ANNALS OF ECONOMICS AND FINANCE **27-1**, 145–168 (2026)

Dynamic Innovation under Model Uncertainty

Dandan Song and Wenwei Wang*

We study the effects of model uncertainty on investment, financing, and risk management for innovative firms. The main results are as follows: (1) Innovation investment can increase firm value and elevate the firm's liquidity demand; (2) Stronger ambiguity aversion towards model uncertainty reduces the innovative firm's liquidity demand, firm value, and payout boundary; (3) As the degree of ambiguity aversion increases, innovative firms become more inclined to pay cash in advance and reduce external financing; (4) Ambiguity-averse innovative firms tend to hedge more than non-ambiguity-averse firms.

Key Words: Model uncertainty; Innovation investment; Hedging; Liquidity management.

JEL Classification Numbers: E21, G11, G32.

1. INTRODUCTION

In the context of business operations, particularly within emerging industries where productivity and profitability are key drivers, firms often seek to invest in innovative projects to enhance their operational efficiency. The successful implementation of such innovations typically results in an immediate boost to firm productivity. However, the process of innovation, especially during the research and development (R&D) phase, is inherently time-consuming and can incur significant costs. Simultaneously, shareholders may exhibit skepticism towards the firm's projected productivity improvements, particularly when these projects are based on the risk-neutral probability measure. This skepticism stems from shareholders' aversion to ambiguity, which leads to concerns regarding the potential misspecification of the model used to forecast the firm's productivity shocks, resulting in distorted beliefs and uncertainty about the true value of the firm's innovation efforts.

* Song: Corresponding author. School of Finance and Statistics, Hunan University, Changsha, China. Email: ddsong@hnu.edu.cn; Wang: School of Finance and Statistics, Hunan University, Changsha, China. Email: martinwang@hnu.edu.cn.

1529-7373/2026

In this study, we adopt the shareholder value maximization model proposed by Jiang et al. (2020) as our baseline framework. In this model, a firm views innovation investment as an expenditure, where the firm's productivity improves by a constant fraction as a result of innovation. As the level of investment in innovation increases, the likelihood of the innovation project's success also rises. To account for the effect of model uncertainty, we employ the methodology suggested by Hansen and Sargent (2001) and Anderson et al. (2003), where the discrepancy between two probability measures is quantified using relative entropy. This approach is well-suited for financial and statistical analyses. The primary objective of this paper is to examine the impact of model uncertainty on an innovative firm's optimal decisions regarding financing, investment, payout boundaries, and hedging strategies, while also exploring how innovation influences firms that are averse to ambiguity.

The main findings can be summarized as follows. First, model uncertainty depresses the innovative firm's liquidity demand, firm value, and payout boundary. When cash holdings are sufficiently high, an ambiguity-averse innovative firm assigns a lower marginal value to internal liquidity relative to its ambiguity-neutral counterpart. This occurs because ambiguity induces shareholders to evaluate policies through a max-min criterion, effectively pricing the firm under the worst-case belief. Consequently, model uncertainty reduces both average and marginal Tobin's q, as an additional unit of capital contributes less to the continuation value under distorted beliefs. Moreover, ambiguity pushes shareholders to prefer holding cash outside the firm rather than inside, thereby tightening the payout boundary and further lowering the marginal value of internal liquidity. By contrast, innovation investment offsets these effects by strengthening future productivity prospects, which raises firm value and expands the payout boundary even in the presence of ambiguity.

Second, when internal liquidity is scarce, the marginal value of cash, as well as the firm's capital and innovation choices, exhibit markedly different behaviors across the liquidation and refinancing regimes. In the liquidation region, model uncertainty depresses the marginal value of cash for innovative firms, whereas in the refinancing region, an ambiguity-averse innovative firm assigns a higher marginal value of cash relative to the benchmark firm. The mechanism is straightforward: model uncertainty lowers the endogenous financing ratio, tightening external financing constraints. As the cost of issuing equity rises when the financing ratio falls, each additional unit of internal liquidity reduces the likelihood of triggering costly external issuance, thereby increasing the marginal value of cash. Moreover, when approaching the liquidation boundary, ambiguity-averse firms tend to overinvest in both physical and R&D capital relative to their ambiguity-neutral counterparts, as the distorted worst-case beliefs reduce the continuation value of cash and make asset sales a less attractive buffer against liquidation. In contrast, under refinancing, model uncertainty induces underinvestment: both equity issuance and asset sales are costly, and the reduced financing ratio further discourages investment. The same logic that governs overinvestment near liquidation and underinvestment under refinancing applies symmetrically to innovation investment.

Third, dynamic hedging enhances firm value and strengthens both capital and innovation investment for innovative firms. Among otherwise identical innovators that adopt hedging, the ambiguity-averse firm optimally chooses a larger hedge position, using financial contracts to counteract the adverse valuation effects induced by worst-case beliefs. Finally, when the firm has access to a credit line, the qualitative implications mirror those in the refinancing environment: credit availability relaxes liquidity constraints, improves investment efficiency, and attenuates the distortions generated by model uncertainty.

1.1. Related Literature

Our paper relates to three main strands of literature: dynamic corporate finance, firm innovation, and model uncertainty. Within the dynamic corporate finance literature, Bolton et al. (2011) develop a unified framework linking investment, financing, and risk management for financially constrained firms, emphasizing the central role of liquidity and hedging. Wang et al. (2012) extend the q-theoretic approach to an incomplete-market entrepreneurial setting, highlighting how market incompleteness shapes firm dynamics. Bolton et al. (2013) build on Bolton et al. (2011) by introducing stochastic financing conditions, while Lin et al. (2018) incorporate stochastic short-term interest rates into the same structural environment.

Relative to this literature, our contribution is twofold. First, we incorporate R&D investment into a dynamic corporate finance framework for an unlevered firm, allowing us to examine how innovation shapes liquidity management, investment behavior, and payout policy. Second, we embed model uncertainty-captured through a robust control structure- to study how ambiguity alters Tobin's q, financing thresholds, and the firm's dynamic distortions. These extensions enable us to offer new insights into how ambiguity interacts with innovation in determining optimal corporate policies.

A second strand of related work examines firm innovation. Berk et al. (2004) and Sundaresan and Wang (2015) incorporate R&D decisions into firm valuation using real-options frameworks, highlighting the nonlinear and state-contingent nature of innovation investment. In contrast, Malamud and Zucchi (2019) and Acemoglu et al. (2018) study innovation in general-equilibrium environments where firms' R&D choices endogenously drive long-run growth. On the financing side, Guney et al. (2017) document how access to credit lines affects firms' R&D spending, underscoring the interaction between liquidity provision and innovation policy.

Bridging dynamic corporate finance and innovation, Jiang et al. (2020) embed R&D investment into the framework of Bolton et al. (2011) to analyze how innovation shapes investment, financing, and risk-management decisions in a dynamic setting. Our paper extends this line of research along two key dimensions. First, we introduce model uncertainty into the innovation-capital accumulation problem, allowing us to characterize how ambiguity distorts innovation incentives, liquidity management, and Tobin's q. Second, we examine environments with credit-line access, showing how external liquidity provision interacts with ambiguity and innovation in determining optimal corporate policies.

