = \theta_t F_n(k_t, n_t), t \geq 1$$

$$k_{t+1} : \theta_t = \beta \theta_{t+1} [F_k(k_{t+1}, n_{t+1}) + 1 - \delta], t \geq 0$$

$$\widehat{m}_{t+1} : [\phi(1-v) u_l(c_t, l_t) - \theta_t F_n(k_t, n_t)] H_{\widehat{m}}(c_t, \widehat{m}_{t+1}) = 0, t \geq 0$$

$$c_0 : U_c(0) = \theta_0 [F_n(k_0, n_0) H_c(c_0, \widehat{m}_1) + 1] + \phi A_c, t = 0$$

$$l_0 : U_l(0) = \theta_0 F_n(0) + \phi A_l, t = 0$$

where

$$U_c(0) = u_c(0) + \phi \begin{pmatrix} u_{cc}(0) c_0 + u_c(0) + u_l(0)(1-v) H_c(0) \\ -u_{lc}(0)(1 - l_0 - (1-v) H(0)) \end{pmatrix},$$

$$U_l(c_0, l_0) = u_l(0) + \phi [u_{cl}(0) c_0 - u_{ll}(0)(1 - l_0 - (1-v) H(0)) + u_l(0)],$$

$$A_c = \frac{[u_{cc}(0) - u_{lc}(0) H_c(0) - u_l(0) H_{cc}(0)]}{[u_c(0) - u_l(0) H_c(0)]} A \\ - [u_c(0) - u_l(0) H_c(0)] (1 - \tau_0^k) F_{kn}(0) H_c(0) k_0,$$

$$A_l = \frac{[u_{cl}(0) - u_{ll}(0) H_c(0)]}{[u_c(0) - u_l(0) H_c(0)]} A - [u_c(0) - u_l(0) H_c(0)] (1 - \tau_0^k) F_{kn}(0) k_0.$$

Combining the above first-order conditions, we have the following optimality conditions:

$$\frac{U_l(c_t, l_t, \widehat{m}_{t+1}, \phi)}{U_c(c_t, l_t, \widehat{m}_{t+1}, \phi)} = \frac{F_n(k_t, n_t)}{F_n(k_t, n_t) H_c(c_t, \widehat{m}_{t+1}) + 1}, t \geq 1,$$

$$\begin{aligned}&\frac{U_c\left(c_t,l_t,\widehat{m}_{t+1},\phi\right)}{\left[F_n\left(k_t,n_t\right)H_c\left(c_t,\widehat{m}_{t+1}\right)+1\right]}\\=&\frac{\beta U_c\left(c_{t+1},l_{t+1},\widehat{m}_{t+2},\phi\right)}{\left[F_n\left(k_{t+1},n_{t+1}\right)H_c\left(c_{t+1},\widehat{m}_{t+2}\right)+1\right]}\left[F_k\left(k_{t+1},n_{t+1}\right)+1-\delta\right],t\geq 1,\end{aligned}$$

$$\left[\left(v\phi+1\right)u_l\left(c_t,l_t\right)+\phi\left(u_{cl}\left(c_t,l_t\right)c_t-u_{ll}\left(c_t,l_t\right)n_t\right)\right]H_{\widehat{m}}\left(c_t,\widehat{m}_{t+1}\right)=0,t\geq 0,$$

$$U_c\left(0\right)-\phi A_{1c}=\beta U_c\left(1\right)\frac{\left[F_k\left(1\right)+1-\delta\right]}{\left[F_n\left(1\right)H_c\left(1\right)+1\right]},t=0,$$

$$U_l\left(0\right)-\phi A_{1l}=\beta U_c\left(1\right)\left[F_k\left(1\right)+1-\delta\right]\frac{F_n\left(0\right)}{F_n\left(1\right)},t=0.$$

Third, the optimality of the Friedman rule and zero capital income taxation is verified in the main tex in Section 2.1. Finally, from the first-order conditions with respect to c_t and l_t in the steady state, we have:

$$\begin{aligned}&u_cF_n-u_l\left(F_nH_c+1\right)\\=&\frac{\phi}{1+\phi}\left[\left(F_nH_c+1\right)\left(u_{cl}c-u_{ll}n\right)-F_n\left(u_{cc}c-u_{lc}n+u_l\left(1-v\right)H_c\right)\right].\end{aligned}$$

