ANNALS OF ECONOMICS AND FINANCE **26-1**, 415–441 (2025)

Robust Non-Zero-Sum Asset Allocation Games Under Relative Wealth Concerns

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This paper considers the non-zero sum stochastic differential asset allocation game problem between two competitive institutional investors, who are concerned with the potential model ambiguity and aim to seek the robust optimal asset allocation strategy. The two investors' decisions influence each other through the investors' relative wealth concerns. By applying the dynamic programming principle, explicit solutions for the robust equilibrium asset allocation strategies are obtained under the representative case of constant relative risk aversion (CRRA) utility. Finally, we provide some numerical studies and draw some economic interpretations.

Key Words: Robust non-zero-sum game; Asset allocation; Ambiguity; Nash equilibrium; Hamilton-Jacobi-Bellman-Isaacs (HJBI) equation.
JEL Classification Numbers: C730, C610, G41.

1. INTRODUCTION

Portfolio selection problem, pioneered by Markowitz (1952) and Merton (1971), is about the optimal wealth allocation among different assets with risk management and stochastic modeling. The model parameters in the dynamics of the asset returns are usually assumed to be fixed and known. Implicitly, the model uncertainty or parameter ambiguity was ignored. Unfortunately, the model uncertainty is unavoidably present due to the estimation error or lack of knowledge of the underlying probability. Some early empirical studies (see Merton, (1980); Cochrane, (1997)) have suggested that the drift parameters of stochastic models, particularly the expected risk premium, are difficult to estimate with precision from historical data. This introduces a significant amount of model ambiguity

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1529-7373/2025

that economic agents have to contend with. Furthermore, Ellsberg (1961) experimentally shows that individuals are not only averse to risk but also to ambiguity. Therefore, it is reasonable to assume that the decision maker is concerned about the model misspecification. Anderson et al. (2003) quantifies model misspecification (uncertainty) with a statistical theory of detection, which links between robustness and model detection. Maenhout (2004) studies the portfolio and consumption problems with ambiguity and proposes an innovative characterization of the ambiguity aversion, which preserves wealth independence and analytical tractability of the decisions. Luo (2017) provides a tractable continuous-time, constant absolute risk aversion-Gaussian framework to explore how the interactions of fundamental uncertainty, model uncertainty and state uncertainty affect strategic consumption-portfolio rules and precautionary savings in the presence of uninsurable labor income. Zeng et al. (2018) discusses a derivative-based optimal investment strategy for an ambiguity-adverse pension investor who faces not only risks from time-varying income and market return volatility but also uncertain economic conditions over a long time horizon. Sun et al. (2019) studies the robust investment and reinsurance problem with a defaultable bond. Yan et al. (2020) solves for the robust time–consistent mean-variance portfolio selection problem on multiple risky assets under a principle component stochastic volatility model. Yang et al. (2020) studies a robust portfolio optimization problem under a multi-factor volatility model. Luo et al. (2023) constructs a recursive utility version of a canonical Merton model with uninsurable labor income and unknown income growth to study how the interaction between two types of uncertainty due to ignorance affects strategic consumption-portfolio rules and precautionary savings. These works consider the equivalent priors which are based on the Girsanov's theorem. Different from them, Rieder and Wopperer (2012) assumes the market price of risk process is unknown and studies the robust consumption-investment strategy under the worst case. While the above literature only considers single-agent optimization problems.

Recently, the studies on dynamic asset allocation with relative wealth concerns have been very popular, (see Basak and Makarov, 2014; Espinosa and Touzi, 2015). There are two natural arguments for studying dynamic asset allocation under relative wealth concerns. On the one hand, interpreting agents as fund managers, relative wealth concerns can describe the competition among fund managers (see Lacker and Zariphopoulou, 2019; Kraft et al., 2020). On the other hand, if we interpret the agents in our model as household investors, then relative wealth concerns fit naturally with models of keeping up with the Joneses (see Abel, 1990); this line of literature directly incorporates the social aspects of investment and consumption decisions.

The individual research on model uncertainty or dynamic asset allocation with relative wealth concerns has already attracted people's attention. However, their incorporation is not studied thoroughly yet. Recently, Wang et al. (2019, 2021) investigate non-zero-sum stochastic differential investment and reinsurance games between two ambiguity-averse insurers under the expected utility maximization and the mean–variance criterion, respectively. Li et al. (2024) consider the non-zero-sum stochastic differential investment and reinsurance game problem between two ambiguity-averse insurers with common shock. These works open up a new path of stochastic differential game theory. However, the worked-out examples with the existing frameworks assume deterministic volatility, which exclude the stochastic volatility models.

In multi-player games, the intertwining of relative wealth concerns and uncertainty makes it difficult for traditional models to effectively deal with complex investment environments in reality, so this paper fills a research gap by investigating the robust non-zero sum stochastic differential asset allocation game under a stochastic volatility model driven by an affine-form square-root factor process. Our model includes relative wealth concerns, model uncertainty, and stochastic volatility as critical features. The investors have access to an incomplete financial market consisting of one risk-free asset and one risky asset described by a stochastic volatility model driven by an affine-form square-root factor process, which is a more generalized model and encompasses the geometric Brownian motion (GBM), constant elasticity of variance (CEV) model, Heston model as special cases. Applying the techniques of stochastic dynamic programming, we derive the HJBI equations for the asset allocation games. Explicit expressions for the robust equilibrium asset allocation strategies that maximize the expected power utility of the terminal wealth relative to that of his competitor and corresponding optimal value functions are obtained. We also provide some special cases of our model and explore the economic implications from numerical examples.

Compared with existing literature, the main contributions of this paper is twofold. First, the impact of model uncertainty on optimal asset allocation strategy is investigated, which is not considered by Kraft et al. (2020), where the authors studied the strategic interaction between two CRRA investors who can invest into stock market. However, it is important for a decision-maker to consider parameter ambiguity and stochastic uncertainty because return levels of investment securities are difficult to obtain. Our numerical studies show that an ambiguity-averse investor would prefer more conservative investment strategy than an ambiguity-neutral investor. Second, we extend the robust asset allocation model in Zeng et al. (2018), where only a single investor was considered, to a continuous-time game framework by taking multiple investors' relative wealth concerns into account. The key reason for formulating non-zero-sum stochastic differential game is that there are always several competing investors in the market in reality, and they often assess their performance against a relative benchmark of their competitors. Therefore, we derive the Nash equilibrium asset allocation strategy of a non-zero-sum game in this paper. Numerical studies demonstrate that the relative wealth concern makes each investor riskier than that without competition, which are reflected in the increased exposure on risky asset. We also find that the investor's optimal strategy is affected by her competitor's ambiguity aversion level. More precisely, the competitor's ambiguity-averse attitude makes the investor more conservative by diminishing the proportion invested in the stock market.

The rest of the paper is organized as follows. Section 2 formulates the robust non-zero-sum stochastic differential game between two CRRA investors. In Section 3, we derive the HJBI equations for the robust optimization problem. Explicit expressions for Nash equilibrium strategies and corresponding optimal value functions are obtained, and the verification theorem is provided as well. Section 4 dicusses some special cases of our model. Detailed numerical simulations are conducted in Section 5 to demonstrate the results. Finally, Section 6 concludes the paper with some suggestions for future research.

2. ASSUMPTION AND PROBLEM FORMULATION

2.1. Preferences and asset price dynamics

We consider a continuous-time setup on the time span $[0, T]$ where two institutional investors optimize their dynamic asset allocation strategies. Uncertainty is represented by a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, where $\mathbb{F} := \{\mathcal{F}_t : t \in [0, T]\}$ is generated by a two-dimensional standard Brownian motion $(W_1(t), W_2(t))$, $T > 0$ is a finite constant representing the investment time horizon; $\mathcal{F}_t$ denotes the information available until time t; and $\mathbb{P}$ is a reference measure.

Each of the two investors aims to maximize expected utility of a weighted average of absolute and relative terminal wealth. Relative terminal wealth is measured with respect to the other investor's wealth. Therefore, this benchmark is endogenous. Possible interpretations of such a setting might be that the investors are competing fund managers that are concerned about the total value of assets under management, both in absolute terms, but also relative to the other fund's assets.

Each investor derives utility from an average over her own wealth and her relative wealth level at time T , aggregated via a constant elasticity Cobb-Douglas function. This average is embedded into a CRRA utility function:

$$U_i(x_i, x_j) = \frac{1}{1-\gamma_i}\left(x_i^{\theta_i}\left(x_i/x_j\right)^{1-\theta_i}\right)^{1-\gamma_i} = \frac{1}{1-\gamma_i}\left(x_i x_j^{\theta_i - 1}\right)^{1-\gamma_i}, \quad (1)$$

where $\gamma_i > 0$, $\gamma_i \neq 1$ denotes the $i-$th investor's relative risk aversion, and $\theta_i \in [0,1]$ measures the weight on terminal wealth. Therefore, $1-\theta_i$ is the weight on the relative wealth concerns. Specifying the investor' utility functions as in Eq.(1) ensures that each investor's terminal wealth is strictly positive, so both versions of relative performance x_i/x_j, for $i \neq j \in \{1,2\}$ are well-defined. Throughout this paper we assume that

$$\gamma_i + \gamma_j > 1. \quad (2)$$

This assumption allows for levels of risk aversion below unity as documented by Koijen (2014), but puts a lower bound on the agents' overall risk aversion to ensure that the portfolio decisions are well-defined. Note that for $\theta_i = \theta_j = 1$ the agents have ordinary CRRA preferences without relative wealth concerns; on the other hand, if $\theta_i = \theta_j = 0$ both investors' utilities are solely determined by their relative performance.

To focus on the effect of relative wealth concerns, there are no informational or skill-related differences between the two investors, and both have access to the same investment opportunities defined by two primitive assets. One is a risk-free asset whose price process evolves over time as

$$\mathrm{d}B(t) = r(t)B(t)\mathrm{d}t, \quad B(0) = 1, \quad (3)$$

where $r(t) > 0$ is the risk-free interest rate at time t, satisfying that $r(\cdot) : [0,T] \to \mathbb{R}^+$ is a deterministic and uniformly bounded function. The other one is a risky asset, whose price process is described by

$$\mathrm{d}S(t) = S(t)\left[\mu(t)\mathrm{d}t + \sigma(t)\mathrm{d}W_1(t)\right], \quad S(0) = s_0 > 0, \quad (4)$$

where $\{W_1(t)\}_{t\in[0,T]}$ is a standard Brownian motion, $\mu(t)$ and $\sigma(t) > 0$ are the appreciation rate and the volatility of the risky asset at time t, respectively. Similar to Shen and Zeng (2015), we assume that $\{\mu(t)\}_{t\in[0,T]}$ and $\{\sigma(t)\}_{t\in[0,T]}$ are $\mathcal{F}-$predictable processes, which are continuous bounded deterministic functions or stochastic processes, and $\{\mu(t)\}_{t\in[0,T]}$, $\{\sigma(t)\}_{t\in[0,T]}$ rely on the market price of risk $\{\vartheta(t)\}_{t\in[0,T]}$ directly, i.e.,

$$\vartheta(t) := \frac{\mu(t) - r(t)}{\sigma(t)}, \ \ \forall t \in [0,T], \quad (5)$$

$\{\vartheta(t)\}_{t\in[0,T]}$ is related to a stochastic factor process $\{\alpha(t)\}_{t\in[0,T]}$ as

$$\vartheta(t) = \lambda\sqrt{\alpha(t)}, \ \ \forall t \in [0,T], \ \ \lambda \in \mathbb{R}_0 := \mathbb{R} \setminus \{0\}. \quad (6)$$

and $\{\alpha(t)\}_{t\in[0,T]}$ satisfies the following Markovian, affine-form and square-root model:

$$\mathrm{d}\alpha(t) = \kappa\left[\delta - \alpha(t)\right]\mathrm{d}t + \sqrt{\alpha(t)}\left[k_1\mathrm{d}W_1(t) + k_2\mathrm{d}W_2(t)\right], \quad \alpha(0) = \alpha_0 \geq 0, \tag{7}$$

where κ, δ, k_1, k_2 are all positive constants and $\{W_2(t)\}_{t\in[0,T]}$ is another standard Brownian motion on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P})$, independent of $\{W_1(t)\}_{t\in[0,T]}$. Besides, we assume that the solution to square-root model (Eq.(7)) is almost surely non negative for any $t \in [0,T]$. Particularly, we do specify the structures of $\{\mu(t)\}_{t\in[0,T]}$ and $\{\sigma(t)\}_{t\in[0,T]}$in several examples to facilitate the understanding of our modeling framework.