A third strand of related research concerns model uncertainty. Since Ellsberg (1961) demonstrated the limitations of subjective expected utility through the Ellsberg paradox, ambiguity-often interpreted as "hard-to-capture model misspecification" has become a central theme in financial economics. Hansen and Sargent (2001) establish the formal connection between max-min expected utility and robust control, providing the foundation for modeling ambiguity-averse decision makers in dynamic environments. Building on this framework, Branger and Larsen (2013) and Aït-Sahalia and Matthys (2019) incorporate model uncertainty into robust portfolio choice settings with jump and diffusion risks, while Miao and Rivera (2016) analyze a continuous-time principal-agent problem in which the principal faces ambiguity about the underlying data-generating process.

Closer to our setting, Wu et al. (2017) and Liu et al. (2021) examine how model uncertainty distorts firms' dynamic financing, investment, payout, and hedging policies under liquidation and refinancing regimes. Luo and Tian (2022) further extend the dynamic corporate finance framework by embedding ambiguity into models of investment, dividends, costly external financing, and liquidation. Our paper adopts a similar robust-control approach for modeling ambiguity, and like these studies focuses on non-levered firms. However, our contribution is distinct: we integrate R&D investment into the ambiguity framework, allowing us to characterize how model uncertainty interacts with innovation, liquidity management, and dynamic investment distortions in a unified setting.

A recent study by Kim (2023) also explores the role of ambiguity in shaping firms' innovation strategies. However, the modeling environments differ fundamentally. Kim (2023) analyzes innovation choices within the real options framework of Grenadier and Weiss (1997), where investment timing and strategic adoption are central. By contrast, our analysis is grounded in a Tobin's q-theoretic dynamic corporate finance model, allowing us to examine how ambiguity interacts with liquidity management, payout policy, and capital adjustment in a unified setting. Moreover, Kim (2023) employs a Choquet-expected-utility formulation to model ambiguity aversion, whereas we rely on a robust control, max-min framework. These conceptual and methodological differences make the two studies complementary but distinct.

The remainder of the paper proceeds as follows. Section 2 introduces the robust innovation model. Section 3 characterizes the firm's optimal policies under liquidation and external financing, exploiting the homogeneity of the capital stock to derive tractable solutions. Section 4 presents the quantitative analysis and compares outcomes across four types of firms. Section 5 incorporates dynamic hedging into the liquidation environment, and Section 6 extends the refinancing framework to include access to a credit line. Section 7 concludes.

2. MODEL SETUP

A. Physical and Innovation Investment

In this section, we distinguish between two forms of investment: physical capital investment and innovation investment. First, to sustain its operations, a firm must maintain a positive stock of physical capital. Consequently, the firm must engage in physical investment to offset the depreciation of its capital stock. We denote the capital stock by K and the gross capital investment by I, respectively. The evolution of the capital stock is governed by the following differential equation:

$$dK_t = (I_t - \delta K_t)dt, \tag{1}$$

where $\delta \geq 0$ represents the depreciation rate of capital.

Second, the firm also invests in innovation, represented by $H_t \geq 0$, which can upgrade its production technology. Specifically, the production process improves once the firm successfully develops a new technology. Intuitively, the likelihood of technological success increases with higher levels of innovation investment. The firm's productivity process dA_t over the time interval $[t, t + dt]$ is given by:

$$dA_t = \mu_t dt + \sigma dZ_t, \tag{2}$$

where Z_t is a standard Brownian motion under the risk-neutral measure $\mathbb{P}$, μ_t is a time-varying drift term, and $\sigma > 0$ measures the volatility of the productivity shock. The time at which innovation succeeds is denoted by τ, at which point the drift of the productivity process increases from μ_0 to $\mu_1 = (1+x)\mu_0$, where x represents the degree of productivity improvement.

This results in the following:

$$\mu_t = \begin{cases} \mu_0, & \text{if } t \in [0, \tau) \\ \mu_1, & \text{if } t \in [\tau, \infty) \end{cases}. \tag{3}$$

The innovation intensity λ is modeled as a power function of the scaled innovation investment $h_t = H_t/K_t$:

$$\lambda(h_t) = \rho h_t^{\zeta}, \tag{4}$$

where ρ and ζ are parameters that determine how innovation investment affects the likelihood of technological success. Intuitively, an increase in λ reduces the time τ until the innovation succeeds. After the innovation is successful, the firm immediately adopts the new technology and ceases further innovation investment, implying that $H_t = 0$ for $t > \tau$.

The firm's incremental operating profit dY_t over the time period $[t, t+dt]$ is given by:

$$dY_t = K_t dA_t - I_t dt - G(I_t, K_t)dt - H_t dt, \tag{5}$$

where $G(I, K)$ represents the adjustment cost incurred by the firm during its capital investment process. Following the approach of Hayashi (1982), we assume the adjustment cost takes the form $G(I, K) = g(i)K$, where $i = I/K$ is the firm's investment-to-capital ratio. For simplicity, we adopt a standard quadratic form for $g(i)$:

$$g(i) = \frac{\beta i^2}{2}, \tag{6}$$

where β is a parameter representing the degree of adjustment cost.

B. Model Uncertainty

We now turn to the issue of model uncertainty, particularly with respect to the productivity process. We assume that the firm is concerned about potential misspecifications in its model and utilizes the risk-neutral probability measure $\mathbb{P}$ as an approximation. To address model misspecification, the firm considers alternative models, with its updated beliefs represented by absolutely continuous probability measures relative to $\mathbb{P}$ over any finite time interval.

Let α_t denote a real-valued process. The density-generating process ξ_t^{α}, associated with α_t, is defined as follows:

$$\xi_t^{\alpha} = \exp\left(-\int_0^t \alpha_s dZ_s - \frac{1}{2}\int_0^t \alpha_s^2 ds\right). \tag{7}$$

Assuming that $\int_0^t \alpha_s^2 ds < \infty$ for all $t > 0$, ξ_t^α is a $(\mathbb{P}, \mathcal{F}_t)$-martingale. By Girsanov's Theorem, ξ_t^α represents the Radon-Nikodym derivative of the new probability measure $\mathbb{P}^\alpha$ with respect to $\mathbb{P}$. Additionally, the process Z_t^α, defined by

$$Z_t^\alpha = Z_t - \int_0^t \alpha_s dZ_s, \tag{8}$$

is a standard Brownian motion under the probability measure $\mathbb{P}^\alpha$. We can characterize any measure $\mathbb{P}^\alpha$ either by its density generator α_t or by its density process ξ_t. Denote the set of density generators by $\mathcal{H}$ and the set of all such measures by $\mathcal{P}_\alpha$.

Under the new probability measure $\mathbb{P}^\alpha$, the cumulative productivity process A_t, as defined in equation (2), is rewritten as:

$$dA_t = \mu_t dt + \sigma \left(dZ_t^\alpha + \alpha_t dt \right). \tag{9}$$

Similarly, the incremental operating profit process Y_t, defined in equation (5), under the measure $\mathbb{P}^\alpha$, follows:

$$dY_t = K_t \left(\mu_t dt + \sigma \alpha_t dt + \sigma dZ_t^\alpha \right) - I_t dt - G(I_t, K_t) dt - H_t dt. \tag{10}$$

C. External Financing, Liquidation, and Payout

When a firm exhausts its cash reserves, it faces the choice between securing external financing or liquidating its assets. If the firm opts for external equity financing, it incurs both a fixed cost Φ and a marginal cost γ. We assume that the fixed financing cost is proportional to the firm's capital stock K, such that $\Phi = \phi K$. Let D_t denote the cumulative amount of external financing raised by the firm up to time t, and X_t represent the cumulative financing costs. Therefore, dD_t and dX_t denote the incremental external financing and financing costs over the time period $[t, t + dt]$, respectively.

