A Wealth-Dependent Investment Opportunity Set: Its Effect on Optimal Consumption and Portfolio Decisions [*]

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We consider a consumption and investment problem where an investor's investment opportunity gets enlarged when she becomes rich enough, i.e., when her wealth touches a critical level. We derive optimal consumption and investment rules assuming that the investor has a time-separable von Neumann-Morgenstern utility function. An interesting feature of optimal rules is that the investor consumes less and takes more risk in risky assets if the investor expects that she will have a better investment opportunity when her wealth reaches a critical level. ⓒ 2003 Peking University Press

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1. INTRODUCTION

It is important to understand investors' behavior in the analysis of financial market. Studies on consumption and investment problem of an economic agent usually assume that her available investment opportunity set is fixed throughout. However, it is not the case in practice that an investor has the same available investment opportunities all the time. As she gets richer, she is likely to find broader investment opportunities. For example, when she has less than $10,000 available for investment, she can usually invest only in banking accounts and mutual fund shares.[1] Mutual fund shares will provide all necessary investment tools for both the rich and the poor if a suitable form of the mutual fund separation theorem is valid. But it is clear that not all the risky or riskless investments in the global economy are covered by the existing array of mutual funds. For instance, many limited partnerships are not covered by public mutual funds. Furthermore, ordinary mutual funds are usually prohibited from using modern investment techniques involving options, futures and other derivative securities. When we are rich enough, however, we are not constrained by the opportunities offered by the mutual funds. we can invest in limited partnerships, hedge funds — these funds are known to be aggressive in employing modern investment techniques — and individual stocks as well as banking accounts and mutual funds. There is also an explicit law which prohibits small investors from trading some securities. In the U.S., the rule 144A stipulates that unregistered securities can be traded only by qualified institutional investors. Therefore, even for institutional investors, the investment opportunity gets better when they have more fund available for investment.

In this paper we study a consumption and investment problem assuming that an investor faces an enlarged investment opportunity set once her wealth level touches a critical level. We derive the value function, optimal consumption and investment rules in closed form under the assumption that the investor's preference can be represented by a time-separable von Neumann-Morgenstern utility function. An interesting feature of the optimal rules is that, before touching the critical wealth level, the investor consumes less and invests more aggressively in risky assets than she does in the case where the investment opportunity set stays constant for all levels of wealth. Hence, this paper shows how the optimal behavior of an investor who expects a better investment opportunity when she becomes rich is different from that of the other investor who does not have such an expectation.

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[1]We can still invest in individual stocks. But in such a case it is hard to have a well-diversified portfolio.

Previous research on consumption and investment problems were done mostly under the assumption that the investment opportunity set does not change. Merton [7] examined the continuous-time consumption investment problem for an individual whose income is generated by capital gains on investment in assets with prices following the *geometric Brownian motion*. He obtained explicit solutions for the optimal consumption and portfolio rules under the assumption of a constant relative or absolute risk aversion utility function. Merton [8] extended these results allowing for more general utility functions, more general price behavior assumptions and additional income generated from non-capital gains sources. Karatzas, Lehoczky, and Shreve [3] studied a general consumption and investment problem in which an agent seeks to maximize total expected discounted utility of both consumption and terminal wealth in a finite time horizon. Under general conditions on the nature of the market and on the utility functions, it was shown that the problem can be solved by decomposing it into two parts, maximizing utility of consumption and maximizing utility of terminal wealth. In the case of a market model with constant coefficients, the optimal consumption and portfolio rules were derived explicitly in feedback form on the current level of wealth. Karatzas and Wang [4] studied a consumption and investment problem allowing the agent to stop freely before or at a pre-specified terminal time, in order to maximize the expected utility of her wealth and/or consumption up to the stopping time. Karatzas, Lehoczky, Sethi, and Shreve [2] solved a general consumption and investment problem in which an investor seeks to maximize total discounted utility of consumption in an infinite time horizon. They determined the value function, optimal consumption and investment policy explicitly under the assumption that the price processes of risky assets follow geometric Brownian motion, but allowing for a general utility function. This paper is different from previous research in the sense that the investment opportunity set gets larger as the investor becomes rich enough.