Solving for (13) and (18) gives rise to

$$u_cF_n-u_l\left(F_nH_c+1\right)=\left(u_c-u_lH_c\right)F_n\tau^n.$$

Combining the above two equations leads to the formula for the limiting labor income tax:

$$\tau^n=\frac{\phi}{1+\phi}\frac{\left(F_nH_c+1\right)\left(u_{cl}c-u_{ll}n\right)-F_n\left[u_{cc}c-u_{lc}n+u_l\left(1-v\right)H_c\right]}{\left(u_c-u_lH_c\right)F_n},$$

which may be positive, negative or zero. □

APPENDIX B

Proof of Proposition 2. From (42), we know that

$$H_{\widehat{m}}\left(t\right)=\frac{\frac{\theta_tw_{1t}}{1-\alpha_2}\left(\frac{\widehat{m}_{t+1}}{k_t-k_{1t}}\right)^{\frac{\alpha_2}{1-\alpha_2}}}{\phi\left(1-v\right)u_l\left(c_t,l_t\right)-\theta_tw_{1t}}.$$

Notice that, as we saw in Section 2.1, $\phi(1-v)\,u_l(c_t,l_t)-\theta_t w_{1t}\neq 0$. Combining the above equation with the necessary condition of the private problem (34) gives us the following equality

$$\frac{\frac{\theta_t w_{1t}}{1-\alpha_2}\left(\frac{\widehat{m}_{t+1}}{k_t-k_{1t}}\right)^{\frac{\alpha_2}{1-\alpha_2}}}{\theta_t w_{1t}-\phi(1-v)\,u_l(c_t,l_t)}=\frac{I_t}{\left(1-\tau_{1t}^n\right)w_{1t}}.$$

If $v=1$, we have $\frac{I_t}{\left(1-\tau_{1t}^n\right)w_{1t}}=\frac{1}{1-\alpha_2}\left(\frac{\widehat{m}_{t+1}}{k_t-k_{1t}}\right)^{\frac{\alpha_2}{1-\alpha_2}}$. Using the no-arbitrage condition for labor mobility and the production function of money (26), we derive that $1-\tau_{2t}^n=I_t$. If $v>1$, then $\frac{I_t}{\left(1-\tau_{1t}^n\right)w_{1t}}<\frac{1}{1-\alpha_2}\left(\frac{\widehat{m}_{t+1}}{k_t-k_{1t}}\right)^{\frac{\alpha_2}{1-\alpha_2}}$. Using a similar procedure, we obtain $1-\tau_{2t}^n>I_t$. Conversely, if $v>1$, then a similar argument yields $1-\tau_{2t}^n<I_t$.

Substituting (38) into (40), we have

$$\frac{U_c(t)}{1+F_{n_1}(t)\,H_c(t)}=\beta\frac{U_c(t+1)}{1+F_{n_1}(t+1)\,H_c(t+1)}\left[F_{k_1}(t+1)+1-\delta_1\right].$$

In the steady state, it turns out to

$$1=\beta\left(r_1+1-\delta_1\right).$$

Meanwhile, equation (32) turns out to

$$1=\beta\left[\left(1-\tau_1^k\right)r_1+1-\delta_1\right].$$

Combining them gives rise to $\tau_1^k=0$. In the steady state, plugging $\tau_1^k=0$ into (29) leads to

$$\tau_2^k=\frac{(r_2-\delta_2)-(r_1-\delta_1)}{r_2},$$

which establishes the results presented in Proposition 3.1. □

APPENDIX C

Proof of Proposition 3. The present-value budget constraint is derived as

$$\begin{pmatrix} \sum_{t=0}^{+\infty} q_t^0\left[c_t-\left(1-\tau_t^n\right)w_t n_t+\frac{i_t}{1+i_t}\widehat{m}_{t+1}\right] \\ +\sum_{t=0}^{+\infty}\left\{q_t^0-q_{t+1}^0\left[\left(1-\tau_{t+1}^k\right)r_{t+1}+1-\delta\right]\right\}k_{t+1} \end{pmatrix}$$
$$=\left[\left(1-\tau_0^k\right)r_0+1-\delta\right]k_0+b_0+\frac{m_0}{p_0}. \quad \text{(C.1)}$$

Combining the no-arbitrage condition (15) and the first-order condition with respect to c, l, and b, we obtain

$$R_t - \left[\left(1-\tau_{t+1}^k\right) r_{t+1} + 1 - \delta\right] = \frac{u_k\left(t+1\right)}{u_c\left(t+1\right) - u_l\left(t+1\right) H_c\left(t+1\right)}.$$

Multiplying both sides of the above equation with q_{t+1}^0 gives rise to

$$q_t^0 - q_{t+1}^0\left[\left(1-\tau_{t+1}^k\right) r_{t+1} + 1 - \delta\right] = q_{t+1}^0 \frac{u_k\left(t+1\right)}{u_c\left(t+1\right) - u_l\left(t+1\right) H_c\left(t+1\right)}. \tag{C.2}$$

Substituting (C.2) into (C.1) leads to the present-value budget constraint (50). Incorporating (20) with the arguments (c, l, k) in the utility function, along with (46) and (49), into (50) results in the implementability condition (51).