However, when the cost of external equity financing becomes prohibitively high, or when the firm's productivity is insufficient, or market liquidity is constrained, the firm may opt for liquidation. We define the firm's endogenous liquidation time by ν. At time ν, the capital stock is sold at a liquidation value of lK, and the firm subsequently declares bankruptcy.

Next, we examine the firm's payout decision. Let U_t represent the cumulative payout to shareholders by time t, with dU_t denoting the incremental payout during the time period $[t, t + dt]$. Dividend payouts directly reduce the firm's cash reserves by dU_t.

D. Firm Optimality

Let W represent the firm's cash holding process, which incorporates operating profits Y, the time value of money, the cumulative external financing process D, and the cumulative dividend payout U. Accordingly, the cash inventory process W_t, accounting for model uncertainty, evolves as follows:

$$\begin{aligned} dW_t = & \left[K_t(\mu_t dt + \sigma\alpha_t) - I_t - G(I_t, K_t) - H_t + (r-c)W_t\right] dt \\ & + K_t \sigma dZ_t^\alpha + dD_t - dU_t. \end{aligned} \tag{11}$$

Here, $c \geq 0$ represents the carrying cost of free cash, and $r \geq 0$ is the risk-free rate.

To account for the firm's concerns regarding model misspecification, and drawing upon the robust control and variational utility models of Anderson et al. (2003), Maccheroni et al. (2006a), and Maccheroni et al. (2006b), the firm optimizes its decisions concerning capital investment I, innovation investment H, cumulative payout U, external financing policy D, and liquidation strategy ν to maximize shareholder value, as defined by the following objective:

$$\inf_{\mathbb{P}^\alpha \in \mathcal{P}_\alpha} \mathbb{E}^{\mathbb{P}^\alpha}\left[\int_0^\nu e^{-rt}\left(dU_t - dD_t - dX_t\right) + e^{-r\nu}\left(lK_\nu + W_\nu\right)\right] + \mathcal{K}(\mathbb{P}^\alpha), \tag{12}$$

where the first term represents the discounted value of continuing net cash payouts to shareholders, and the second term is the present value of liquidation. The additional term $\mathcal{K}(\mathbb{P}^\alpha)$ represents the entropy penalty, which reflects the firm's concern for model misspecification. Intuitively, the firm seeks to make decisions that are robust to the worst-case model $\mathbb{P}^\alpha$.

Building on the relative entropy cost framework of Maccheroni et al. (2006a), Maccheroni et al. (2006b), and Miao and Rivera (2016), we specify the penalty term under measure $\mathbb{P}$ as:

$$\mathcal{K}\left(\mathbb{P}^\alpha\right) = \mathbb{E}^{\mathbb{P}}\left[\int_0^\tau e^{-rs}\Theta_s d\phi\left(\frac{\xi_s^\alpha}{\xi_0^\alpha}\right)\right], \tag{13}$$

where $\phi(x) \equiv x \ln x$ and $\Theta_t > 0$ measures the size of the entropy cost. Following Ling et al. (2021), we obtain the following specification:

$$\mathcal{K}\left(\mathbb{P}^\alpha\right) = \frac{1}{2}\mathbb{E}^{\mathbb{P}^\alpha}\left[\int_0^\tau e^{-rs}\Theta_s \alpha_s^2 ds\right]. \tag{14}$$

In line with the intuition that larger firms face a greater entropy penalty, we assume that Θ_t is proportional to the capital stock K_t, such that:

$$\Theta_t = \Theta(K_t) = \theta K_t, \tag{15}$$

where $\theta > 0$ is the ambiguity aversion parameter[1]. It is important to note that the homogeneity assumption on Θ_t allows us to reduce the two-dimensional problem to a one-dimensional one. Specifically, a smaller θ (larger $1/\theta$) indicates higher ambiguity aversion: the firm assigns a larger entropy penalty to model misspecification and adopts a more conservative "max-min" decision criterion. While a larger θ (smaller $1/\theta$) indicates lower ambiguity aversion: the firm relies more on the baseline risk-neutral probability measure $\mathbb{P}$, and the entropy penalty for alternative models diminishes (approaching the ambiguity-neutral case when $\theta \to +\infty$).

3. MODEL SOLUTION

In this section, we derive the solution to the robust control model. The firm's value is determined by two key state variables: its capital stock K and its cash holdings W. Let $P^0(K, W)$ and $P^1(K, W)$ denote the firm's value before and after the technological breakthrough, respectively. Upon the successful completion of innovation, the firm's productivity improves immediately, causing a transition from $P^0(K, W)$ to $P^1(K, W)$. The firm's optimal decisions and value depend on its position within one of the following three regions: (i) the internal financing region, (ii) the payout region, and (iii) the external financing/liquidation region. As we will elaborate below, there are two critical endogenous boundaries: the upper boundary $\overline{W}$ and the lower boundary $\underline{W}$. The firm is in the internal financing region when its cash reserves lie between $\underline{W}$ and $\overline{W}$. If its cash holdings exceed $\overline{W}$, the firm enters the payout region, while if its cash holdings fall below $\underline{W}$, the firm enters the external financing or liquidation region.

A. Internal Financing Region

In this region, the firm value during the innovation phase, denoted by $P^0(K, W)$, satisfies the following Hamilton-Jacobi-Bellman (HJB) equation:

$$\begin{aligned} rP^0(K,W) = \sup_{I,H} \inf_{\alpha} \Big(&(I - \delta K) P_K^0(K,W) \\ &+ [(r-c)W + \mu_0 K - I - H - G(I,K) + \sigma\alpha K]\, P_W^0(K,W) \\ &+ \frac{\sigma^2 K^2}{2} P_{WW}^0(K,W) + \lambda(h)\left[P^1(K,W) - P^0(K,W)\right] + \frac{1}{2}\Theta(K)\alpha^2 \Big) \end{aligned} \tag{16}$$

[1]Here, the degree of ambiguity aversion describes the firm's preference toward ambiguity, while the amount of model uncertainty (such as detection error probability defined in Hansen and Sargent (2001), Luo (2017) and Luo et al. (2022)) describes the objective ambiguity inherent in the environment. It measures the statistical distance between the baseline model and alternative misspecified models. These concepts are closely related but distinct.

The first term on the right-hand side of equation (16) represents the marginal effect of net capital investment on firm value, $P^0(K,W)$. The second term captures the impact of the expected cash increment on $P^0(K,W)$. The third term measures the effect of cash holding volatility, W, while the fourth term reflects the instantaneous jump in firm value from $P^0(K,W)$ to $P^1(K,W)$ once the innovation succeeds. The final term, $\frac{1}{2}\Theta(K)\alpha^2$, accounts for the quadratic entropy penalty, reflecting the firm's aversion to model uncertainty.