Example 2.1. [CEV model] If $r(t) = r$, $\mu(t) = \mu$, and $\sigma(t) = \sigma S^{\ell}(t)$, where $r > 0$, $\mu > 0$, $\sigma > 0$, and $\ell \in \mathbb{R}$ such that $\mu \neq r$, then the risky asset price is given by the CEV model:

$$\mathrm{d}S(t) = S(t)\left[\mu\mathrm{d}t + \sigma S^{\ell}(t)\mathrm{d}W_1(t)\right], \quad S(0) = s_0 > 0. \tag{8}$$

Here ℓ is called the elasticity parameter of the risky asset. Applying Itô's formula to $S^{-2\ell}(t)$, then

$$\mathrm{d}S^{-2\ell}(t) = 2\ell\mu\left[\left(\ell + \frac{1}{2}\right)\frac{\sigma^2}{\mu} - S^{-2\ell}(t)\right]\mathrm{d}t - 2\ell\sigma S^{-\ell}(t)\mathrm{d}W_1(t). \tag{9}$$

If we set $\alpha(t) = S^{-2\ell}(t)$, $\kappa = 2\ell\mu$, $\delta = \left(\ell + \frac{1}{2}\right)\frac{\sigma^2}{\mu}$, $k_1 = -2\ell\sigma$, $k_2 = 0$, and $\lambda = \frac{\mu - r}{\sigma}$, then it is not difficult to see that the CEV model is a special case of the model given by Eqs.(4)-(7). In this case, the market price of risk is $\vartheta(t) = \frac{\mu-r}{\sigma}\sqrt{\alpha(t)} = \frac{\mu-r}{\sigma S^{\ell}(t)}$. Note that the risky asset price is always non negative in the CEV model and the reachability of the boundary $\{0\}$ depends on the value of ℓ. Hence, we do not need to impose any additional conditions on $\ell \in \mathbb{R}$ such that the stochastic factor $\alpha(t) = S^{-2\ell}(t)$ is well defined and non negative. Particularly, when $\ell = 0$, the price model reduces to the GBM motion model or the Black–Scholes model.

Example 2.2. [Heston model] If $r(t) = r$, $\mu(t) = r + \lambda\alpha(t)$, $\sigma(t) = \sqrt{\alpha(t)}$, $k_1 = \sigma\rho$, and $k_2 = \sigma\sqrt{1-\rho^2}$, where $r > 0$, $\lambda \in \mathbb{R}_0$, $\sigma > 0$, and $\rho \in (-1, 1)$, then the risky asset's price is governed by the Heston model:

$$\begin{cases} \mathrm{d}S(t) = S(t)\left[(r + \lambda\alpha(t))\,\mathrm{d}t + \sqrt{\alpha(t)}\mathrm{d}W_1(t)\right], \quad S(0) = s_0 > 0, \\ \mathrm{d}\alpha(t) = \kappa\left[\delta - \alpha(t)\right]\mathrm{d}t + \sigma\sqrt{\alpha(t)}\left[\rho\mathrm{d}W_1(t) + \sqrt{1-\rho^2}\mathrm{d}W_2(t)\right], \quad \alpha(0) = \alpha_0 > 0, \end{cases} \tag{10}$$

which is also a special case of the model given by Eqs.(4)-(7). Here, the stochastic factor process $\{\alpha(t)\}_{t\in[0,T]}$ is the variance process, $\kappa > 0$ is the mean-reversion rate, $\delta > 0$ is the long-run level, σ_0 is the volatility of volatility, and ρ is the correlation coefficient between the risky asset's price and the variance. In this case, the market price of risk is $\vartheta(t) = \lambda\sqrt{\alpha(t)}$. It is required that the Feller condition is satisfied, i.e., $2\kappa\delta \geq \sigma_0^2$, such that the variance $\alpha(t) > 0$ for any $t \in [0, T]$.

2.2. Ambiguity

The above-mentioned framework is a traditional asset allocation model, where each investor is assumed to be ambiguity neutral. However, in reality, the investor is usually ambiguity averse and wants to guard herself against worst-case scenarios. To incorporate ambiguity aversion into the investor's asset allocation problem, we assume that the reference model capturing the knowledge of the investor's ambiguity is described by the probability measure $\mathbb{P}$, but she is skeptical of this reference model and is willing to consider some alternative models, which are defined by a class of probability measures equivalent to $\mathbb{P}$ as follows:

$$\mathcal{Q} := \{\mathbb{Q}|\mathbb{Q} \sim \mathbb{P}\}.$$

For each $i \in \{1, 2\}$, define $\phi_i := \{\phi_i(t) := (\phi_{i1}(t), \phi_{i2}(t))\}_{t\in[0,T]}$, which satisfies two conditions: (i) $\phi_{i1}(t)$ and $\phi_{i2}(t)$ are $\mathcal{F}_t-$measurable for each $t \in [0, T]$; and (ii) $\mathbb{E}\left\{\exp\left\{\frac{1}{2}\int_0^T \left[\phi_{i1}^2(t) + \phi_{i2}^2(t)\right] \mathrm{d}t\right\}\right\} < \infty$. We denote $\mathit{\Phi}_i$ for the space of all such processes ϕ_i. Furthermore, we define a real-valued process $\left\{\Lambda^{\phi_i}(t)\right\}_{t\in[0,T]}$ as

$$\Lambda^{\phi_i}(t) = \exp\left\{-\int_0^t \phi_{i1}(s)\mathrm{d}W_1(s) - \frac{1}{2}\int_0^T \phi_{i1}^2(s)\mathrm{d}s - \int_0^t \phi_{i2}(s)\mathrm{d}W_2(s) - \frac{1}{2}\int_0^T \phi_{i2}^2(s)\mathrm{d}s\right\}. \tag{11}$$

Accordingly, $\Lambda^{\phi_i}(t)$ is a $\mathbb{P}-$martingale. For each ϕ_i, a new alternative measure $\mathbb{Q}_i$ that is absolutely continuous with $\mathbb{P}$ on $\mathcal{F}_T$ is defined by

$$\left.\frac{\mathrm{d}\mathbb{Q}_i}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_T} = \Lambda^{\phi_i}(T).$$

By Girsanov's theorem, under the alternative measure $\mathbb{Q}_i$, we have

$$\mathrm{d}W_1^{\mathbb{Q}_i}(t) = \mathrm{d}W_1(t) + \phi_{i1}(t)\mathrm{d}t,$$
$$\mathrm{d}W_2^{\mathbb{Q}_i}(t) = \mathrm{d}W_2(t) + \phi_{i2}(t)\mathrm{d}t,$$

where $W_1^{\mathbb{Q}_i}(t)$ and $W_2^{\mathbb{Q}_i}(t)$ are one-dimensional standard Brownian motions. Furthermore, the dynamic price process $S(t)$ of the risky asset and the stochastic factor process $\alpha(t)$ under $\mathbb{Q}_i$ can be written as

$$\mathrm{d}S(t) = S(t)\left[(\mu(t) - \phi_{i1}(t)\sigma(t))\,\mathrm{d}t + \sigma(t)\mathrm{d}W_1^{\mathbb{Q}_i}(t)\right], \tag{12}$$

$$\begin{aligned}\mathrm{d}\alpha(t) = &\left[\kappa\left(\delta - \alpha(t)\right) - \sqrt{\alpha(t)}\left(k_1\phi_{i1}(t) + k_2\phi_{i2}(t)\right)\right]\mathrm{d}t \\ &+ \sqrt{\alpha(t)}\left[k_1\mathrm{d}W_1^{\mathbb{Q}_i}(t) + k_2\mathrm{d}W_2^{\mathbb{Q}_i}(t)\right].\end{aligned} \tag{13}$$

2.3. Robust asset allocation game and optimality

Each investor aims to maximize the expected utility of her wealth at terminal time T relative to that of her competitor under the worst-case scenario of the alternative measure. Thus the robust optimization problem for investor $i \in \{1, 2\}$ is given by

$$\sup_{\pi_i \in \Pi_i} \inf_{\mathbb{Q}_i \in \mathcal{Q}} \mathbb{E}^{\mathbb{Q}_i}\left[U_i\left(X_i(T), X_j(T)\right) + \int_0^T \left(\frac{\phi_{i1}^2(s)}{2\Psi_{i1}(s)} + \frac{\phi_{i2}^2(s)}{2\Psi_{i2}(s)}\right)\mathrm{d}s\right], \tag{14}$$

subject to the dynamic budget constraint given by the stochastic differential equation (SDE):

$$\begin{aligned}\frac{\mathrm{d}X_i(t)}{X_i(t)} = &\left[r(t) + \pi_i(t)\left(\mu(t) - r(t) - \phi_{i1}(t)\frac{\mu(t) - r(t)}{\lambda\sqrt{\alpha(t)}}\right)\right]\mathrm{d}t \\ &+ \pi_i(t)\frac{\mu(t) - r(t)}{\lambda\sqrt{\alpha(t)}}\mathrm{d}W_1^{\mathbb{Q}_i}(t).\end{aligned} \tag{15}$$

In Eqs.(14) and (15), $X_i = \{X_i^{\pi_i}(t)\}_{t\in[0,T]}$ is investor i's assets wealth, and the set Π_i denotes investor i's admissible dynamic portfolio strategies $\pi_i = \{\pi_i(t)\}_{t\in[0,T]}$. For each $t \in [0, T]$, the process $\pi_i(t)$ represents the proportions of wealth invested in the risky asset at time t. The perturbations $\phi_{i1}(t)$ and $\phi_{i2}(t)$ in the penalty term are scaled by $\Psi_{i1}(t)$ and $\Psi_{i2}(t)$, respectively. $\Psi_{i1}(t)$ and $\Psi_{i2}(t)$ represent the preference parameters for ambiguity aversion and measure the degree of confidence in the reference model $\mathbb{P}$ at time t; and deviations from the reference measure are penalized by the last integral term in the expectation, which depends on the relative entropy arising from the diffusion risks. According to Maenhout (2004), the larger $\Psi_{i1}(t)$ and $\Psi_{i2}(t)$ are, the less the deviations from the reference model are penalized. Furthermore, the investor has less faith in the reference model, such that she is more likely to consider alternative models. Hence, the investor's ambiguity aversion is increasing with respect to $\Psi_{i1}(t)$ and $\Psi_{i2}(t)$.

Definition 2.1. [Admissible strategies] A strategy $\pi_i = \{\pi_i(t)\}_{t\in[0,T]}$ is said to be admissible for investor i, for $i = 1, 2$, if

(i) $\pi_i(t)$ is a $\mathcal{F}_t-$progressively measurable process and $\mathbb{E}^{\mathbb{Q}_i^*}\left[\int_0^T ||\pi_i(t)||^2 \mathrm{d}t\right] < \infty$, where $\mathbb{Q}_i^*$ is the chosen probability measure to describe the worst-case scenario and will be determined later;

(ii) For any $(t, x, \alpha) \in [0, T]\times\mathbb{R}\times\mathbb{R}^+$, Eq.(15) has a pathwise unique solution $\{X_i(t)\}_{t\in[0,T]}$ with $\mathbb{E}^{\mathbb{Q}_i^*}_{t,x_i,x_j,\alpha}\left[U_i\left(X_i(T), X_j(T)\right)\right] < \infty$, where $\mathbb{E}^{\mathbb{Q}_i^*}_{t,x_i,x_j,\alpha}[\cdot] = \mathbb{E}^{\mathbb{Q}_i^*}[\cdot|X_i(t) = x_i, X_j(t) = x_j, \alpha(t) = \alpha]$.