The paper proceeds as follows. Section 2 sets up the consumption and investment problem. Section 3 derives the properties of the optimal value functions, optimal consumption and portfolio decisions under the assumption that the investor's preference can be represented by a time-separable von Neumann-Morgenstern utility function. Some examples in the HARA utility class are given in Section 4. Section 5 concludes and Appendix contains proofs of all the results.

2. AN INVESTMENT PROBLEM

We consider a market in which there is a riskless asset and $m + n$ risky assets. We assume that the risk-free rate is a constant $r > 0$ and the price

$p_0(t)$ of the riskless asset follows a deterministic process

$$dp_0(t) = p_0(t)rdt, \quad p_0(0) = p_0.$$

The price $p_j(t)$ of the j-th risky asset, as in Karatzas, Lehoczky, Sethi, and Shreve [2] and Merton [7], [8], follows geometric Brownian motion

$$dp_j(t) = p_j(t)\big\{\alpha_j dt + \sum_{k=1}^{m+n} \sigma_{jk} dw_k(t)\big\}, \quad p_j(0) = p_j, \quad j = 1, \ldots, m+n,$$

where $\mathbf{w}(t) = (w_1(t), \ldots, w_{m+n}(t))$ is a $(m+n)$ dimensional standard Brownian motion defined on the underlying probability space $(\Omega, \mathcal{F}, P)$. The market parameters, α_j's and σ_{jk}'s for $j, k = 1, \ldots, m+n$, are assumed to be constants. We assume that the matrix $D = (\sigma_{ij})_{i,j=1}^{m+n}$ is nonsingular, i.e., there is no redundant asset among the $m+n$ risky assets. Hence $\Sigma \equiv DD^T$ is positive definite. Let D_m denote the first m by $m+n$ submatrix of D, and $\Sigma_m = D_m D_m^T$, then Σ_m is the first m by m submatrix of Σ and is also positive definite.

The problem in this paper is different from an ordinary consumption and investment problem in the following sense: let x_t be the investor's wealth at time t and T_ξ the first time that her wealth reaches ξ. There exists a critical wealth level z such that if time t is less than T_z then the investor is restricted to invest only in the riskless asset and the first m risky assets, but if $t \geq T_z$ then the investor can invest in all $m+n+1$ assets.

The investor's wealth process x_t, $0 \leq t < \infty$, evolves according to

$$dx_t = (\alpha - r\mathbf{1}_{m+n})\pi_t^T x_t dt + (rx_t - c_t)dt + x_t \pi_t D d\mathbf{w}^T(t), \quad x_0 = x > 0, \ (1)$$

where c_t is the consumption rate at time t, $\alpha = (\alpha_1, \ldots, \alpha_{m+n})$ the row vector of returns of risky assets, $\pi_t = (\pi_{1,t}, \ldots, \pi_{m+n,t})$ the row vector of fractions of wealth invested in the risky assets at time t, and $\mathbf{1}_{m+n} = (1, \ldots, 1)$ the row vector of $m+n$ ones. The superscript T denotes the transpose of a matrix or vector. The investor faces a nonnegative wealth constraint

$$x_t \geq 0, \quad \text{for all } t \geq 0 \text{ a.s.} \tag{2}$$

The controls are the consumption rate $\mathbf{c} = (c_t)_{t=0}^\infty$ and the vector process $\pi = (\pi_t)_{t=0}^\infty$. The consumption rate process $(c_t)_{t=0}^\infty$ is a nonnegative process adapted to $(\mathcal{F}_t)_{t=0}^\infty$, the augmentation under P of the natural filtration generated by the standard Brownian motion $(\mathbf{w}(t))_{t=0}^\infty$, and satisfies $\int_0^t c_s ds < \infty$, for all $t \geq 0$, a.s.