By exploiting the homogeneity property, we can reduce the firm's two-state optimization problem to a one-state problem, as follows:

$$P^0(K,W) = K \cdot p_0(w), \tag{17}$$

where $w = W/K$ represents the firm's cash-to-capital ratio. Rather than solving for the firm value, $P^0(K,W)$, we focus on solving for the firm's value-capital ratio, $p_0(w)$. In this case, the marginal q is given by $P^0_K(K,W) = p_0(w) - wp_0'(w)$, the marginal value of cash by $P^0_W(K,W) = p_0'(w)$, and $P_{WW} = p_0''(w)/K$. Simplifying equation (16), we obtain the following ordinary differential equation for $p_0(w)$:

$$\begin{aligned} rp_0(w) = &\sup_{i,h}\inf_{\alpha}\Big[(i(w)-\delta)\left(p_0(w) - wp_0'(w)\right) \\ &+ \left((r-c)w + \mu_0 - i(w) - h(w) - g(i(w)) + \sigma\alpha\right)p_0'(w) \\ &+ \frac{\sigma^2}{2}p_0''(w) + \lambda(h(w))\left[p_1(w) - p_0(w)\right] + \frac{1}{2}\theta\alpha^2\Big]. \end{aligned} \tag{18}$$

Solving the first-order conditions yields the optimal innovation investment:

$$h(w) = \left(\frac{p_0'(w)}{\left[p_1(w) - p_0(w)\right]\rho\zeta}\right)^{\frac{1}{\zeta-1}}, \tag{19}$$

the optimal physical capital investment:

$$i(w) = \frac{1}{\beta}\left(\frac{p_0(w)}{p_0'(w)} - w - 1\right), \tag{20}$$

and the density generator α:

$$\alpha = -\frac{\sigma p_0'(w)}{\theta}. \tag{21}$$

Similarly, the HJB equation for the firm value after a technological breakthrough is given by:

$$
\begin{aligned}
rP^1(K,W) = \sup_I \inf_\alpha & \Big[(I-\delta K)P^1_K(K,W) \\
& + ((r-c)W + \mu_1 K - I - G(I,K) + \sigma\alpha K)\, P^1_W(K,W) \\
& + \frac{\sigma^2 K^2}{2} P^1_{WW}(K,W) + \frac{1}{2}\Theta(K)\alpha^2\Big].
\end{aligned}
\tag{22}
$$

Since the firm ceases innovation investment after the technology breakthrough (i.e., $H=0$), the value gap term $\lambda(h(w))\,[p_1(w)-p_0(w)]$ vanishes from equation (22). The homogeneity property reduces equation (22) to:

$$
\begin{aligned}
rp_1(w) = & (i(w)-\delta)\,(p_1(w)-wp_1'(w)) \\
& + ((r-c)w + \mu_1 - i(w) - g(i(w)) + \sigma\alpha)\, p_1'(w) \\
& + \frac{\sigma^2}{2} p_1''(w) + \frac{1}{2}\theta\alpha^2.
\end{aligned}
\tag{23}
$$

Under the first-order conditions, the optimal capital investment is given by:

$$
i(w) = \frac{1}{\beta}\left(\frac{p_1(w)}{p_1'(w)} - w - 1\right), \tag{24}
$$

and the process α is determined by:

$$
\alpha = -\frac{\sigma p_1'(w)}{\theta}. \tag{25}
$$

B. Payout Region

Intuitively, when the firm's cash holdings are substantial, it becomes optimal for the firm to distribute excess cash to shareholders in order to mitigate the cost associated with holding idle cash. Let $\overline{w} = \overline{W}/K$ denote the endogenous payout boundary. Once the firm's cash-to-capital ratio exceeds $\overline{w}$, it is optimal for the firm to distribute the excess cash as a lump-sum payout, thereby reducing the cash-to-capital ratio, w, to $\overline{w}$. Therefore, for $w > \overline{w}$, the following condition holds:

$$
p(w) = p(\overline{w}) + (w - \overline{w}), \quad w > \overline{w}. \tag{26}
$$

The smooth-pasting condition mandates that, at the payout boundary $\overline{w}$, the firm is indifferent between retaining or distributing an additional dollar of cash. This implies that the marginal value of cash must be unity at the boundary, which leads to the condition $p'(\overline{w}) = 1$. Furthermore, to ensure the optimality of the payout boundaries, the "super-contact" conditions must be satisfied, which require that the second derivative of the firm value with respect to the cash-capital ratio equals zero at the payout boundary, i.e., $p''(\overline{w}) = 0$.

C. Liquidation/External Financing Region

When the firm's cash holdings become severely depleted, the firm faces the choice between external equity financing or liquidation. Intuitively, the firm will continue operating until its cash holdings are exhausted. On the one hand, given the efficiency of its production technology, the firm prefers to avoid premature liquidation. On the other hand, due to the positive adjustment costs associated with investment and the high costs of external financing, the firm will refrain from issuing equity until its cash reserves are fully depleted, i.e., when $\underline{w} = 0$.

If the firm opts for liquidation, the firm value at the time of liquidation is given by $p(0)K = lK$, where l represents the liquidation value. Thus, we have:

$$p(0) = l. \tag{27}$$

Alternatively, if the firm chooses to raise external equity, it will issue a lump-sum amount of equity whenever its cash holdings are exhausted. In the context of homogeneity, we assume that the equity issuance amount is proportional to the firm's capital, mK, where $m > 0$ is endogenously determined. The firm's value remains continuous before and after the issuance of external equity, implying the following condition:

$$p(0) = p(m) - \phi - (1 + \gamma)m. \tag{28}$$

Here, the right-hand side represents the firm value, adjusted for the fixed cost, ϕ, and the marginal cost of external financing, $(1 + \gamma)m$.

To ensure the optimality of the equity issuance amount, m, the marginal value of the last dollar raised must satisfy the smooth-pasting condition, which dictates that:

$$p'(m) = 1 + \gamma. \tag{29}$$

4. QUANTITATIVE ANALYSIS

In this section, we present the quantitative analysis of the model. Our framework consists of two distinct stages: the innovation investment phase, prior to the invention of the new technology, and the post-innovation phase, once the new technology is operational. Since the dynamics in the post-innovation phase align with those presented in Bolton et al. (2011), our primary focus will be on the analysis of the innovation stage. We set the mean and volatility of the productivity shock at $\mu_0 = 16\%$ and $\sigma = 9\%$, respectively. The relatively high value of μ_0 ensures that the firm will remain operational until its cash reserves are exhausted. The parameters governing the innovation success rate are set to $\rho = 0.7$ and $\xi = 0.2$, while the productivity improvement factor is specified as $x = 12\%$. Following Whited (1992), we set the depreciation rate to $\delta = 10.07\%$ and the adjustment cost parameter to $\beta = 1.5$. Consistent with Bolton et al. (2011), we assume a risk-free rate of $r = 6\%$ and a cash-carrying cost of $c = 1\%$. For the liquidation scenario, we assume a recovery rate of $l = 0.9$. In the external financing region, the proportional and fixed financing costs are set to $\gamma = 6\%$ and $\phi = 1\%$, respectively. When incorporating dynamic hedging, we adopt the same parameters as Bolton et al. (2011) for comparative purposes. To account for model misspecification, according to Wu et al. (2017) and Liu et al. (2021), we set the ambiguity aversion parameter θ to 1/3. Each curve in the following plots is terminated at the payout boundary. Table 1 provides a summary of the key parameters used in our model.

TABLE 1.