The Ramsey problem is to maximize expression (44) subject to the implementability condition (51) and the feasibility constraint (5). The first-order conditions for this problem are:

$$c_t : U_c\left(t\right) = \theta_t\left[F_n\left(t\right) H_c\left(c_t, \widehat{m}_{t+1}\right) + 1\right], t \geq 1 \tag{C.3}$$

$$l_t : U_l\left(t\right) = \theta_t F_n\left(t\right), t \geq 1 \tag{C.4}$$

$$k_{t+1} : \theta_t = \beta\left\{U_k\left(t+1\right) + \theta_{t+1}\left[F_k\left(t+1\right) + 1 - \delta\right]\right\}, t \geq 0 \tag{C.5}$$

$$\widehat{m}_{t+1} : \left[\phi\left(1-v\right) u_l\left(c_t, l_t, k_t\right) - \theta_t F_n\left(t\right)\right] H_{\widehat{m}}\left(c_t, \widehat{m}_{t+1}\right) = 0, t \geq 0 \tag{C.6}$$

$$c_0 : U_c\left(0\right) = \theta_0\left[F_n\left(0\right) H_c\left(0\right) + 1\right] + \phi A_c,$$

$$l_0 : U_l\left(0\right) = \theta_0 F_n\left(0\right) + \phi A_l,$$

$$k_0 : U_k\left(0\right) = \phi A_k - \theta_0\left[F\left(0\right) + \left(1-\delta\right)\right],$$

where

$$U\left(t\right) = u\left(c_t, l_t, k_t\right) + \phi\left[u_c\left(c_t, l_t, k_t\right) c_t - u_l\left(c_t, l_t, k_t\right)\left[1 - l_t - \left(1-v\right) H\left(c_t, \widehat{m}_{t+1}\right)\right] + u_k\left(c_t, l_t, k_t\right) k_t\right],$$

$$U_c\left(t\right) = u_c\left(t\right) + \phi\left[u_{cc}\left(t\right) c_t + u_c\left(t\right) - u_{lc}\left(t\right) n_t + u_l\left(t\right)\left(1-v\right) H_c\left(t\right) + u_{kc}\left(t\right) k_t\right],$$

$$U_l\left(t\right) = u_l\left(t\right) + \phi\left[u_{cl}\left(t\right) c_t - u_{ll}\left(t\right) n_t + u_l\left(t\right) + u_{kl}\left(t\right) k_t\right],$$

$$U_k\left(t+1\right) = u_k\left(t+1\right) + \phi\left[u_{ck}\left(t+1\right) c_{t+1} - u_{lk}\left(t+1\right) n_{t+1} + u_{kk}\left(t+1\right) k_{t+1} + u_k\left(t+1\right)\right],$$

$$U_c(0) = u_c(0) + \phi \begin{pmatrix} u_{cc}(0)\, c_0 + u_c(0) + u_l(0)(1-v) H_c(0) \\ -u_{lc}(0)(1 - l_0 - (1-v) H(0)) \end{pmatrix},$$

$$U_l(0) = u_l(0) + \phi \left[u_{cl}(0)\, c_0 - u_{ll}(0)(1 - l_0 - (1-v) H(0)) + u_l(0) \right],$$

$$\begin{aligned} A_{3c} = & \frac{[u_{cc}(0) - u_{lc}(0) H_c(0) - u_l(0) H_{cc}(0)]}{[u_c(0) - u_l(0) H_c(0)]} A_3 \\ & - [u_c(0) - u_l(0) H_c(0)] \left(1 - \tau_0^k\right) F_{kn}(0) H_c(0)\, k_0, \end{aligned}$$

$$A_{3l} = \frac{[u_{cl}(0) - u_{ll}(0) H_c(0)]}{[u_c(0) - u_l(0) H_c(0)]} A_3 - [u_c(0) - u_l(0) H_c(0)] \left(1 - \tau_0^k\right) F_{kn}(0)\, k_0.$$

Combining these conditions, we have

$$\frac{U_l(t)}{U_c(t)} = \frac{F_n(t)}{F_n(t) H_c(t) + 1}, t \geq 1$$

$$\frac{U_c(t)}{[F_n(t) H_c(t) + 1]} = \beta \frac{U_c(t+1)}{[F_n(t+1) H_c(t+1) + 1]} \left[F_k(t+1) + 1 - \delta \right], t \geq 1$$

$$\left\{ (1 + v\phi) u_l(t) + \phi \left[u_{cl}(t)\, c_t - u_{ll}(t)\, n_t + u_{kl}(t)\, k_t \right] \right\} H_{\widehat{m}}(t) = 0, t \geq 0 \tag{C.7}$$

$$U_c(0) - \phi A_{3c} = \beta U_c(1) \frac{[F_k(1) + 1 - \delta]}{[F_n(1) H_c(1) + 1]}, t = 0$$

$$U_l(0) - \phi A_{3l} = \beta U_c(1) \left[F_k(1) + 1 - \delta \right] \frac{F_n(0)}{F_n(1)}, t = 0.$$

From equalities (C.6) and (C.7), by the similar procedure to that in the proof of Proposition 1, we conclude that the Friedman rule is optimal, namely, $I_t = i_t = 0$.