Let $T(\pi) \equiv \sup\{t \geq 0 : \int_0^t \pi_s \Sigma \pi_s^T ds < \infty\}$. Then we require that $T(\pi) = \infty$ or $T_0 < T(\pi)$ or $\lim_{t \uparrow T(\pi)} x_t = 0$ so that the stochastic differential equation (1) has a unique strong solution.

As described above, the investor faces the restriction

$$\pi_{m+1,t} = \cdots = \pi_{m+n,t} = 0, \tag{3}$$

for $t < T_z$. We define an admissible set $A(x)$ by the set of control processes satisfying the above conditions with $x_0 = x$. The investor wishes to choose $(\mathbf{c}, \pi) \in A(x)$ to maximize the expected total reward

$$V_{(\mathbf{c},\pi)}(x) \equiv E_x \int_0^\infty \exp{(-\beta t)} U(c_t) dt$$

for $0 < x \equiv x_0 < z$, where E_x denotes the expectation operator conditioned on $x_0 = x$, the function U, called a utility function, is real-valued on $(0, \infty)$ and $\beta > 0$ is a discount rate. We assume that U is strictly increasing, strictly concave and three times continuously differentiable. We also assume that $\lim_{c \to \infty} U'(c) = 0$. For later use, we let $I(\cdot)$ be the inverse function of $U'(\cdot)$.

We let

$$V^*(x) \equiv \sup \{V_{(\mathbf{c},\pi)}(x) : (\mathbf{c}, \pi) \in A(x)\} \tag{4}$$

be the optimal expected reward or the optimal value at wealth x. Put

$$\kappa_1 \equiv \frac{1}{2}(\tilde{\alpha} - r\mathbf{1}_m)\Sigma_m^{-1}(\tilde{\alpha} - r\mathbf{1}_m)^T,$$

where $\mathbf{1}_m$ is the row vector of m ones and $\tilde{\alpha}$ denotes the m dimensional row vector consisting of the first m components of α. If we assume that $\kappa_1 > 0$, then the quadratic equation of λ

$$\kappa_1 \lambda^2 - (r - \beta - \kappa_1)\lambda - r = 0 \tag{5}$$

has two distinct solutions $\lambda_- < -1$ and $\lambda_+ > 0$. When the investment opportunity set consists constantly of the one riskless asset and the first m risky assets, i.e. when $\pi_{m+1,t} = \cdots = \pi_{m+n,t} = 0$ for all $t \geq 0$, the optimal value at x, say $V_m(x)$, is finite and attainable by a strategy for all $x > 0$, as is shown in Karatzas, Lehoczky, Sethi, and Shreve [2], if it is assumed that

$$\int_c^\infty \frac{d\theta}{(U'(\theta))^{\lambda_-}} < \infty \tag{6}$$

for all $c > 0$. If the utility function is given by $U(c) = \frac{c^{1-\gamma}}{1-\gamma}$, $0 < \gamma \neq 1$ for $c > 0$, condition (6) is equivalent to $-\gamma\lambda_- > 1$, which is again equivalent to

$$K_1 > 0, \tag{7}$$

where

$$K_1 \equiv r + \frac{\beta - r}{\gamma} + \frac{\gamma - 1}{\gamma^2}\kappa_1, \tag{8}$$

since λ_- is the negative solution of the equation (5). This condition (7) is equivalent to condition (40) in Merton [7]. If the utility function is given by $U(c) = \log c$ or $U(c) = -\exp(-ac), \quad a > 0$ for $c > 0$, condition (6) is automatically satisfied. Similarly, put

$$\kappa_2 \equiv \frac{1}{2}(\alpha - r\mathbf{1}_{m+n})\Sigma^{-1}(\alpha - r\mathbf{1}_{m+n})^T.$$