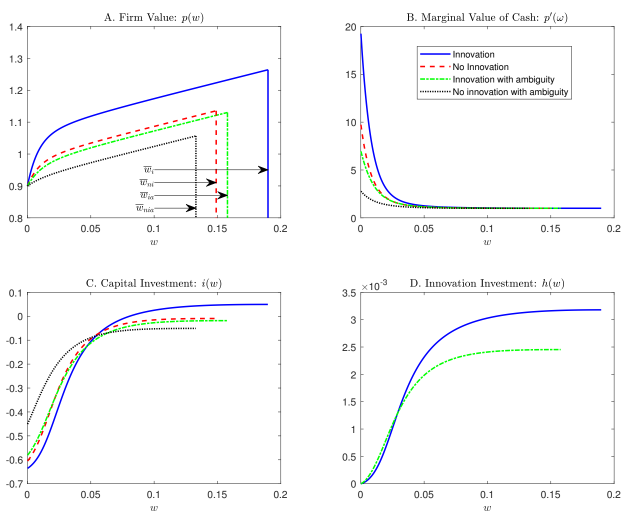
Parameters for quantitative analysis

Parameter	Symbol	Values
Risk-free rate	r	0.06
Mean productivity shock for non-innovative firm	μ_0	0.16
Productivity improvement	x	0.12
Monotonic parameter for innovation intensity	ρ	0.7
Concavity parameter for innovation intensity	ζ	0.2
Rate of depreciation	δ	0.1007
Volatility of productivity shock	σ	0.09
Ambiguity parameter	θ	$\frac{1}{3}$
Capital liquidation rate	l	0.9
Adjustment cost parameter	β	1.5
Proportional cash-carrying cost	c	1%
Fixed financing cost	ϕ	1%
Proportional financing cost	γ	6%
Correlation between market and firm	ρ_m	0.8
Margin requirement	π	5
Flow cost in margin account	ϵ	0.5%
Market volatility	σ_m	20%

A. Firm Value and Investment Strategy

Panels A and B of Figure 1 present the scaled firm value, $p(w)$, and the marginal value of cash, $p'(w)$, under four scenarios: innovation, no inno-

FIG. 1. Firm's value and investment decisions



vation, innovation with ambiguity, and no innovation with ambiguity. For clarity, the payoff boundaries are denoted by $\overline{w}_i$, $\overline{w}_{ni}$, $\overline{w}_{ia}$, and $\overline{w}_{nia}$, respectively. The firm value, $p(w)$, starts at $l = 0.9$ when its cash holdings are zero. As expected, in both cases (whether with or without ambiguity) the firm value with innovation exceeds that without innovation. This is attributable to the potential productivity increase driven by technological breakthroughs. Moreover, the firm's concerns about model misspecifications significantly reduce its value, $p(w)$, in both scenarios, with or without innovation.

Panel A of Figure 1 further demonstrates that the endogenous payoff boundary for an innovative firm is higher than that for a non-innovative firm. Notably, ambiguity aversion leads to a substantial reduction in the payoff boundary. For instance, the payoff boundary for an innovative firm declines from 0.19 to 0.158, while for a non-innovative firm, it drops from 0.149 to 0.133. This observation reflects that ambiguity-averse shareholders prefer to hold cash in hand, reducing uncertainty, rather than retaining cash within the firm. Additionally, the payout is accelerated because the marginal value of cash, $p'(w)$, is lower for firms with ambiguity than for those without. This is evident in Panel B of Figure 1, where the gap in $p'(w)$ between the two types of firms widens as the cash reserve, w, approaches zero, while the gap narrows when w becomes sufficiently large.

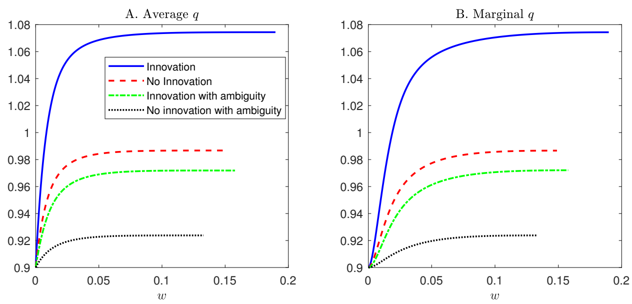
By combining the results from Panels A and B, it is clear that innovation investment enhances both firm value and the marginal value of cash.

Panel C of Figure 1 illustrates the optimal investment, $i(w)$, across the four scenarios. Comparing the cases of ambiguity versus no ambiguity, we observe a critical point in optimal physical investment, regardless of whether the firm chooses innovation or not. When cash holdings are relatively low, the ambiguity-averse firm tends to invest more than the non-ambiguity firm. Furthermore, when comparing innovation with no innovation, whether the firm is ambiguity-averse or not, we see a similar critical point in investment decisions. As the firm approaches the liquidation boundary, it becomes optimal for an innovative firm to sell more capital stock in order to delay liquidation, given the higher marginal value of cash. In contrast, ambiguity aversion reduces both the firm's continuation value and the marginal value of cash, leading shareholders to be less willing to support the firm, which in turn reduces asset sales in this scenario. When cash holdings are sufficiently high, the firm benefits from the production innovation, which increases both the firm's value and capital investment. Particularly, ambiguity-averse shareholders are more inclined toward cash payouts, which results in lower capital investment due to model uncertainty.

Panel D shows the optimal level of innovation investment for firms with and without ambiguity, respectively. The analysis reveals that innovation investment increases with the cash-to-capital ratio. When the ratio is low, ambiguity aversion leads to overinvestment due to the lower marginal value of cash. However, as the cash balance rises, the firm with model ambiguity tends to maintain a relatively lower level of innovation investment compared to the firm without ambiguity.

B. Average q and Marginal q

FIG. 2. Firm's average q and marginal q



We now turn our attention to the firm's average and marginal q. The firm's average q, denoted by $q_a(w)$, is defined as the firm value net of cash divided by its capital stock, i.e.:

$$q_a(w) = \frac{P(K, W) - W}{K} = p(w) - w. \tag{30}$$

The firm's marginal q, $q_m(w)$, is given by:

$$q_m(w) = \frac{d(P(K, W) - W)}{dK} = p(w) - wp'(w). \tag{31}$$

Figure 2 illustrates the firm's average q and marginal q across four different scenarios. Similar to Figure 1, each curve is truncated at the payoff boundary. From the figure, it is evident that both the average and marginal q for innovative firms exceed those of non-innovative firms. Furthermore, Figure 2 shows that ambiguity aversion significantly reduces both the average and marginal q. This is intuitive, as the same increment in capital, K, results in a smaller increase in firm value due to the ambiguity. This observation also provides further insight into why firms with ambiguity aversion tend to underinvest in capital stock, particularly when they are no longer facing a financial crisis.

C. Refinancing

TABLE 2.

Financing ratios and payout boundaries when $\theta = 1/3$

	Innovation	No innovation	Innovation with ambi	No innovation with ambi
Financing ratio	0.053	0.044	0.046	0.043
Payout boundary	0.163	0.130	0.138	0.126

Figure 3 presents the results for the refinancing scenario under the same four conditions as Figure 1. For clarity, the financing ratios across these four scenarios are denoted by m_i, m_{ni}, m_{ia}, and m_{nia}, respectively. Panel A of Figure 3 shows that the effects of ambiguity aversion and innovation investment on firm values and payout boundaries are similar to those observed in the liquidation case. Specifically, Table 2 provides details of the firm's payout boundaries and financing ratios when the ambiguity aversion parameter, $\theta = 1/3$. As illustrated, ambiguity aversion reduces both the financing ratio and the payout boundary, which aligns with the findings in Liu et al. (2021). Figure 4 further corroborates that both the financing ratio and payout boundary decrease as the degree of ambiguity aversion increases. To facilitate interpretation, we plot $\frac{1}{\theta}$ on the horizontal axis, which ensures that as $\frac{1}{\theta}$ increases, the firm becomes more ambiguity-averse.