Note that, due to the relative wealth terms in their utility functions, the investor' asset allocation problems are linked and must be solved simultaneously. The state variables relevant for investor i are time, her own wealth, her peer's wealth, and the square root factor. Since the portfolio decision depends on her peer's choice π_j, this decision implicitly also determines investor i's risky portfolio share, π_i. In fact, problem (14) is a typical example of the non-cooperative, non-zero-sum stochastic differential game between two competing ambiguity-averse investors. Consequently, the solution to problem (14) is the Nash equilibrium of the non-zero-sum game between two competing ambiguity-averse investors, where the Nash equilibrium is the strategy profile $(\pi_i^*, \pi_j^*) \in \Pi_i \times \Pi_j$ such that, for all $(\pi_1, \pi_2) \in \Pi_i \times \Pi_j$,

$$\begin{cases} \inf\limits_{\mathbb{Q}_1\in\mathcal{Q}} \mathbb{E}^{\mathbb{Q}_1}\left[U_1\left(X_1^{\pi_1^*}(T), X_2^{\pi_2}(T)\right) + \int_0^T \left(\frac{\phi_{11}^2(s)}{2\Psi_{11}(s)} + \frac{\phi_{12}^2(s)}{2\Psi_{12}(s)}\right)\mathrm{d}s\right] \\ \quad \leq \inf\limits_{\mathbb{Q}_1\in\mathcal{Q}} \mathbb{E}^{\mathbb{Q}_1}\left[U_1\left(X_1^{\pi_1^*}(T), X_2^{\pi_2^*}(T)\right) + \int_0^T \left(\frac{\phi_{11}^2(s)}{2\Psi_{11}(s)} + \frac{\phi_{12}^2(s)}{2\Psi_{12}(s)}\right)\mathrm{d}s\right], \\ \inf\limits_{\mathbb{Q}_2\in\mathcal{Q}} \mathbb{E}^{\mathbb{Q}_2}\left[U_2\left(X_2^{\pi_2^*}(T), X_1^{\pi_1}(T)\right) + \int_0^T \left(\frac{\phi_{21}^2(s)}{2\Psi_{21}(s)} + \frac{\phi_{22}^2(s)}{2\Psi_{22}(s)}\right)\mathrm{d}s\right] \\ \quad \leq \inf\limits_{\mathbb{Q}_2\in\mathcal{Q}} \mathbb{E}^{\mathbb{Q}_2}\left[U_2\left(X_2^{\pi_2^*}(T), X_1^{\pi_1^*}(T)\right) + \int_0^T \left(\frac{\phi_{21}^2(s)}{2\Psi_{21}(s)} + \frac{\phi_{22}^2(s)}{2\Psi_{22}(s)}\right)\mathrm{d}s\right]. \end{cases} \tag{16}$$

3. SOLUTION TO THE GAME

In this section, we will first derive the robust Nash equilibrium asset allocation strategy. Then we shall present the verification theorem.

3.1. Nash equilibrium

We use a stochastic control approach and denote the candidates for investor i's value function by $V^i(t, x_i, x_j, \alpha)$, for $i \neq j \in \{1, 2\}$. The goal is to show that

$$V^i(t,x_i,x_j,\alpha)=\sup_{\pi_i\in\Pi_i}\inf_{\phi_i\in\Phi_i}\mathbb{E}^{\mathbb{Q}_i}\left[U_i\left(X_i(T),X_j(T)\right)+\int_t^T\left(\tfrac{\phi_{i1}^2(s)}{2\Psi_{i1}(s)}+\tfrac{\phi_{i2}^2(s)}{2\Psi_{i2}(s)}\right)\mathrm{d}s \right. \\ \left. |X_i(t)=x_i,X_j(t)=x_j,\alpha(t)=\alpha\right]. \tag{17}$$

According to the dynamic programming principle, the robust Hamilton-Jacobi-Bellman-Isaacs (HJBI) equation for investor i's problem can be derived as follows (for the sake of brevity, we omit the variable (t,x_i,x_j,α) in some functions):

$$\sup_{\pi_i\in\Pi_i}\inf_{\phi_i\in\Phi_i}\left\{\mathcal{L}^{\pi_i,\pi_j^*,\phi_i,\phi_j^*}W^i+\tfrac{\phi_{i1}^2}{2\Psi_{i1}}+\tfrac{\phi_{i2}^2}{2\Psi_{i2}}\right\}=0, \tag{18}$$

where the operator $\mathcal{L}$ is defined as

$$\begin{aligned}\mathcal{L}^{\pi_i,\pi_j,\phi_i,\phi_j}W^i \triangleq\ & W_t^i+\left[\kappa(\delta-\alpha)-\sqrt{\alpha}\left(k_1\phi_{i1}+k_2\phi_{i2}\right)\right]V_\alpha^i+\tfrac{1}{2}\alpha\left(k_1^2+k_2^2\right)V_{\alpha\alpha}^i\\ &+\left[r+\pi_i\left(\mu-r-\phi_{i1}\tfrac{\mu-r}{\lambda\sqrt{\alpha}}\right)\right]x_iV_{x_i}^i+\tfrac{1}{2}\pi_i^2(t)\tfrac{(\mu-r)^2}{\lambda^2\alpha}x_i^2V_{x_ix_i}^i\\ &+\left[r+\pi_j(t)\left(\mu-r-\phi_{j1}\tfrac{\mu-r}{\lambda\sqrt{\alpha}}\right)\right]x_jV_{x_j}^i+\tfrac{1}{2}\pi_j^2(t)\tfrac{(\mu-r)^2}{\lambda^2\alpha}x_j^2V_{x_jx_j}^i\\ &+\pi_i\pi_j\tfrac{(\mu-r)^2}{\lambda^2\alpha}x_ix_jV_{x_ix_j}^i+\pi_i\tfrac{k_1}{\lambda}\left(\mu-r\right)x_iV_{x_i\alpha}^i+\pi_j\tfrac{k_1}{\lambda}\left(\mu-r\right)x_jV_{x_j\alpha}^i.\end{aligned}$$

For analytical tractability, refer to the method of Maenhout (2004), we assume that the preference functions $\Psi_{i1}(t)$ and $\Psi_{i2}(t)$ are state dependent and have the form

$$\Psi_{i1}(t)=\frac{\beta_{i1}}{(1-\gamma_i)V^i(t,x_i,x_j,\alpha)},\quad \Psi_{i2}(t)=\frac{\beta_{i2}}{(1-\gamma_i)V^i(t,x_i,x_j,\alpha)}, \tag{19}$$

where β_{i1} and β_{i2} are nonnegative parameters representing the ambiguity-averse level of investor i to the diffusion risk from risky asset and the stochastic factor process, respectively. When $\beta_{i1}=\beta_{i2}=0$, investor i is ambiguity-neutral for the diffusion risk.

The following theorem provides the candidates for the solution of the investors' robust asset allocation game:

THEOREM 1 (Solution to HJBI (18)). *Suppose that $B_1(t)$ and $B_2(t)$ solve the following coupled system of ordinary differential equations (ODEs):*

$$\dot{B}_1(t)=\xi_{11}+\xi_{12}B_1(t)+\xi_{13}B_1^2(t)+\xi_{14}B_2(t)+\xi_{15}B_2^2(t)+\xi_{16}B_1(t)B_2(t), \tag{20}$$

$$\dot{B}_2(t)=\xi_{21}+\xi_{22}B_2(t)+\xi_{23}B_2^2(t)+\xi_{24}B_1(t)+\xi_{25}B_1^2(t)+\xi_{26}B_2(t)B_1(t), \tag{21}$$

subject to the boundary conditions $B_1(T)=B_2(T)=0$, where the constants $\xi_{i1},\xi_{i2},\cdots,\xi_{i6}$, $i=1,2$, are given by (A.13)-(A.18). *Then the solution to* *the HJBI equation* (18) *is* $V^i(t,x_i,x_j,\alpha)$, *for* $i=1,2$, *which admits the following explicit form:*

$$V^i(t,x_i,x_j,\alpha)=\frac{1}{1-\gamma_i}\left(x_i x_j^{\theta_i-1}\right)^{1-\gamma_i}\exp\left[A_i(t)-B_i(t)\alpha\right], \tag{22}$$

where $A_i(t)$ *is given by* (A.19). *The associated maximizer for the HJBI equation is given by*

$$\pi_i^*(t)=\frac{\lambda\alpha(t)}{(1-\nu_i\nu_j)(\gamma_i+\beta_{i1})(\mu(t)-r(t))}\left[\lambda(1+\nu_i)-k_1\left(\frac{1-\gamma_i-\beta_{i1}}{1-\gamma_i}B_i(t)+\frac{1-\gamma_j-\beta_{j1}}{1-\gamma_j}\nu_i B_j(t)\right)\right], \tag{23}$$

where $\nu_i=(1-\gamma_i)(\theta_i-1)/(\gamma_j+\beta_{j1})$, $i\neq j\in\{1,2\}$, *satisfy* $1+\nu_i>0$ *and* $1-\nu_i\nu_j>0$.

Proof. See Appendix 1. ▮

Remark 3.1. (The case of ambiguity-neutral). For $i=1,2$, if all of the ambiguity-aversion coefficients β_{i1} and β_{i2} equal to 0, i.e., $\beta_{i1}=\beta_{i2}=0$, our model reduces to a classical non-zero-sum stochastic differetnial game between two competing CRRA investors. Investor i solves the optimization problem

$$\sup_{\hat{\pi}_i\in\Pi_i}\mathbb{E}^{\mathbb{P}}\left[\frac{1}{1-\gamma_i}\left(\widehat{X}_i(T)\widehat{X}_j^{\theta_i-1}(T)\right)^{1-\gamma_i}\right], \tag{24}$$

where $\left\{\widehat{X}_i(t)\right\}_{t\in[0,T]}$ and $\left\{\widehat{X}_j(t)\right\}_{t\in[0,T]}$ have dynamics

$$\begin{cases}\mathrm{d}\widehat{X}_i(t)=\widehat{X}_i(t)\left[r(t)+\hat{\pi}_i(t)\left(\mu(t)-r(t)\right)\right]\mathrm{d}t+\hat{\pi}_i(t)\widehat{X}_i(t)\frac{\mu(t)-r(t)}{\lambda\sqrt{\alpha(t)}}\mathrm{d}W_1(t),\\ \mathrm{d}\widehat{X}_j(t)=\widehat{X}_j(t)\left[r(t)+\hat{\pi}_j(t)\left(\mu(t)-r(t)\right)\right]\mathrm{d}t+\hat{\pi}_j(t)\widehat{X}_j(t)\frac{\mu(t)-r(t)}{\lambda\sqrt{\alpha(t)}}\mathrm{d}W_1(t).\end{cases} \tag{25}$$