If $\kappa_1 > 0$, then $\kappa_2 > 0$ and the quadratic equation of η

$$\kappa_2\eta^2 - (r - \beta - \kappa_2)\eta - r = 0 \tag{9}$$

has two distinct solutions $\eta_- < -1$ and $\eta_+ > 0$. When the investment opportunity set consists constantly of all the $m + n + 1$ assets, i.e. when the constraint (3) is not imposed, the optimal value at x, say $V_{m+n}(x)$, is finite and attainable by a strategy for all $x > 0$ if it is assumed that

$$\int_c^\infty \frac{d\theta}{(U'(\theta))^{\eta_-}} < \infty \tag{10}$$

for all $c > 0$. If the utility function is given by $U(c) = \frac{c^{1-\gamma}}{1-\gamma}, \quad 0 < \gamma \neq 1$ for $c > 0$, condition (10) is equivalent to $-\gamma\eta_- > 1$, which is again equivalent to

$$K_2 > 0, \tag{11}$$

where

$$K_2 \equiv r + \frac{\beta - r}{\gamma} + \frac{\gamma - 1}{\gamma^2}\kappa_2, \tag{12}$$

since η_- is the negative solution of the equation (9). This condition (11) is also equivalent to condition (40) in Merton [7]. If the utility function is given by $U(c) = \log c$ or $U(c) = -\exp(-ac), \quad a > 0$ for $c > 0$, condition (10) is automatically satisfied. Thus, we assume

ASSUMPTION 1.

$$\kappa_1 > 0, \tag{13}$$

$$\int_c^\infty \frac{d\theta}{(U'(\theta))^{\lambda_-}} < \infty \tag{14}$$

for all $c > 0$, and

$$\int_c^\infty \frac{d\theta}{(U'(\theta))^{\eta-}} < \infty \tag{15}$$

for all $c > 0$.

It is obvious that the optimal value at wealth z with the investment opportunity set consisting constantly of all the $m+n+1$ assets is larger than or equal to that with the investment opportunity set consisting constantly of the one riskless asset and the first m risky assets. That is, $V_{m+n}(z) \geq V_m(z)$. However, if $V_{m+n}(z) = V_m(z)$, then the optimal value at wealth before touching the critical level z is the same as that of the case where the investment opportunity set consists constantly of the one riskless asset and the first m risky assets and it is attainable with the same strategy. (See Karatzas, Lehoczky, Sethi and Shreve [2].) Therefore we consider only the case

$$V_{m+n}(z) > V_m(z), \tag{16}$$

which implies that the investment opportunity set consisting constantly of all the $m+n+1$ assets is better than that of the one riskless asset and the first m risky assets for the investor's optimal performance.

3. OPTIMAL POLICIES AND THEIR PROPERTIES

In this section we state optimal policies and value functions with general utility function class. We also propose properties of the optimal policies compared to the case when the investment opportunity set consists constantly of the one riskless asset and the first m risky assets. Proofs for all the assertions and statements in this section are deferred to Appendix.

For optimal policies and the value function, we first consider the case when $U'(0) = \infty$: For $\hat{B} \geq 0$, define

$$X(c; \hat{B}) = \hat{B}(U'(c))^{\lambda-} + X_0(c) \tag{17}$$

for $c > 0$, where

$$X_0(c) = \frac{c}{r} - \frac{1}{\kappa_1(\lambda_+ - \lambda_-)}\Big\{ \frac{(U'(c))^{\lambda_+}}{\lambda_+} \int_0^c \frac{d\theta}{(U'(\theta))^{\lambda_+}}$$
$$+ \frac{(U'(c))^{\lambda_-}}{\lambda_-} \int_c^\infty \frac{d\theta}{(U'(\theta))^{\lambda_-}} \Big\}. \tag{18}$$