FIG. 3. Firm value and investment decisions when refinancing is allowed

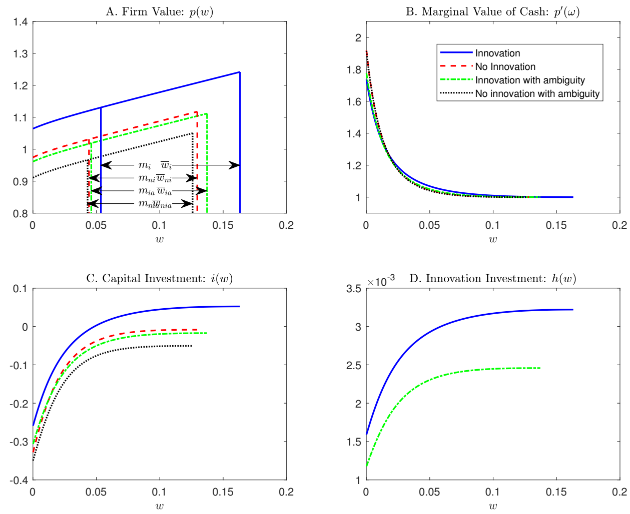
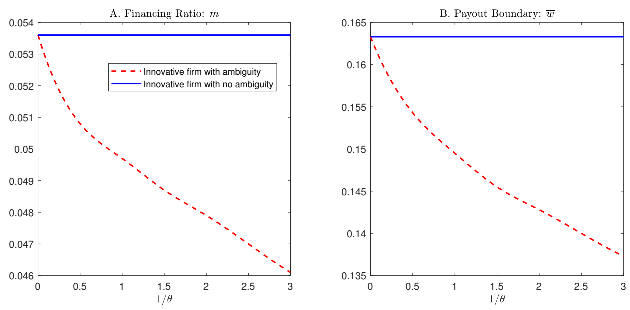


FIG. 4. Innovative firm's financing ratio and payout boundary



In particular, as the firm becomes more ambiguity-averse, the financing threshold m decreases from 0.053 to 0.046.

Panel B of Figure 3 shows that when cash holdings are low, a firm concerned about model misspecification exhibits a higher marginal value of cash, $p'(w)$, than a firm that is ambiguity-neutral. As cash holdings increase, the ambiguity-averse firm's marginal value of cash declines more quickly than that of the benchmark firm. This pattern holds for both innovative and non-innovative firms. Conversely, when cash holdings are low, the marginal value of cash is higher for the non-innovative firm than for the innovative firm. Notably, the non-innovative firm experiences a more rapid decrease in $p'(w)$ as cash reserves rise. These findings differ from those observed in the liquidation scenario. The elevated marginal value of cash for both the non-innovative and ambiguity-averse firms can be attributed to the option of refinancing. Both model uncertainty and the non-innovative nature of the firm contribute to lower financing ratios, resulting in tighter financing constraints. The reduction in the financing ratio increases the cost of equity issuance, such that each additional unit of cash further reduces the probability of issuing equity or making payouts, thereby increasing the marginal value of cash. This result is consistent with the findings in Liu et al. (2021).

Panel C of Figure 3 suggests that both ambiguity aversion and the absence of innovation investment lead to lower capital investment. The reasoning behind this differs slightly from that in the liquidation scenario, as shown in Figure 1. When refinancing is allowed, an innovative firm, which possesses more productive technology, opts for equity financing to continue operations rather than liquidating by selling capital stock. Panel D of Figure 3 reaffirms that ambiguity aversion results in reduced innovation investment. This outcome is intuitive, as the reduction in both capital and innovation investment due to ambiguity aversion towards model uncertainty diminishes the need for cash flow, thereby leading to a decrease in both the financing ratio and the payout boundary.

5. DYNAMIC HEDGING

In addition to cash inventory management, and following the framework of Bolton et al. (2011) and Liu et al. (2021), we assume that the firm can engage in financial contracts, such as options or futures contracts, to mitigate its cash flow risk. Specifically, we consider a futures contract on the market index. Let F_t denote the futures price of the market index under the risk-neutral probability, with its dynamics governed by the following stochastic differential equation:

$$dF_t = \sigma_m F_t dB_t, \tag{32}$$

where σ_m represents the volatility of the market portfolio, and B_t is a standard Brownian motion, partially correlated with Z_t with correlation ρ. Let $\psi_t \le 0$ denote the hedge ratio, and margin requirements are imposed. We also assume that the firm's futures position cannot exceed a multiple, π, of the amount of cash, $\kappa_t W_t$, in the margin account, as given by the constraint:

$$|\psi_t W_t| \le \pi \kappa_t W_t. \tag{33}$$

Let ϵ denote the proportional cash flow cost in the margin account. Under the probability measure $\mathbb{P}^{\alpha}$, the dynamics of cash are given by:

$$\begin{aligned} dW_t =& K_t\left(\mu dt + \sigma\left(dZ_t^{\alpha} - \alpha_t dt\right)\right) - \left(I_t + G_t + H_t\right)dt + dD_t - dU_t + (r-c)W_t dt \\ & - \epsilon\kappa_t W_t dt + \psi_t \sigma_m W_t\left(\rho dZ_t^{\alpha} + \sqrt{1-\rho^2}d\tilde{Z}_t - \rho\alpha_t dt\right), \end{aligned} \tag{34}$$

where $\tilde{Z}_t$ is a Brownian motion orthogonal to Z_t and Z_t^{α}. Consequently, the HJB equation for firm value following a technological breakthrough becomes:

$$\begin{aligned} rP^1(K,W) = \max_{I,\psi,K}\inf_{\alpha} & \left(I - \delta K\right)P_K^1(K,W) \\ & + \left[(r-c)W + \mu K - I - G(I,K) - H - \epsilon\kappa W + K\sigma\alpha + \psi\sigma_m W\rho_m\alpha\right]P_W^1(K,W) \\ & + \frac{1}{2}\left(\sigma^2K^2 + \psi^2\sigma_m^2W^2 + 2\rho_m\sigma_m\sigma\psi WK\right)P_{WW}^1(K,W) + \frac{\theta K}{2}\alpha^2. \end{aligned} \tag{35}$$

The HJB equation for firm value in the innovation stage is:

$$\begin{aligned} rP^0(K,W) = \max_{I,\psi,K}\inf_{\alpha} & \left(I - \delta K\right)P_K^0(K,W) \\ & + \left[(r-c)W + \mu K - I - G(I,K) - H - \epsilon\kappa W + K\sigma\alpha + \psi\sigma_m W\rho_m\alpha\right]P_W^0(K,W) \\ & + \frac{1}{2}\left(\sigma^2K^2 + \psi^2\sigma_m^2W^2 + 2\rho_m\sigma_m\sigma\psi WK\right)P_{WW}^0(K,W) \\ & + \frac{\theta K}{2}\alpha^2 + \lambda(h)\left[P^1(K,W) - P^0(K,W)\right], \end{aligned} \tag{36}$$

subject to the constraint:

$$\kappa = \min\left(\frac{|\psi|}{\pi}, 1\right). \tag{37}$$

As in the previous cases, for either p_0 or p_1, the first-order condition for α is given by:

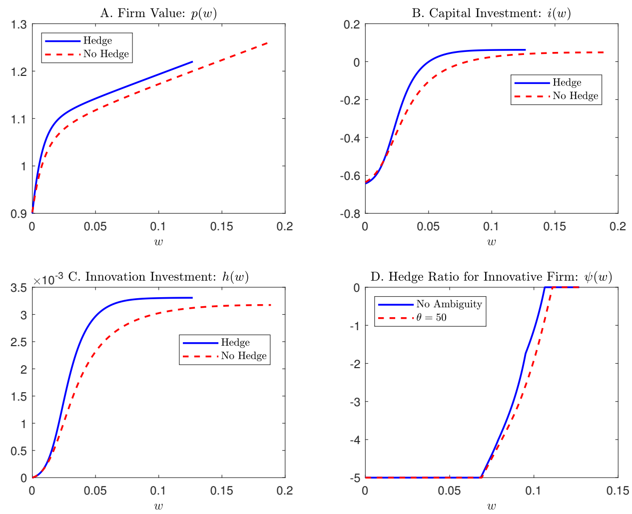
$$\alpha = -\frac{(\sigma + \psi\sigma_m w\rho_m)p'(w)}{\theta}. \tag{38}$$

The first-order condition for the hedge ratio ψ is:

$$\psi = \frac{\frac{1}{\theta}\rho_m\sigma\sigma_m p'^2 - \frac{\epsilon}{\pi}p' - \rho_m\sigma_m\sigma p''}{\sigma_m^2 wp'' - \frac{1}{\theta}\sigma_m^2 w\rho_m^2 p'^2}. \tag{39}$$

In this section, we focus on two key aspects: First, we examine how hedging affects the firm value and investment decisions of innovative firms that avoid ambiguity; second, we investigate how ambiguity aversion influences the optimal hedging ratio of innovative firms. For simplicity, we focus on the liquidation case and set the ambiguity aversion parameter $\theta = 50$, which implies a negligible discrepancy between the two probability measures $\mathbb{P}$ and $\mathbb{P}^{\alpha}$. The results are presented in Figure 5.

FIG. 5. Firm's dynamic hedging strategy and investment decisions



Panel A illustrates that, holding the productivity enhancement parameter x and the ambiguity aversion parameter θ constant, the firm value increases and the firm opts for earlier payouts when hedging is available. Both outcomes are intuitive, since dynamic hedging allows the firm to manage its cash flow risk more effectively. Panels B and C demonstrate that hedging promotes both capital and innovation investment. This occurs because a reduction in operational risk ensures that the firm is less likely to face the costly consequences of asset sales, thereby encouraging investment. Panel D compares the hedge ratios for two innovative firms: one that relies on the risk-neutral probability and another that exhibits slight ambiguity aversion. Let w_+ denote the zero-hedging boundary. For the ambiguity-averse firm, the non-hedging threshold w_+ is higher than that for the non-ambiguity-averse firm, and the ambiguity-averse firm engages in more hedging in the partial hedging region. This result is driven by the interaction between the optimal hedge ratio and firm value discrepancy, as described in equations (38) and (39), where the firm mitigates the negative effects of model uncertainty by increasing its hedge ratio. The findings in Panel D align with the results in Liu et al. (2021). As shown in Panel D, even a slight discrepancy in firm value can lead to a substantial difference in hedge ratios between the two firms.

6. CREDIT LINE

In this section, we extend the model in Section 2 to incorporate the possibility of the firm accessing a credit line. Following the intuition of Bolton et al. (2011) and Guney et al. (2017), we assume that the firm's cash inventory can be negative, and that its debt capacity is proportional to its capital, given by dK, where $d > 0$ is a constant. Additionally, we assume that the firm faces a constant spread, ν, over the risk-free rate on the credit it utilizes. For simplicity, we set the parameters $d = 0.2$ and $\nu = 1.5\%$, consistent with Bolton et al. (2011). To generate sufficiently large positive payout boundaries, $\overline{w}$, we set the ambiguity aversion parameter $\theta = 1/2$.

When the credit line serves as the marginal source of financing ($w < 0$), the firm value, $p_1(w)$, satisfies the following ordinary differential equation (ODE):

$$\begin{aligned} rp_1(w) =& (i(w) - \delta)\left(p_1(w) - wp_1'(w)\right) \\ &+ \left[(r+\nu)w + \mu_1 - i(w) - g(i(w)) + \sigma\alpha\right] p_1'(w) \\ &+ \frac{\sigma^2}{2} p_1''(w) + \frac{1}{2}\theta\alpha^2. \end{aligned} \tag{40}$$

In the innovation stage, the firm value, $p_0(w)$, satisfies the corresponding ODE:

$$\begin{aligned} rp_0(w) =& (i(w) - \delta)\left(p_0(w) - wp_0'(w)\right) \\ &+ \left[(r+\nu)w + \mu_0 - i(w) - h(w) - g(i(w)) + \sigma\alpha\right] p_0'(w) \\ &+ \frac{\sigma^2}{2} p_0''(w) + \lambda(h(w))\left[p_1(w) - p_0(w)\right] + \frac{1}{2}\theta\alpha^2. \end{aligned} \tag{41}$$

When cash is the marginal source of financing ($w > 0$), $p_0(w)$ and $p_1(w)$ satisfy equations (18) and (23) respectively. Consistent with Bolton et al. (2011), we have the boundary condition $p(-d) = p(m) - \phi - (1+\gamma)(m+d)$ and the smooth-pasting condition $p'(m) = 1 + \gamma$ for both $p_0(w)$ and $p_1(w)$. Additionally, $p(w)$ is continuous and smooth at $w = 0$.

The results presented in Figure 6 indicate that when the firm has access to a credit line, the optimal firm values, payout boundaries, marginal value of cash, scaled investment ratios, and R&D expenditure exhibit the same trends as in the refinancing case discussed in Section 4. Therefore, these results require no further elaboration. What is of particular interest,

FIG. 6. Firm value and investment decisions with credit lines

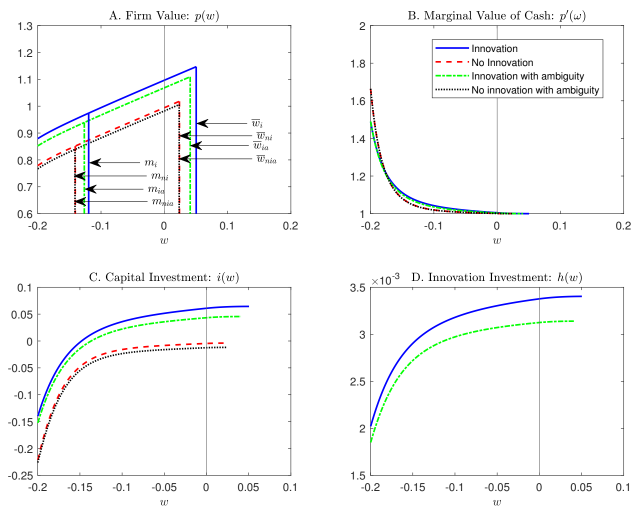
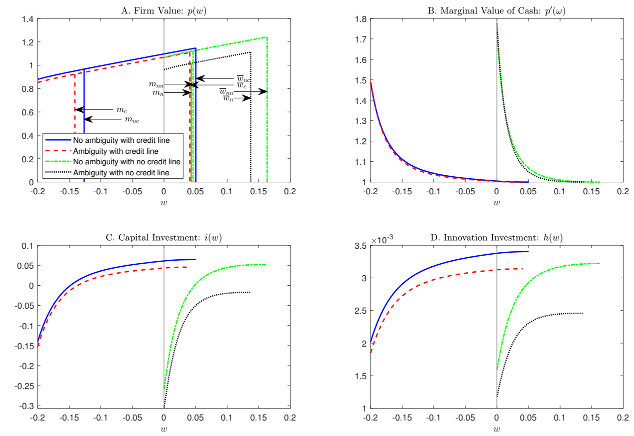