Using the method similar to solve HJBI equation (18), we can derive the Nash equilibrium strategy profile $(\hat{\pi}_1^*,\hat{\pi}_2^*)$ as

$$\begin{cases}\hat{\pi}_1^*(t)=\frac{\lambda\alpha(t)}{\gamma_1(1-\nu_1\nu_2)(\mu(t)-r(t))}\left[\lambda(1+\nu_1)-k_1\left(\widehat{B}_1(t)+\widehat{B}_2(t)\hat{\nu}_1\right)\right],\\ \hat{\pi}_2^*(t)=\frac{\lambda\alpha(t)}{\gamma_2(1-\nu_1\nu_2)(\mu(t)-r(t))}\left[\lambda(1+\nu_1)-k_1\left(\widehat{B}_2(t)+\widehat{B}_1(t)\hat{\nu}_2\right)\right],\end{cases} \tag{26}$$

with $\hat{\nu}_1=(1-\gamma_1)(\theta_1-1)/\gamma_2$, $\hat{\nu}_2=(1-\gamma_2)(\theta_2-1)/\gamma_1$ satisfying $1+\hat{\nu}_1>0$, $1+\hat{\nu}_2>0$, $1-\hat{\nu}_1\hat{\nu}_2>0$. And $\widehat{B}_1(t)$ and $\widehat{B}_2(t)$ are the solutions of the following coupled system of ODEs:

$$\dot{\widehat{B}}_1(t)=\hat{\xi}_{11}+\hat{\xi}_{12}\widehat{B}_1(t)+\hat{\xi}_{13}\widehat{B}_1^2(t)+\hat{\xi}_{14}\widehat{B}_2(t)+\hat{\xi}_{15}\widehat{B}_2^2(t)+\hat{\xi}_{16}\widehat{B}_1(t)\widehat{B}_2(t), \tag{27}$$

$$\dot{\widehat{B}}_2(t) = \hat{\xi}_{21} + \hat{\xi}_{22}\widehat{B}_2(t) + \hat{\xi}_{23}\widehat{B}_2^2(t) + \hat{\xi}_{24}\widehat{B}_1(t) + \hat{\xi}_{25}\widehat{B}_1^2(t) + \hat{\xi}_{26}\widehat{B}_2(t)\widehat{B}_1(t), \tag{28}$$

with the terminal condition $\widehat{B}_1(T) = \widehat{B}_2(T) = 0$, where the constants $\hat{\xi}_{i1}, \hat{\xi}_{i2}, \cdots, \hat{\xi}_{i6}$, $i = 1, 2$, are given by

$$\begin{cases} \hat{\xi}_{i1} = \frac{\lambda^2\hat{\nu}_i(1+\hat{\nu}_j)}{1-\hat{\nu}_i\hat{\nu}_j} + \frac{\lambda^2(1+\hat{\nu}_i)^2(1-\gamma_i)}{2(1-\hat{\nu}_i\hat{\nu}_j)^2\gamma_i} - \frac{\lambda^2\hat{\nu}_i(1+\hat{\nu}_j)^2[1-(\theta_i-1)(1-\gamma_i)]}{2(1-\hat{\nu}_i\hat{\nu}_j)^2\gamma_j}, \\ \hat{\xi}_{i2} = \kappa - \frac{\lambda k_1\hat{\nu}_i(1+\hat{\nu}_j)}{1-\hat{\nu}_i\hat{\nu}_j} - \frac{\lambda k_1\hat{\nu}_i\hat{\nu}_j}{(1-\hat{\nu}_i\hat{\nu}_j)} - \frac{\lambda k_1(1+\hat{\nu}_i)(1-\gamma_i)}{(1-\hat{\nu}_i\hat{\nu}_j)^2\gamma_i} + \frac{\lambda k_1\hat{\nu}_i\hat{\nu}_j(1+\hat{\nu}_j)[1-(\theta_i-1)(1-\gamma_i)]}{(1-\hat{\nu}_i\hat{\nu}_j)^2\gamma_j}, \\ \hat{\xi}_{i3} = \frac{k_1^2+k_2^2}{2} + \frac{k_1^2\hat{\nu}_i\hat{\nu}_j}{(1-\hat{\nu}_i\hat{\nu}_j)} + \frac{k_1^2(1-\gamma_i)}{2(1-\hat{\nu}_i\hat{\nu}_j)^2\gamma_i} - \frac{k_1^2\hat{\nu}_i\hat{\nu}_j^2[1-(\theta_i-1)(1-\gamma_i)]}{2(1-\hat{\nu}_i\hat{\nu}_j)^2\gamma_j}, \\ \hat{\xi}_{i4} = -\frac{\lambda k_1\hat{\nu}_i}{(1-\hat{\nu}_i\hat{\nu}_j)} - \frac{\lambda k_1\hat{\nu}_i(1+\hat{\nu}_i)(1-\gamma_i)}{(1-\hat{\nu}_i\hat{\nu}_j)^2\gamma_i} + \frac{\lambda k_1\hat{\nu}_i(1+\hat{\nu}_j)[1-(\theta_i-1)(1-\gamma_i)]}{(1-\hat{\nu}_i\hat{\nu}_j)^2\gamma_j}, \\ \hat{\xi}_{i5} = \frac{k_1^2\hat{\nu}_i^2(1-\gamma_i)}{2(1-\hat{\nu}_i\hat{\nu}_j)^2\gamma_i} - \frac{k_1^2\hat{\nu}_i[1-(\theta_i-1)(1-\gamma_i)]}{2(1-\hat{\nu}_i\hat{\nu}_j)^2\gamma_j}, \\ \hat{\xi}_{i6} = \frac{k_1^2\hat{\nu}_i}{1-\hat{\nu}_i\hat{\nu}_j} + \frac{k_1^2\hat{\nu}_i(1-\gamma_i)}{(1-\hat{\nu}_i\hat{\nu}_j)^2\gamma_i} - \frac{k_1^2\hat{\nu}_i\hat{\nu}_j[1-(\theta_i-1)(1-\gamma_i)]}{(1-\hat{\nu}_i\hat{\nu}_j)^2\gamma_j}. \end{cases} \tag{29}$$

Moreover, the optimal value function of investor i is

$$\widehat{V}^i(t, x_i, x_j, \alpha) = \frac{1}{1-\gamma_i}\left(x_i x_j^{\theta_i - 1}\right)^{1-\gamma_i} \exp\left[\widehat{A}_i(t) - \widehat{B}_i(t)\alpha\right], \tag{30}$$

where $\widehat{A}_i(t)$ is given by

$$\widehat{A}_i(t) = \theta_i\left(1-\gamma_i\right)\int_t^T r(s)\mathrm{d}s - \kappa\delta\int_t^T \widehat{B}_i(s)\mathrm{d}s.$$

3.2. Verification

Based on the HJBI system identified in Section 3.1, we provide a verification theorem to show that the solution V^i of the HJBI system and corresponding policy π_i^* coincide with the investors' value functions and optimal asset allocation strategies for the robust asset allocation game, respectively.

THEOREM 2. *Suppose there exists a function* $\widetilde{W}^i(t, x_i, x_j, \alpha) \in C^{1,2}\left([0,T]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^+\right)$ *and a Markov control* $(\pi_i^*, \phi_i^*) \in \Pi_i \times \Phi_i$ *such that*

(i). for any $\phi_i \in \Phi_i$, $\mathcal{L}^{\pi_i^*,\pi_j^*,\phi_i,\phi_j^*}\widetilde{W}^i + \frac{\phi_{i1}^2}{2\Psi_{i1}} + \frac{\phi_{i2}^2}{2\Psi_{i2}} \geq 0$;

(ii). for any $\pi_i \in \Pi_i$, $\mathcal{L}^{\pi_i,\pi_j^*,\phi_i^*,\phi_j^*}\widetilde{W}^i + \frac{\phi_{i1}^{*2}}{2\Psi_{i1}} + \frac{\phi_{i2}^{*2}}{2\Psi_{i2}} \leq 0$;

(iii). $\mathcal{L}^{\pi_i^*,\pi_j^*,\phi_i^*,\phi_j^*}\widetilde{W}^i + \frac{\phi_{i1}^{*2}}{2\Psi_{i1}} + \frac{\phi_{i2}^{*2}}{2\Psi_{i2}} = 0$;

(iv). for all $(\pi_i, \phi_i) \in \Pi_i\times\Phi_i$, $\lim_{t\to T}\widetilde{W}^i\left(t, X_i^{\pi_i}(t), X_j^{\pi_j^*}(t), \alpha(t)\right) = U_i\left(X_i^{\pi_i}(T), X_j^{\pi_j^*}(T)\right)$;

(v). $\left\{\widetilde{W}^i\left(\tau, X_i^{\pi_i}(\tau), X_j^{\pi_j^*}(\tau), \alpha(\tau)\right)\right\}_{\tau\in\mathcal{T}}$ *and* $\left\{\frac{\phi_{i1}^2(\tau)}{2\Psi_{i1}(\tau)}+\frac{\phi_{i2}^2(\tau)}{2\Psi_{i2}(\tau)}\right\}_{\tau\in\mathcal{T}}$ *are uniformly integrable, where* $\mathcal{T}$ *denotes the set of all stopping times satisfying* $\tau \leq \mathcal{T}$.

Then π_i^* *is the optimal strategies and* $\widetilde{W}^i\left(t, x_i, x_j, \alpha\right) = V^i\left(t, x_i, x_j, \alpha\right)$ *is the associated value function.*

Proof. The proof of this theorem is standard, and thus omitted for simplicity. One can refer to Theorem 4.1 in Zeng and Taksar (2013), Theorem 3.2 in Mataramvura and Øksendal (2008) and Proposition 2.4 in Kraft et al. (2020). ▮

4. SPECIAL CASES

The square-root model used to describe the price process of the risky asset can reduce to some well-known models, such as GBM, CEV model, and Heston model. In this section, we provide the solutions under the CEV and Heston models, respectively.

4.1. Nash equilibrium strategy under the CEV model

In this case, we discuss the optimization problem under the CEV model in Example 1. Then the robust optimization problem for investor $i \in \{1,2\}$ becomes

$$\begin{cases} \sup\limits_{\pi_i\in\Pi_i}\inf\limits_{\mathbb{Q}_i\in\mathcal{Q}}\mathbb{E}^{\mathbb{Q}_i}\left[\frac{1}{1-\gamma_i}\left(X_i(T)X_j^{\theta_i-1}(T)\right)^{1-\gamma_i}+\int_0^T\left(\frac{\phi_{i1}^2(s)}{2\Psi_{i1}(s)}+\frac{\phi_{i2}^2(s)}{2\Psi_{i2}(s)}\right)\mathrm{d}s\right], \\ \text{subject to the budget constraint} \\ \mathrm{d}X_i(t) = X_i(t)\left[r+\pi_i(t)\left(\mu-r-\phi_{i1}(t)\sigma S^{\ell}(t)\right)\right]\mathrm{d}t+\pi_i(t)X_i(t)\sigma S^{\ell}(t)\mathrm{d}W_1^{\mathbb{Q}_i}(t). \end{cases} \tag{31}$$

THEOREM 3. *For the robust asset allocation game between two competing ambiguity-averse investors under the CEV model, the Nash equilibrium strategy* π_i^* *of investor* i*, for* $i=1,2$*, is given by*

$$\pi_i^*(t) = \frac{(\mu-r)(1+\nu_i)+2\ell\sigma^2\left[\frac{1-\gamma_i-\beta_{i1}}{1-\gamma_i}B_i(t)+\frac{1-\gamma_j-\beta_{j1}}{1-\gamma_j}B_j(t)\nu_i\right]}{(1-\nu_i\nu_j)(\gamma_i+\beta_{i1})\sigma^2 s^{2\ell}}, \tag{32}$$

where $\nu_i = (1-\gamma_i)(\theta_i-1)/(\gamma_j+\beta_{j1})$, $i \neq j \in \{1,2\}$, *satisfy* $1+\nu_i>0$, $1-\nu_i\nu_j>0$; $B_i(t)$ *and* $B_j(t)$ *are the solutions of the following coupled* *system of ODEs:*

$$\begin{cases} \dot{B}_i(t) = \xi_{i1} + \xi_{i2}B_i(t) + \xi_{i3}B_i^2(t) + \xi_{i4}B_j(t) + \xi_{i5}B_j^2(t) + \xi_{i6}B_i(t)B_j(t), \\ \dot{B}_j(t) = \xi_{j1} + \xi_{j2}B_j(t) + \xi_{j3}B_j^2(t) + \xi_{j4}B_i(t) + \xi_{j5}B_i^2(t) + \xi_{j6}B_j(t)B_i(t), \end{cases} \tag{33}$$

with the terminal condition $B_i(T) = B_j(T) = 0$*, where the constants* $\xi_{i1}, \xi_{i2}, \cdots, \xi_{i6}$*,* $i = 1, 2$ *are given by*