To examine the optimal tax rates, we consider the special case where $T \geq 0$ such that $g_t = g$ for all $t \geq T$. Assume that there exists a stationary solution to the Ramsey problem, which converges to a time-invariant allocation, so that c, l, $\widehat{m}$, and k remain constant after some time. The steady state of the economy can be found by solving the steady-state version of equations (C.3)-(C.5):

$$u_c + \phi \left[u_{cc} c + u_c - u_{lc} n + u_l (1-v) H_c + u_{kc} k \right] = \theta \left(F_n H_c + 1 \right), \tag{C.8}$$

$$\theta = \beta \left[u_k + \phi \left(u_{ck} c - u_{lk} n + u_{kk} k + u_k \right) + \theta \left(F_k + 1 - \delta \right) \right], \tag{C.9}$$

$$u_l + \phi \left(u_{cl} c - u_{ll} n + u_l + u_{kl} k \right) = \theta F_n, \tag{C.10}$$

Equations (C.8)-(C.10) are rewritten as

$$F_n H_c + 1 = \frac{1+\phi}{\theta} u_c + \frac{\phi}{\theta} \underbrace{\left(u_{cc} c - u_{lc} n + u_l \left(1 - v \right) H_c + u_{kc} k \right)}_{\equiv \eta_1}, \tag{C.11}$$

$$1 - \beta \left(F_k + 1 - \delta \right) = \beta u_k \frac{1+\phi}{\theta} + \beta \frac{\phi}{\theta} \underbrace{\left(u_{ck} c - u_{lk} n + u_{kk} \right) k}_{\equiv \eta_2}, \tag{C.12}$$

$$F_n = \frac{1+\phi}{\theta} u_l + \frac{\phi}{\theta} \underbrace{\left(u_{cl} c - u_{ll} n + u_{kl} k \right)}_{\equiv \eta_3}. \tag{C.13}$$

We solve equations (C.11) and (C.13) for $(1+\phi)/\theta$ and ϕ/θ as follows:

$$\frac{1+\phi}{\theta} = \frac{\left(F_n H_c + 1 \right) \eta_3 - F_n \eta_1}{u_c \eta_3 - u_l \eta_1}, \tag{C.14}$$

$$\frac{\phi}{\theta} = \frac{u_c F_n - u_l \left(F_n H_c + 1 \right)}{u_c \eta_3 - u_l \eta_1}. \tag{C.15}$$

The steady-state version of consumption Euler equation (47) is changed as

$$\left[1 - \beta \left(F_k + 1 - \delta \right) \right] \left(u_c - u_l H_c \right) = \beta u_k - \beta \left(u_c - u_l H_c \right) \tau^k F_k. \tag{C.16}$$

Substituting (C.14)-(C.16) into (C.12) yields us the formula for the capital income tax rate (55), namely,

$$\tau^k = \frac{1}{F_k \left(u_c - u_l H_c \right)} \frac{u_c F_n - u_l \left(F_n H_c + 1 \right)}{u_c \eta_3 - u_l \eta_1} \left[u_k \left(\eta_1 - \eta_3 H_c \right) - \eta_2 \left(u_c - u_l H_c \right) \right].$$

From equation (C.15), the term $\frac{u_c F_n - u_l (F_n H_c + 1)}{u_c \eta_3 - u_l \eta_1} = \frac{\phi}{\theta}$ is nonnegative because the Lagrange multiplier ϕ is nonnegative, while the insatiable utility function implies that θ is strictly positive. Note that F_k and $(u_c - u_l H_c)$ are both nonnegative. Hence, the sign of the limiting capital income tax is determined completely by the sign of the term $\left[u_k \left(\eta_1 - H_c \eta_3 \right) - \eta_2 \left(u_c - u_l H_c \right) \right]$.

From equalities (C.11) and (C.13), we have:

$$u_c F_n - u_l \left(F_n H_c + 1 \right) = \frac{\phi}{1+\phi} \left[\left(F_n H_c + 1 \right) \eta_3 - F_n \eta_1 \right]. \tag{C.17}$$

Equation (46) yields us

$$F_n\left(u_c - u_l H_c\right)\tau^n = u_c F_n - u_l\left(F_n H_c + 1\right). \qquad \text{(C.18)}$$

Combining equations (C.17) and (C.18), we derive the formula for the optimal labor income tax, denoted as (56), whose sign is also indeterminate. □

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