FIG. 7. Innovative firms' value and investment decisions with and without credit lines



however, is the impact of credit lines on innovative firms. Figure 7 plots the payoff boundaries under four scenarios: no ambiguity with credit line, ambiguity with credit line, no ambiguity without credit line, and ambiguity without credit line, denoted by $\overline{w}_{nc}$, $\overline{w}_c$, $\overline{w}_{nn}$, and $\overline{w}_n$, respectively. The financing ratios in these four cases are denoted by m_{nc}, m_c, m_{nn}, and m_n, respectively. Figure 7 reveals the following key findings: First, access to a credit line increases the firm's value. Second, the payout boundary decreases significantly when the firm is able to borrow. Third, the availability of a credit line reduces the marginal value of cash and mitigates the underinvestment problem, including both capital and innovation investment.

7. CONCLUSIONS

In this paper, we incorporate model uncertainty and innovation investment into a tractable model to study their effects on firm value, the marginal value of cash, investment levels, hedge ratios, and other aspects. The results show that innovation investment can increase firm value and elevate the firm's liquidity demand. Ambiguity aversion towards model uncertainty reduces the innovative firm's liquidity demand, firm value, and payout boundary. In addition, some interesting findings emerge regarding the impact of ambiguity aversion on investment and the marginal value of cash when cash holdings are low: there is a critical threshold in either capital or innovation investment in the liquidation case, and a similar threshold exists in the marginal value of cash in the external financing case. Hedging actions increase firm value, capital, and innovation investment; meanwhile, ambiguity aversion encourages firms to hedge more. There are several directions for future research. Lin et al. (2018) incorporate stochastic interest rates into a standard q-theory framework, making it natural to extend our model by relaxing the assumption that the risk-free rate is constant. Luo and Tian (2022) assume that the homogeneity property implies the ambiguity parameter is scaled by firm value rather than total capital stock. Future research could explore innovative firms under this assumption.

REFERENCES

Acemoglu, Daron, Ufuk Akcigit, Harun Alp, Nicholas Bloom, and William Kerr, 2018. Innovation, reallocation, and growth. *American Economic Review* **108(11)**, 3450-3491.

Aït-Sahalia, Yacine, and Frans Matthys, 2019. Robust consumption and portfolio policies when asset prices can jump. *Journal of Economic Theory* **179**, 1-56.

Anderson, Evan W., Lars Peter Hansen, and Thomas J. Sargent, 2003. A quartet of semigroups for model specification, robustness, prices of risk, and model detection. *Journal of the European Economic Association* **1**, 68-123.

Berk, Jonathan B., Richard C. Green, and Vasant Naik, 2004. Valuation and return dynamics of new ventures. *Review of Financial Studies* **17(1)**, 1-35.

Bolton, Patrick, Hui Chen, and Neng Wang, 2011. A unified theory of Tobin's q, corporate investment, financing, and risk management. *The Journal of Finance* **66(5)**, 1545-1578.

Bolton, Patrick, Hui Chen, and Neng Wang, 2013. Market timing, investment, and risk management. *Journal of Financial Economics* **109(1)**, 40-62.

Branger, Nicole, and Linda Sandris Larsen, 2013. Robust portfolio choice with uncertainty about jump and diffusion risk. *Journal of Banking & Finance* **37(12)**, 5036-5047.

Ellsberg, Daniel, 1961. Risk, ambiguity, and the Savage axioms. *Quarterly Journal of Economics* **75(4)**, 643-669.

Grenadier, Steven R., and Allen M. Weiss, 1997. Investment in technological innovations: An option pricing approach. *Journal of Financial Economics* **44(3)**, 397-416.

Guney, Yilmaz, Ahmet Karpuz, and Neslihan Ozkan, 2017. R&D investments and credit lines. *Journal of Corporate Finance* **46**, 261-283.

Hansen, Lars Peter, and Thomas J. Sargent, 2001. Robust control and model uncertainty. *American Economic Review* **91(2)**, 60-66.

Hayashi, Fumio, 1982. Tobin's marginal q and average q: A neoclassical interpretation. *Econometrica* **50**, 213-224.

Jiang, Ruoqing, Bo Liu, Yang Liu, Hongli Wang, and Jinqiang Yang, 2020. Theory of dynamic innovation. Working paper.

Kim, Hwa-Sung, 2023. Effects of ambiguity on innovation strategies. *Financial Innovation* **9(1)**, 68.

Lin, Xiaoji, Chong Wang, Neng Wang, and Jinqiang Yang, 2018. Investment, Tobin's q, and interest rates. *Journal of Financial Economics* **130(3)**, 620-640.

Ling, Aifan, Jianjun Miao, and Neng Wang, 2021. Robust financial contracting and investment. NBER Working Paper 28367.

Liu, Bo, Hongli Wang, and Jinqiang Yang, 2021. Dynamic financing and hedging under model uncertainty. *European Journal of Finance* **27(8)**, 740-751.

Luo, Pengfei, and Yu Tian, 2022. Investment, payout, and cash management under risk and ambiguity. *Journal of Banking & Finance* **141**, 106551.

Luo, Yulei, 2017. Robustly Strategic Consumption-Portfolio Rules with Informational Frictions. *Management Science* **63(12)**, 4158-4174.

Luo, Yulei, Jun Nie, and Haijun Wang, 2022, Ignorance, Pervasive Uncertainty, and Household Finance. *Journal of Economic Theory* **199**, 105204.

Maccheroni, Fabio, Massimo Marinacci, and Aldo Rustichini, 2006. Ambiguity aversion, robustness, and the variational representation of preferences. *Econometrica* **74(6)**, 1447-1498.

Maccheroni, Fabio, Massimo Marinacci, and Aldo Rustichini, 2006. Dynamic variational preferences. *Journal of Economic Theory* **128(1)**, 4-44.

Malamud, Semyon, and Francesca Zucchi, 2019. Liquidity, innovation, and endogenous growth. *Journal of Financial Economics* **132**, 519-541.

Miao, Jianjun, and Alejandro Rivera, 2016. Robust contracts in continuous time. *Econometrica* **84(4)**, 1405-1440.

Sundaresan, Suresh, and Neng Wang, 2015. Dynamic investment, capital structure, and debt overhang. *Review of Corporate Finance Studies* **4**, 1-42.

Wang, Chong, Neng Wang, and Jinqiang Yang, 2012. A unified model of entrepreneurship dynamics. *Journal of Financial Economics* **106(1)**, 1-23.

Whited, Toni, 1992. Debt, liquidity constraints, and corporate investment: Evidence from panel data. *The Journal of Finance* **47(4)**, 1425-1460.

Wu, Yaoyao, Jinqiang Yang, and Zhentao Zou, 2017. Dynamic corporate investment and liquidity management under model uncertainty. *Economics Letters* **155**, 9-13.