$$\begin{cases} \xi_{i1} = \frac{(\mu-r)^2\nu_i(1+\nu_j)}{\sigma^2(1-\nu_i\nu_j)} + \frac{(\mu-r)^2(1+\nu_i)^2(1-\gamma_i)}{2\sigma^2(1-\nu_i\nu_j)^2(\gamma_i+\beta_{i1})} - \frac{(\mu-r)^2\nu_i(1+\nu_j)^2[2\beta_{j1}-(\theta_i-1)(1-\gamma_i)+1]}{2\sigma^2(1-\nu_i\nu_j)^2(\gamma_j+\beta_{j1})}, \\ \xi_{i2} = 2\ell\mu + \frac{2(\mu-r)\ell\sigma\nu_i(1+\nu_j)}{\sigma(1-\nu_i\nu_j)} + \frac{2(\mu-r)\ell\sigma\nu_i\nu_j(1-\gamma_i-\beta_{i1})}{\sigma(1-\nu_i\nu_j)(1-\gamma_i)} + \frac{2(\mu-r)\ell(1+\nu_i)(1-\gamma_i-\beta_{i1})}{(1-\nu_i\nu_j)^2(\gamma_i+\beta_{i1})} \\ \quad - \frac{2(\mu-r)\ell\nu_i\nu_j(1+\nu_j)(1-\gamma_i-\beta_{i1})[2\beta_{j1}-(\theta_i-1)(1-\gamma_i)+1]}{(1-\nu_i\nu_j)^2(\gamma_j+\beta_{j1})(1-\gamma_i)}, \\ \xi_{i3} = \frac{2\ell^2\sigma^2(1-\gamma_i-\beta_{i1})}{1-\gamma_i} + \frac{4\ell^2\sigma^2\nu_i\nu_j(1-\gamma_i-\beta_{i1})}{(1-\nu_i\nu_j)(1-\gamma_i)} + \frac{2\ell^2\sigma^2(1-\gamma_i-\beta_{i1})^2}{(1-\nu_i\nu_j)^2(\gamma_i+\beta_{i1})(1-\gamma_i)} \\ \quad - \frac{2\ell^2\sigma^2\nu_i\nu_j^2(1-\gamma_i-\beta_{i1})^2[2\beta_{j1}-(\theta_i-1)(1-\gamma_i)+1]}{(1-\nu_i\nu_j)^2(\gamma_j+\beta_{j1})(1-\gamma_i)^2}, \\ \xi_{i4} = -\frac{2\ell\sigma\nu_i(2\beta_{j1}+\beta_{j1}\nu_j-1+\gamma_j)}{(1-\nu_i\nu_j)(1-\gamma_j)} + \frac{2(\mu-r)\ell\nu_i(1+\nu_i)(1-\gamma_j-\beta_{j1})(1-\gamma_i)}{(1-\nu_i\nu_j)^2(\gamma_i+\beta_{i1})(1-\gamma_j)} \\ \quad - \frac{2(\mu-r)\ell\nu_i(1+\nu_j)(1-\gamma_j-\beta_{j1})[2\beta_{j1}-(\theta_i-1)(1-\gamma_i)+1]}{(1-\nu_i\nu_j)^2(\gamma_j+\beta_{j1})(1-\gamma_j)}, \\ \xi_{i5} = -\frac{4\ell^2\sigma^2\nu_i\beta_{j1}(1-\gamma_j-\beta_{j1})}{(1-\nu_i\nu_j)(1-\gamma_j)^2} + \frac{4\ell^2\sigma^2\nu_i^2(1-\gamma_j-\beta_{j1})^2(1-\gamma_i)}{2(1-\nu_i\nu_j)^2(\gamma_i+\beta_{i1})(1-\gamma_j)^2} - \frac{4\ell^2\sigma^2\nu_i(1-\gamma_j-\beta_{j1})^2[2\beta_{j1}-(\theta_i-1)(1-\gamma_i)+1]}{2(1-\nu_i\nu_j)^2(\gamma_j+\beta_{j1})(1-\gamma_j)^2}, \\ \xi_{i6} = \frac{4\ell^2\sigma^2\nu_i(1-\gamma_j-\beta_{j1})}{(1-\nu_i\nu_j)(1-\gamma_j)} - \frac{4\ell^2\sigma^2\nu_i\nu_j\beta_{j1}(1-\gamma_i-\beta_{i1})}{(1-\nu_i\nu_j)(1-\gamma_i)(1-\gamma_j)} + \frac{4\ell^2\sigma^2\nu_i(1-\gamma_i-\beta_{i1})(1-\gamma_j-\beta_{j1})}{(1-\nu_i\nu_j)^2(\gamma_i+\beta_{i1})(1-\gamma_j)} \\ \quad - \frac{4\ell^2\sigma^2\nu_i\nu_j(1-\gamma_i-\beta_{i1})(1-\gamma_j-\beta_{j1})[2\beta_{j1}-(\theta_i-1)(1-\gamma_i)+1]}{(1-\nu_i\nu_j)^2(\gamma_j+\beta_{j1})(1-\gamma_i)(1-\gamma_j)}. \end{cases}$$

Furthermore, the equilibrium value function is

$$V^i(t, x_i, x_j, s) = \frac{1}{1-\gamma_i}\left(x_i x_j^{\theta_i - 1}\right)^{1-\gamma_i} \exp\left[A_i(t) - B_i(t)s^{-2\ell}\right], \tag{34}$$

where $A_i(t)$ *is given by*

$$A_i(t) = \theta_i(1-\gamma_i)\int_t^T r(s)\mathrm{d}s - \kappa\delta\int_t^T B_i(s)\mathrm{d}s.$$

The proof of Theorem 3 is similar to that of Theorem 1, so we omit it here.

4.2. Nash equilibrium strategy under the Heston model

In this case, we assume that the risky asset $S(t)$ follows the Heston model described by Example 2, and then the robust optimization problem for investor $i \in \{1,2\}$ becomes

$$\begin{cases} \sup\limits_{\pi_i \in \Pi_i} \inf\limits_{\mathbb{Q}_i \in \mathcal{Q}} \mathbb{E}^{\mathbb{Q}_i} \left[\frac{1}{1-\gamma_i} \left(X_i(T) X_j^{\theta_i - 1} \right)^{1-\gamma_i} + \int_0^T \left(\frac{\phi_{i1}^2(s)}{2\Psi_{i1}(s)} + \frac{\phi_{i2}^2(s)}{2\Psi_{i2}(s)} \right) \mathrm{d}s \right], \\ \text{subject to the budget constraint} \\ \mathrm{d}X_i(t) = X_i(t) \left[r + \pi_i(t) \left(\lambda\alpha(t) - \phi_{i1}(t)\sqrt{\alpha(t)} \right) \right] \mathrm{d}t + \pi_i(t) X_i(t) \sqrt{\alpha(t)} \mathrm{d}W_1^{\mathbb{Q}_i}(t). \end{cases} \tag{35}$$

THEOREM 4. *For the robust asset allocation game between two competing ambiguity-averse investors under the Heston model, the Nash equilibrium strategy π_i^* of investor i, for $i = 1, 2$, is given by*

$$\pi_i^*(t) = \frac{\lambda (1 + \nu_i) - \sigma\rho \left[\frac{1-\gamma_i-\beta_{i1}}{1-\gamma_i} B_i(t) + \frac{1-\gamma_j-\beta_{j1}}{1-\gamma_j} B_j(t)\nu_i \right]}{(1 - \nu_i\nu_j)(\gamma_i + \beta_{i1})}, \tag{36}$$

where $\nu_i = (1-\gamma_i)(\theta_i - 1)/(\gamma_j + \beta_{j1})$, $i \neq j \in \{1,2\}$, satisfy $1 + \nu_i > 0$ and $1 - \nu_i\nu_j > 0$; $B_i(t)$ and $B_j(t)$ are the solutions of the following coupled system of ODEs:

$$\begin{cases} \dot{B}_i(t) = \xi_{i1} + \xi_{i2} B_i(t) + \xi_{i3} B_i^2(t) + \xi_{i4} B_j(t) + \xi_{i5} B_j^2(t) + \xi_{i6} B_i(t) B_j(t), \\ \dot{B}_j(t) = \xi_{j1} + \xi_{j2} B_j(t) + \xi_{j3} B_j^2(t) + \xi_{j4} B_i(t) + \xi_{j5} B_i^2(t) + \xi_{j6} B_j(t) B_i(t), \end{cases} \tag{37}$$

with the terminal condition $B_i(T) = B_j(T) = 0$, where the constants $\xi_{i1}, \xi_{i2}, \cdots, \xi_{i6}$, $i = 1, 2$ are given by

$$\begin{cases} \xi_{i1} = \frac{\lambda^2\nu_i(1+\nu_j)}{1-\nu_i\nu_j} + \frac{\lambda^2(1+\nu_i)^2(1-\gamma_i)}{2(1-\nu_i\nu_j)^2(\gamma_i+\beta_{i1})} - \frac{\lambda^2\nu_i(1+\nu_j)^2[2\beta_{j1}-(\theta_i-1)(1-\gamma_i)+1]}{2(1-\nu_i\nu_j)^2(\gamma_j+\beta_{j1})}, \\ \xi_{i2} = \kappa - \frac{\lambda\sigma\rho\nu_i(1+\nu_j)}{1-\nu_i\nu_j} - \frac{\lambda\sigma\rho\nu_i\nu_j(1-\gamma_i-\beta_{i1})}{(1-\nu_i\nu_j)(1-\gamma_i)} - \frac{\lambda\sigma\rho(1+\nu_i)(1-\gamma_i-\beta_{i1})}{(1-\nu_i\nu_j)^2(\gamma_i+\beta_{i1})} \\ \quad + \frac{\lambda\sigma\rho\nu_i\nu_j(1+\nu_j)(1-\gamma_i-\beta_{i1})[2\beta_{j1}-(\theta_i-1)(1-\gamma_i)+1]}{(1-\nu_i\nu_j)^2(\gamma_j+\beta_{j1})(1-\gamma_i)}, \\ \xi_{i3} = \frac{\sigma^2\rho^2(1-\gamma_i-\beta_{i1})+\sigma^2\left(1-\rho^2\right)(1-\gamma_i-\beta_{i2})}{2(1-\gamma_i)} + \frac{\sigma^2\rho^2\nu_i\nu_j(1-\gamma_i-\beta_{i1})}{(1-\nu_i\nu_j)(1-\gamma_i)} + \frac{\sigma^2\rho^2(1-\gamma_i-\beta_{i1})^2}{2(1-\nu_i\nu_j)^2(\gamma_i+\beta_{i1})(1-\gamma_i)} \\ \quad - \frac{\sigma^2\rho^2\nu_i\nu_j^2(1-\gamma_i-\beta_{i1})^2[2\beta_{j1}-(\theta_i-1)(1-\gamma_i)+1]}{2(1-\nu_i\nu_j)^2(\gamma_j+\beta_{j1})(1-\gamma_i)^2}, \\ \xi_{i4} = \frac{\lambda\sigma\rho\nu_i(2\beta_{j1}+\beta_{j1}\nu_j-1+\gamma_j)}{(1-\nu_i\nu_j)(1-\gamma_j)} - \frac{\lambda\sigma\rho\nu_i(1+\nu_i)(1-\gamma_j-\beta_{j1})(1-\gamma_i)}{(1-\nu_i\nu_j)^2(\gamma_i+\beta_{i1})(1-\gamma_j)} \\ \quad + \frac{\lambda\sigma\rho\nu_i(1+\nu_j)(1-\gamma_j-\beta_{j1})[2\beta_{j1}-(\theta_i-1)(1-\gamma_i)+1]}{(1-\nu_i\nu_j)^2(\gamma_j+\beta_{j1})(1-\gamma_j)}, \\ \xi_{i5} = -\frac{\sigma^2\rho^2\nu_i\beta_{j1}(1-\gamma_j-\beta_{j1})}{(1-\nu_i\nu_j)(1-\gamma_j)^2} + \frac{\sigma^2\rho^2\nu_i^2(1-\gamma_j-\beta_{j1})^2(1-\gamma_i)}{2(1-\nu_i\nu_j)^2(\gamma_i+\beta_{i1})(1-\gamma_j)^2} - \frac{\sigma^2\rho^2\nu_i(1-\gamma_j-\beta_{j1})^2[2\beta_{j1}-(\theta_i-1)(1-\gamma_i)+1]}{2(1-\nu_i\nu_j)^2(\gamma_j+\beta_{j1})(1-\gamma_j)^2}, \\ \xi_{i6} = \frac{\sigma^2\rho^2\nu_i(1-\gamma_j-\beta_{j1})}{(1-\nu_i\nu_j)(1-\gamma_j)} - \frac{\sigma^2\rho^2\nu_i\nu_j\beta_{j1}(1-\gamma_i-\beta_{i1})}{(1-\nu_i\nu_j)(1-\gamma_i)(1-\gamma_j)} + \frac{\sigma^2\rho^2\nu_i(1-\gamma_i-\beta_{i1})(1-\gamma_j-\beta_{j1})}{(1-\nu_i\nu_j)^2(\gamma_i+\beta_{i1})(1-\gamma_j)} \\ \quad - \frac{\sigma^2\rho^2\nu_i\nu_j(1-\gamma_i-\beta_{i1})(1-\gamma_j-\beta_{j1})[2\beta_{j1}-(\theta_i-1)(1-\gamma_i)+1]}{(1-\nu_i\nu_j)^2(\gamma_j+\beta_{j1})(1-\gamma_i)(1-\gamma_j)}. \end{cases}$$

Furthermore, the equilibrium value function is

$$V^i(t, x_i, x_j, \alpha) = \frac{1}{1-\gamma_i} \left(x_i x_j^{\theta_i - 1} \right)^{1-\gamma_i} \exp\left[A_i(t) - B_i(t)\alpha \right], \tag{38}$$

where $A_i(t)$ is given by

$$A_i(t) = \theta_i\left(1-\gamma_i\right)\int_t^T r(s)\mathrm{d}s - \kappa\delta\int_t^T B_i(s)\mathrm{d}s.$$

The proof of Theorem 4 is similar to that of Theorem 1, and thus, we omit it here.

Remark 4.1. If $\beta_{i1}=\beta_{i2}=0$, the optimization problem (35) degenerated to a classical non-zero-sum stochastic differetnial game between two competing ambiguity neutral investors. The result becomes that in Kraft et al. (2020)

Remark 4.2. From (36), for $i=1,2$, we can rewrite ith investor's optimal asset allocation strategy as

$$\begin{aligned}\pi_i^*(t) = {} & \underbrace{\frac{\lambda}{\gamma_i+\beta_{i1}}}_{Myopic} + \underbrace{\frac{\lambda\nu_i\left(1+\nu_j\right)}{\left(1-\nu_i\nu_j\right)\left(\gamma_i+\beta_{i1}\right)}}_{Myopic\ benchmarking} - \underbrace{\frac{1-\gamma_i-\beta_{i1}}{\gamma_i+\beta_{i1}}\frac{\sigma\rho}{1-\gamma_i}B_i(t)}_{Hedge} \\ & \underbrace{-\frac{\sigma\rho}{\left(1-\nu_i\nu_j\right)\left(\gamma_i+\beta_{i1}\right)}\left[\frac{1-\gamma_i-\beta_{i1}}{1-\gamma_i}B_i(t)(-\nu_i\nu_j) + \frac{1-\gamma_j-\beta_{j1}}{1-\gamma_j}B_j(t)\nu_i\right]}_{Hedge\ benchmarking}.\end{aligned}$$

From the above equation, we can find that the optimal asset allocation strategy consist of a myopic term, an adjustment term for relative wealth concerns, an hedging term, and an additional hedging term for relative wealth concerns. In the utility function (1), if $\theta_i=1$, the optimal strategy $\pi_i^*(t)$ of investor i would simlpy be the optimal asset allocation strategy arising from classical (single-agent) utility maximization problem. In this case, the optimal asset allocation strategy only consist of a myopic term and an hedging term, terms generated by relative wealth concerns do not appear, this is consistent with the optimal portfolio choice given by Eq. (33) in Xu et al. (2011).

5. NUMERICAL STUDIES

In this section, we provide several numerical examples to illustrate the effects of model parameters on the equilibrium asset allocation strategies. We consider two special cases for the CEV model and Heston model, respectively. To improve the credibility of our results, we fix a set of basic parameters for CEV model and Heston model (Table 1) using data from Shen and Zeng (2015) and Kraft et al. (2020).

TABLE 1.

Model parameters.

CEV model							
r	μ	σ	ℓ	s_0	T		
0.05	0.12	0.2	0.3	0.5	10		
Heston model							
r	λ	κ	δ	σ	ρ	α_0	T
0.05	0.5	2	0.4	0.3	0.3	0.04	10
Investor 1				Investor 2			
γ_1	θ_1	β_{11}	β_{12}	γ_2	θ_2	β_{21}	β_{22}
5	0.4	3	1	7	0.6	4	2

Example 5.1. In this example, numerical analyses of the equilibrium asset allocation strategy under the CEV model are shown in Figures 1–3.

FIG. 1. Effects of parameters μ and r on π_i, for $i = 1, 2$

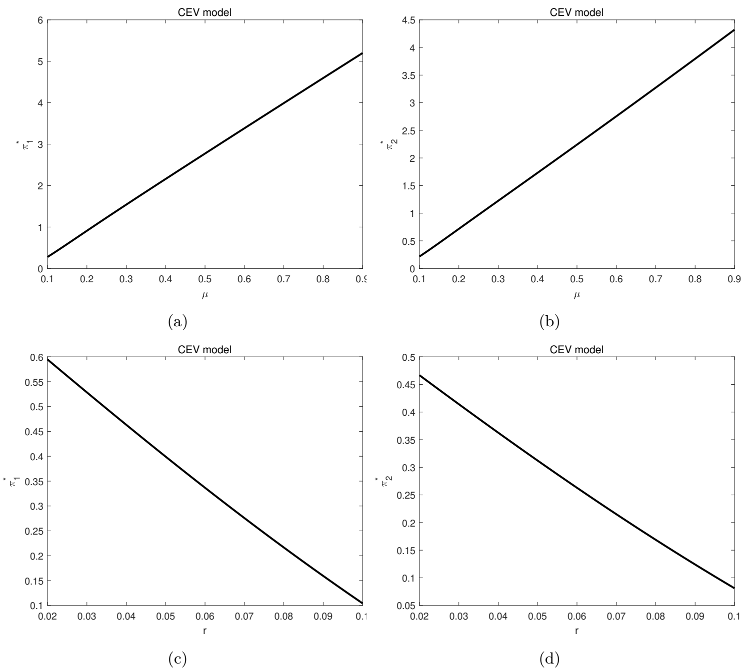


FIG. 2. Effects of parameters γ_i and θ_i on π_i, for $i = 1, 2$

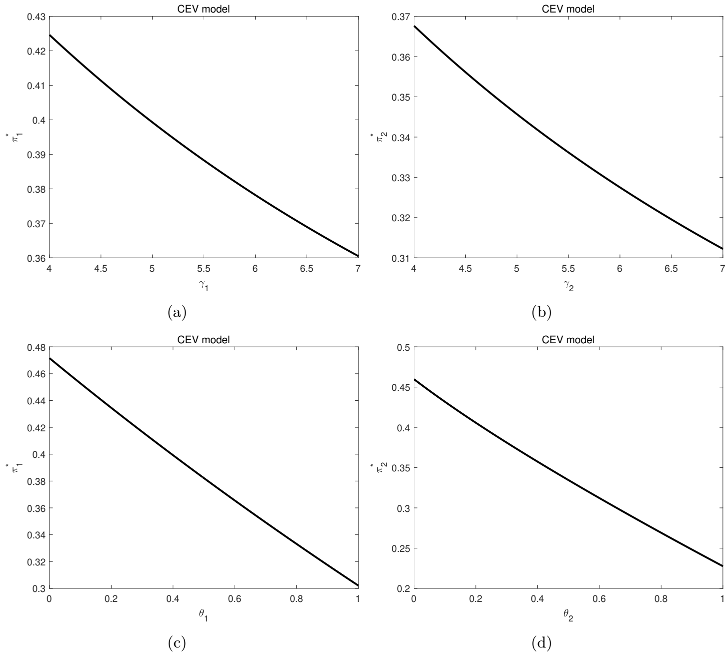
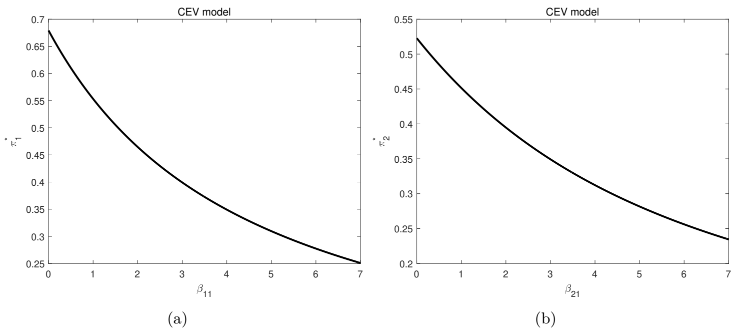


FIG. 3. Effects of parameter β_{i1} on π_i, for $i = 1, 2$



In Figure 1, we plot the impact of appreciation rate of risky asset μ and interest rate r on the equilibrium asset allocation strategy π_i^*. The proportion invest in risky asset increases with μ and decreases with r. A greater value of μ yields to more profits from risky asset. Thus, investor $i \in \{1, 2\}$ will put more money in the risky asset to gain more profits. In addition, with the increase of r, the risk-free asset becomes more attractive, investor $i \in \{1, 2\}$ would like to invest more money in the risk-free asset. Thus, the proportion invested in the risky asset becomes less.

From Figure 2, we find that the equilibrium asset allocation strategy π_i^* decreases with the risk aversion coefficient γ_i and relative wealth concerns parameter θ_i. Investor $i \in \{1, 2\}$ is risk averse, she will invest less in risky asset as the risk aversion coefficient γ_i becomes larger. What's more, a higher θ_i may lead to the investor is less concerned with her relative wealth levels. This induces her to allocate less wealth to the risky asset.

Figure 3 presents the effect of ambiguity averse parameter β_{i1} on the equilibrium asset allocation strategy π_i^*. With the increase of the β_{i1}, the optimal investment proportion gradually decreases. The explanation for this phenomenon is because of the misspecification of the model parameter, which causes that investor i, for $i = 1, 2$ adopts more conservative strategy, i.e.,she wolud allocate less wealth to the risky asset.

Example 5.2. In this example, numerical analyses of the equilibrium asset allocation strategy under the Heston model are shown in Figures 4–7.

FIG. 4. Effects of parameters κ and σ on π_i, for $i = 1, 2$

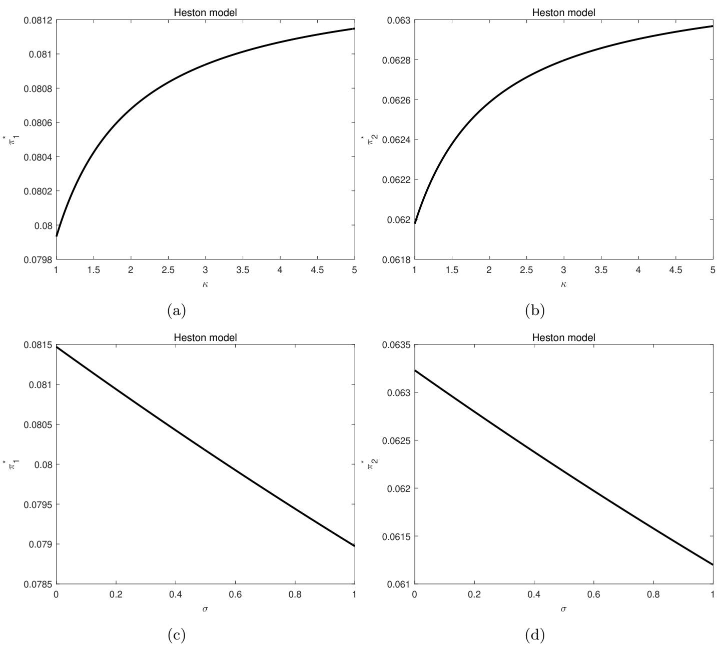
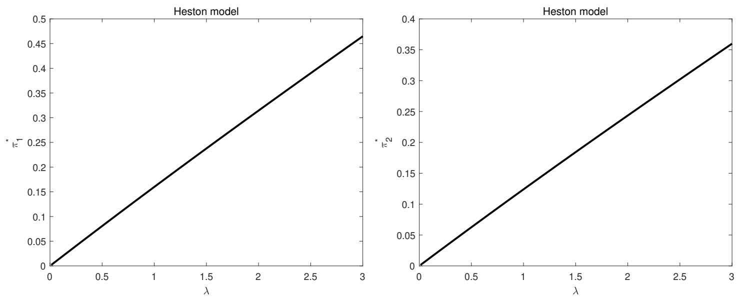


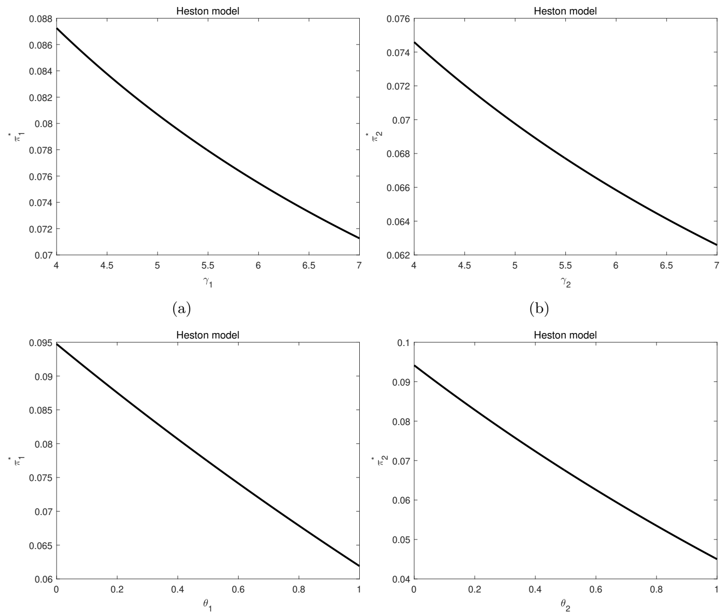
Figure 4 demonstrates the effects of κ and σ on π_i^*. From Figures 4a and 4b we can find that the equilibrium asset allocation strategy π_i^* increases

FIG. 5. Effects of parameter λ on π_i, for $i = 1, 2$



(a) (b)

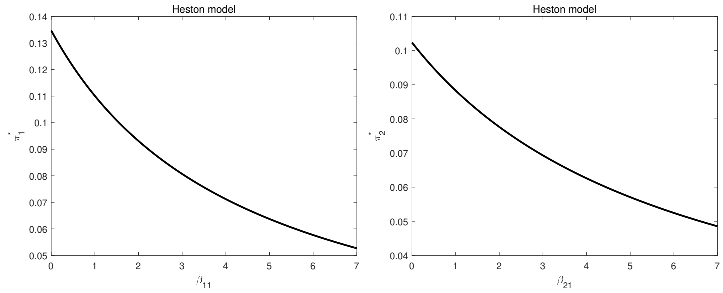
FIG. 6. Effects of parameters γ_i and θ_i on π_i, for $i = 1, 2$



(c) (d)

with respect to κ. For the Heston model, ρ represents the correlation of the risky asset S and its volatility α, when $\rho > 0$, the uncertainties of the risky asset's price and its volatility change in the same way. A larger κ leads to a more stable volatility of the risky asset. Thus, investor $i \in \{1, 2\}$ will invest more in the risky asset. Instead, Figures 4c and 4d show that σ exerts a negative effect on the equilibrium asset allocation strategy π_i^*. Because when $\rho > 0$, as σ increases, the volatility of the risky asset will

FIG. 7. Effects of parameter β_{i1} on π_i, for $i = 1, 2$



(a) (b)

fluctuate drastically. Thus, investor $i \in \{1, 2\}$ prefers to put less money in the risky asset.

Figure 5 demonstrates that the equilibrium asset allocation strategy π_i^* increases with respect to λ. A larger λ leads to a higher appreciation rate of the risky asset. Thus, investor $i \in \{1, 2\}$ will invest more in the risky asset when λ becomes larger.

Figure 6 depicts the sensitivity of the equilibrium asset allocation strategy π_i^* to the risk aversion coefficient γ_i and relative wealth concerns parameter θ_i. We find that γ_i and θ_i exert a negative effect on π_i^*. As γ_i becomes larger, the investor is more risk averse. Thus, she will reduce the proportion invested in the risky asset to avoid investment risk. Moreover, a higher θ_i may lead to the investor's being less concerned with her relative wealth levels. This induces her to allocate less wealth to the risky asset.

Figure 7 captures the same effects of Figure 3 under the CEV model in Example 1. Note that the patterns of the equilibrium asset allocation strategy, π_i^*, for $i = 1, 2$, are same as those in Figure 3, albeit with difference values.

6. CONCLUDING REMARKS

In this paper, we study a class of non-zero-sum asset allocation games between two institutional investors who want to handle model misspecification or model uncertainty by developing robust optimal strategies. Specifically, we allow each investor to allocate her wealth to one risk-free asset and one risky asset whose price dynamics follows a stochastic volatility model driven by an affine-form square-root factor process, where the price process of the risky asset in this paper can be reduced to the GBM, CEV model, Heston model, etc. Applying the techniques of stochastic dynamic programming, we derive the HJBI equations for the asset allocation games. Explicit expressions for the robust equilibrium asset allocation strategies that maximize the expected power utility of the terminal wealth relative to that of her competitor and corresponding optimal value functions are obtained. We also provide some special cases of our model and explore the economic implications from numerical examples. Results indicate that the relative wealth concerns of the investor increase the proportion invested in the risky asset, which implies that the competition would lead the investors to be much more risk-seeking. Besides, the equilibrium strategies of an ambiguity-averse investor are significantly affected by her attitudes towards model ambiguity. The ambiguity-averse investor would choose more conservative strategies than the ambiguity-neutral investor, which is reflected in transferring more risks by reducing the risky-asset investment. Overall, the investor's optimal strategies are influenced by her competitor's attitudes towards model ambiguity. That is to say, the strategies of two ambiguity-averse investors in a game would be more conservative than these of a game consisting two ambiguity-neutral investors.

There are some possible extensions of this paper. The first one is to apply other utility functions in establishing objective function in the game framework. Under such formulation, explicit expressions for Nash equilibrium strategies might be difficult to derive. However, we could apply suitable numerical approximation methods (cf. Vamvoudakis and Lewis, 2011; Bui et al. 2019) when solving the system of HJBI equations. The second one is to extend the framework and method in the paper to the case with multiple risky assets. First of all, a suitable multi-dimensional market price of risk model should be chosen. Multivariate Ornstein-Uhlenbeck and Feller processes, such as co-integrated model and Wishard model, may be candidate models. We leave them for future research.

ACKNOWLEDGEMENT

The authors are very grateful to the anonymous referees and editors for their helpful suggestions. This research was supported by the National Social Science Foundation of China (No.22FJYB003), Guangdong Basic and Applied Basic Research Foundation (No.2023A1515012335).

APPENDIX A

Proof of Theorem 1. According to the first-order optimality conditions, the functions ϕ_{i1}^* and ϕ_{i2}^*, which realize the infimum part of Eq. (18), are given by

$$\begin{cases} \phi_{i1}^*(t) = \frac{\beta_{i1}}{(1-\gamma_i)V^i}\left(\sqrt{\alpha}k_1 V_\alpha^i + \pi_i(t)\frac{\mu(t)-r(t)}{\lambda\sqrt{\alpha}}x_i V_{x_i}^i\right), \\ \phi_{i2}^*(t) = \frac{\beta_{i2}}{(1-\gamma_i)V^i}\sqrt{\alpha}k_2 V_\alpha^i. \end{cases} \tag{A.1}$$

Substituting Eq. (A.1) into Eq. (18), we have

$$\sup_{\pi_i\in\Pi_i}\left\{V_t^i+\kappa(\delta-\alpha)V_\alpha^i+\tfrac{1}{2}\alpha k_1^2\left[V_{\alpha\alpha}^i-\tfrac{\beta_{i1}\left(V_\alpha^i\right)^2}{(1-\gamma_i)V^i}\right]+\tfrac{1}{2}\alpha k_2^2\left[V_{\alpha\alpha}^i-\tfrac{\beta_{i2}\left(V_\alpha^i\right)^2}{(1-\gamma_i)V^i}\right]+r(t)x_iV_{x_i}^i\right.$$
$$+r(t)x_jV_{x_j}^i+\pi_j^*(t)\left(\mu(t)-r(t)\right)x_j\left[V_{x_j}^i+\tfrac{k_1}{\lambda}V_{x_j\alpha}^i-\tfrac{k_1}{\lambda}\tfrac{\beta_{i1}V_{x_j}^iV_\alpha^i}{(1-\gamma_i)V^i}\right]$$
$$-\tfrac{1}{2}\pi_j^{*2}(t)\tfrac{(\mu(t)-r(t))^2}{\lambda^2\alpha}x_j^2\left[\tfrac{\beta_{j1}\left(V_{x_j}^i\right)^2}{(1-\gamma_i)V^i}-V_{x_jx_j}^i\right]+\pi_i(t)\left(\mu(t)-r(t)\right)x_iV_{x_i}^i$$
$$+\pi_i(t)\tfrac{k_1}{\lambda}\left(\mu(t)-r(t)\right)x_iV_{x_i\alpha}^i-\pi_i(t)\tfrac{k_1}{\lambda}\left(\mu(t)-r(t)\right)\tfrac{\beta_{i1}}{(1-\gamma_i)V^i}x_iV_{x_i}^iV_\alpha^i$$
$$\left.+\pi_i(t)\pi_j^*(t)\tfrac{(\mu(t)-r(t))^2}{\lambda^2\alpha}x_ix_jV_{x_ix_j}^i-\tfrac{1}{2}\pi_i^2(t)\tfrac{(\mu(t)-r(t))^2}{\lambda^2\alpha}x_i^2\left[\tfrac{\beta_{i1}\left(V_{x_i}^i\right)^2}{(1-\gamma_i)V^i}-V_{x_ix_i}^i\right]\right\}=0. \tag{A.2}$$

Differentiating Eq. (A.2) with respect to π_i implies

$$\pi_i^*(t)=\frac{V_{x_i}^i+\frac{k_1}{\lambda}V_{x_i\alpha}^i-\frac{k_1}{\lambda}\frac{\beta_{i1}}{(1-\gamma_i)V^i}V_{x_i}^iV_\alpha^i+\pi_j^*(t)\frac{(\mu(t)-r(t))}{\lambda^2\alpha}x_jV_{x_ix_j}^i}{\frac{\mu(t)-r(t)}{\lambda^2\alpha}x_i\left[\frac{\beta_{i1}\left(V_{x_i}^i\right)^2}{(1-\gamma_i)V^i}-V_{x_ix_i}^i\right]}. \tag{A.3}$$

Plugging Eq. (A.3) into Eq. (A.2) implies

$$V_t^i+\kappa(\delta-\alpha)V_\alpha^i+\tfrac{1}{2}\alpha k_1^2\left[V_{\alpha\alpha}^i-\tfrac{\beta_{i1}\left(V_\alpha^i\right)^2}{(1-\gamma_i)V^i}\right]+\tfrac{1}{2}\alpha k_2^2\left[V_{\alpha\alpha}^i-\tfrac{\beta_{i2}\left(V_\alpha^i\right)^2}{(1-\gamma_i)V^i}\right]+r(t)x_iV_{x_i}^i$$
$$+r(t)x_jV_{x_j}^i+\pi_j^*(t)\left(\mu(t)-r(t)\right)x_j\left[V_{x_j}^i+\tfrac{k_1}{\lambda}V_{x_j\alpha}^i-\tfrac{k_1}{\lambda}\tfrac{\beta_{i1}V_{x_j}^iV_\alpha^i}{(1-\gamma_i)V^i}\right]$$
$$-\tfrac{1}{2}\pi_j^{*2}(t)\tfrac{(\mu(t)-r(t))^2}{\lambda^2\alpha}x_j^2\left[\tfrac{\beta_{j1}\left(V_{x_j}^i\right)^2}{(1-\gamma_i)V^i}-V_{x_jx_j}^i\right]$$
$$+\frac{\left[V_{x_i}^i+\frac{k_1}{\lambda}V_{x_i\alpha}^i-\frac{k_1}{\lambda}\frac{\beta_{i1}}{(1-\gamma_i)V^i}V_{x_i}^iV_\alpha^i+\pi_j^*(t)\frac{(\mu(t)-r(t))}{\lambda^2\alpha}x_jV_{x_ix_j}^i\right]^2\lambda^2\alpha}{2\left[\frac{\beta_{i1}\left(V_{x_i}^i\right)^2}{(1-\gamma_i)V^i}-V_{x_ix_i}^i\right]}=0. \tag{A.4}$$

To solve Eq. (A.4), we attempt to conjecture the solution in the following form:

$$V^i(t,x_i,x_j,\alpha)=\frac{1}{1-\gamma_i}\left(x_ix_j^{\theta_i-1}\right)^{1-\gamma_i}\exp\left[A_i(t)-B_i(t)\alpha\right],\quad A_i(T)=B_i(T)=0, \tag{A.5}$$

the partial derivatives of which are

$$\begin{cases} V_t^i = \left[\dot{A}_i(t) - \dot{B}_i(t)\alpha\right] V^i, \;\; V_\alpha^i = -B_i(t)V^i, \;\; V_{\alpha\alpha}^i = B_i^2(t)V^i, \\ V_{x_i}^i = \frac{1-\gamma_i}{x_i} V^i, \;\; V_{x_j}^i = \frac{(\theta_i-1)(1-\gamma_i)}{x_j} V^i, \;\; V_{x_ix_i}^i = -\frac{\gamma_i(1-\gamma_i)}{x_i^2} V^i, \\ V_{x_jx_j}^i = \frac{(\theta_i-1)(1-\gamma_i)[(\theta_i-1)(1-\gamma_i)-1]}{x_j^2} V^i, V_{x_ix_j}^i = \frac{(\theta_i-1)(1-\gamma_i)^2}{x_ix_j} V^i, \\ V_{x_i\alpha}^i = -\frac{1-\gamma_i}{x_i} B_i(t)V^i, \;\; V_{x_j\alpha}^i = -\frac{(\theta_i-1)(1-\gamma_i)}{x_j} B_i(t)V^i. \end{cases} \tag{A.6}$$

Substituting Eqs. (A.5) and (A.6) into Eq. (A.4), we have

$$\begin{aligned} &\dot{A}_i(t) - \dot{B}_i(t)\alpha - \kappa(\delta-\alpha)B_i(t) + \tfrac{1}{2}\alpha k_1^2 \tfrac{1-\gamma_i-\beta_{i1}}{1-\gamma_i} B_i^2(t) + \tfrac{1}{2}\alpha k_2^2 \tfrac{1-\gamma_i-\beta_{i2}}{1-\gamma_i} B_i^2(t) + r(t)\theta_i(1-\gamma_i) \\ &+\pi_j^*(t)(\mu(t)-r(t))(\theta_i-1)\left[(1-\gamma_i) + \tfrac{k_1}{\lambda}\beta_{j1}B_i(t) - \tfrac{k_1}{\lambda}(1-\gamma_i)B_i(t)\right] \\ &+\tfrac{1}{2}\pi_j^{*2}(t)\tfrac{(\mu(t)-r(t))^2}{\lambda^2\alpha}(\theta_i-1)(1-\gamma_i)\left[(\theta_i-1)(\gamma_i+2\beta_{i1}-1)+1\right] \\ &+\frac{(1-\gamma_i)\left[1-\frac{k_1}{\lambda}B_i(t)+\frac{k_1}{\lambda}\frac{\beta_{i1}}{1-\gamma_i}B_i(t)+\pi_j^*(t)\frac{\mu(t)-r(t)}{\lambda^2\alpha}(\theta_i-1)(1-\gamma_i)\right]^2\lambda^2\alpha}{2(\gamma_i+\beta_{i1})} = 0. \end{aligned} \tag{A.7}$$

By separating the variables with and without α, we can derive the following equations:

$$\dot{A}_i(t) + \theta_i(1-\gamma_i)r(t) - \kappa\delta B_i(t) = 0, \tag{A.8}$$

$$\begin{aligned} &-\dot{B}_i(t) + \kappa B_i(t) + \tfrac{1}{2}k_1^2\tfrac{1-\gamma_i-\beta_{i1}}{1-\gamma_i}B_i^2(t) + \tfrac{1}{2}k_2^2\tfrac{1-\gamma_i-\beta_{i2}}{1-\gamma_i}B_i^2(t) \\ &+\pi_j^*(t)(\mu(t)-r(t))(\theta_i-1)\left[(1-\gamma_i)+\tfrac{k_1}{\lambda}\beta_{j1}B_i(t)-\tfrac{k_1}{\lambda}(1-\gamma_i)B_i(t)\right] \\ &+\tfrac{1}{2}\pi_j^{*2}(t)\tfrac{(\mu(t)-r(t))^2}{\lambda^2\alpha}(\theta_i-1)(1-\gamma_i)\left[(\theta_i-1)(\gamma_i+2\beta_{i1}-1)+1\right] \\ &+\frac{(1-\gamma_i)\left[1-\frac{k_1}{\lambda}B_i(t)+\frac{k_1}{\lambda}\frac{\beta_{i1}}{1-\gamma_i}B_i(t)+\pi_j^*(t)\frac{\mu(t)-r(t)}{\lambda^2\alpha}(\theta_i-1)(1-\gamma_i)\right]^2\lambda^2}{2(\gamma_i+\beta_{i1})} = 0. \end{aligned} \tag{A.9}$$

Substituting Eqs. (A.5) and (A.6) into Eq. (A.3), we can derive $\pi_i^*(t)$ as

$$\pi_i^*(t) = \frac{1-\frac{k_1}{\lambda}B_i(t)+\frac{k_1}{\lambda}\frac{\beta_{i1}}{1-\gamma_i}B_i(t)}{\frac{\mu(t)-r(t)}{\lambda^2\alpha}(\gamma_i+\beta_{i1})} + \frac{(\theta_i-1)(1-\gamma_i)}{(\gamma_i+\beta_{i1})}\pi_j^*(t). \tag{A.10}$$

Explicit expressions for the equilibrium portfolio strategy of each investor can be obtained by solving the following system of equations:

$$\begin{cases} \pi_1^*(t) = \frac{1-\frac{k_1}{\lambda}B_1(t)+\frac{k_1}{\lambda}\frac{\beta_{11}}{1-\gamma_1}B_1(t)}{\frac{\mu(t)-r(t)}{\lambda^2\alpha}(\gamma_1+\beta_{11})} + \frac{(\theta_1-1)(1-\gamma_1)}{(\gamma_1+\beta_{11})}\pi_2^*(t), \\ \pi_2^*(t) = \frac{1-\frac{k_1}{\lambda}B_2(t)+\frac{k_1}{\lambda}\frac{\beta_{21}}{1-\gamma_2}B_2(t)}{\frac{\mu(t)-r(t)}{\lambda^2\alpha}(\gamma_2+\beta_{21})} + \frac{(\theta_2-1)(1-\gamma_2)}{(\gamma_2+\beta_{21})}\pi_1^*(t). \end{cases} \tag{A.11}$$

Substituting $\pi_j^*(t)$ displayed in Eq. (A.10) into Eq. (A.9) yields

$$\dot{B}_i(t) = \xi_{i1} + \xi_{i2}B_i(t) + \xi_{i3}B_i^2(t) + \xi_{i4}B_j(t) + \xi_{i5}B_j^2(t) + \xi_{i6}B_i(t)B_j(t), \tag{A.12}$$

with

$$\xi_{i1} = \frac{\lambda^2\nu_i(1+\nu_j)}{1-\nu_i\nu_j} + \frac{\lambda^2(1+\nu_i)^2(1-\gamma_i)}{2(1-\nu_i\nu_j)^2(\gamma_i+\beta_{i1})} - \frac{\lambda^2\nu_i(1+\nu_j)^2[2\beta_{j1}-(\theta_i-1)(1-\gamma_i)+1]}{2(1-\nu_i\nu_j)^2(\gamma_j+\beta_{j1})}, \tag{A.13}$$

$$\xi_{i2} = \kappa - \frac{\lambda k_1\nu_i(1+\nu_j)}{1-\nu_i\nu_j} - \frac{\lambda k_1\nu_i\nu_j(1-\gamma_i-\beta_{i1})}{(1-\nu_i\nu_j)(1-\gamma_i)} - \frac{\lambda k_1(1+\nu_i)(1-\gamma_i-\beta_{i1})}{(1-\nu_i\nu_j)^2(\gamma_i+\beta_{i1})} + \frac{\lambda k_1\nu_i\nu_j(1+\nu_j)(1-\gamma_i-\beta_{i1})[2\beta_{j1}-(\theta_i-1)(1-\gamma_i)+1]}{(1-\nu_i\nu_j)^2(\gamma_j+\beta_{j1})(1-\gamma_i)}, \tag{A.14}$$

$$\xi_{i3} = \frac{k_1^2(1-\gamma_i-\beta_{i1})+k_2^2(1-\gamma_i-\beta_{i2})}{2(1-\gamma_i)} + \frac{k_1^2\nu_i\nu_j(1-\gamma_i-\beta_{i1})}{(1-\nu_i\nu_j)(1-\gamma_i)} + \frac{k_1^2(1-\gamma_i-\beta_{i1})^2}{2(1-\nu_i\nu_j)^2(\gamma_i+\beta_{i1})(1-\gamma_i)} - \frac{k_1^2\nu_i\nu_j^2(1-\gamma_i-\beta_{i1})^2[2\beta_{j1}-(\theta_i-1)(1-\gamma_i)+1]}{2(1-\nu_i\nu_j)^2(\gamma_j+\beta_{j1})(1-\gamma_i)^2}, \tag{A.15}$$

$$\xi_{i4} = \frac{\lambda k_1\nu_i(2\beta_{j1}+\beta_{j1}\nu_j-1+\gamma_j)}{(1-\nu_i\nu_j)(1-\gamma_j)} - \frac{\lambda k_1\nu_i(1+\nu_i)(1-\gamma_j-\beta_{j1})(1-\gamma_i)}{(1-\nu_i\nu_j)^2(\gamma_i+\beta_{i1})(1-\gamma_j)} + \frac{\lambda k_1\nu_i(1+\nu_j)(1-\gamma_j-\beta_{j1})[2\beta_{j1}-(\theta_i-1)(1-\gamma_i)+1]}{(1-\nu_i\nu_j)^2(\gamma_j+\beta_{j1})(1-\gamma_j)}, \tag{A.16}$$

$$\xi_{i5} = -\frac{k_1^2\nu_i\beta_{j1}\left(1-\gamma_j-\beta_{j1}\right)}{\left(1-\nu_i\nu_j\right)\left(1-\gamma_j\right)^2} + \frac{k_1^2\nu_i^2\left(1-\gamma_j-\beta_{j1}\right)^2\left(1-\gamma_i\right)}{2\left(1-\nu_i\nu_j\right)^2\left(\gamma_i+\beta_{i1}\right)\left(1-\gamma_j\right)^2} - \frac{k_1^2\nu_i\left(1-\gamma_j-\beta_{j1}\right)^2\left[2\beta_{j1}-\left(\theta_i-1\right)\left(1-\gamma_i\right)+1\right]}{2\left(1-\nu_i\nu_j\right)^2\left(\gamma_j+\beta_{j1}\right)\left(1-\gamma_j\right)^2}, \tag{A.17}$$

$$\xi_{i6} = \frac{k_1^2\nu_i(1-\gamma_j-\beta_{j1})}{(1-\nu_i\nu_j)(1-\gamma_j)} - \frac{k_1^2\nu_i\nu_j\beta_{j1}(1-\gamma_i-\beta_{i1})}{(1-\nu_i\nu_j)(1-\gamma_i)(1-\gamma_j)} + \frac{k_1^2\nu_i(1-\gamma_i-\beta_{i1})(1-\gamma_j-\beta_{j1})}{(1-\nu_i\nu_j)^2(\gamma_i+\beta_{i1})(1-\gamma_j)} - \frac{k_1^2\nu_i\nu_j(1-\gamma_i-\beta_{i1})(1-\gamma_j-\beta_{j1})[2\beta_{j1}-(\theta_i-1)(1-\gamma_i)+1]}{(1-\nu_i\nu_j)^2(\gamma_j+\beta_{j1})(1-\gamma_i)(1-\gamma_j)}. \tag{A.18}$$

Given $B_i(t)$, the function $A_i(t)$ is defined by

$$A_i(t) = \theta_i\left(1-\gamma_i\right)\int_t^T r(s)\mathrm{d}s - \kappa\delta\int_t^T B_i(s)\mathrm{d}s. \tag{A.19}$$

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