Then, $X'(c; \hat{B}) > 0$ for all $c > 0$ and $X(\cdot; \hat{B})$ maps $[0, \infty)$ onto itself if we let $X(0) \equiv \lim_{c \downarrow 0} X(c)$. Let $C(\cdot; \hat{B})$ be the inverse function of $X(\cdot; \hat{B})$ and let $C_0 \equiv C(\cdot; 0)$, that is, C_0 is the inverse function of X_0. For $\hat{A} \geq 0$, we also define

$$J(c; \hat{A}) = \hat{A}(U'(c))^{\rho_-} + J_0(c) \tag{19}$$

for $c > 0$, where

$$\begin{aligned} J_0(c) &= \frac{U(c)}{\beta} - \frac{1}{\kappa_1(\rho_+ - \rho_-)} \Big\{ \frac{(U'(c))^{\rho_+}}{\rho_+} \int_0^c \frac{d\theta}{(U'(\theta))^{\lambda_+}} \\ &+ \frac{(U'(c))^{\rho_-}}{\rho_-} \int_c^\infty \frac{d\theta}{(U'(\theta))^{\lambda_-}} \Big\}, \end{aligned} \tag{20}$$

where $\rho_+ = 1 + \lambda_+$ and $\rho_- = 1 + \lambda_-$. If the investment opportunity set consists constantly of the one riskless asset and the first m risky assets, then as is shown in Karatzas, Lehoczky, Sethi and Shreve [2], the optimal value at x becomes

$$V_m(x) = J_0(C_0(x)), \tag{21}$$

for $x > 0$ and an optimal strategy is given by

$$c_t = C_0(x_t), \quad \tilde{\pi}_t = \frac{V_m'(x_t)}{-x_t V_m''(x_t)} (\tilde{\alpha} - r\mathbf{1}_m)\Sigma_m^{-1}, \tag{22}$$

for $t \geq 0$ where $\tilde{\pi}_t$ denote the vector of fractions of wealth invested in the first m risky assets at time t. Define

$$\begin{aligned} X_{m+n}(c) &= \frac{c}{r} - \frac{1}{\kappa_2(\eta_+ - \eta_-)} \Big\{ \frac{(U'(c))^{\eta_+}}{\eta_+} \int_0^c \frac{d\theta}{(U'(\theta))^{\eta_+}} \\ &+ \frac{(U'(c))^{\eta_-}}{\eta_-} \int_c^\infty \frac{d\theta}{(U'(\theta))^{\eta_-}} \Big\}, \end{aligned} \tag{23}$$

for $c > 0$. Then, $X'_{m+n}(c) > 0$ for all $c > 0$ and $X_{m+n}(\cdot)$ maps $[0, \infty)$ onto itself if we let $X_{m+n}(0) \equiv \lim_{c \downarrow 0} X_{m+n}(c)$. Let $C_{m+n}(\cdot)$ be the inverse function of $X_{m+n}(\cdot)$. We also define

$$\begin{aligned} J_{m+n}(c) &= \frac{U(c)}{\beta} - \frac{1}{\kappa_2(\rho_+ - \rho_-)} \Big\{ \frac{(U'(c))^{\nu_+}}{\nu_+} \int_0^c \frac{d\theta}{(U'(\theta))^{\eta_+}} \\ &+ \frac{(U'(c))^{\nu_-}}{\nu_-} \int_c^\infty \frac{d\theta}{(U'(\theta))^{\eta_-}} \Big\}, \end{aligned} \tag{24}$$

where $\nu_+ = \eta_+ + 1$ and $\nu_- = \eta_- + 1$. Similarly to above, if the investment opportunity set consists constantly of all the $m + n + 1$ assets, then the

optimal value at x becomes

$$V_{m+n}(x) = J_{m+n}(C_{m+n}(x)), \tag{25}$$

for $x > 0$ and an optimal strategy is given by

$$c_t = C_{m+n}(x_t), \quad \pi_t = \frac{V'_{m+n}(x_t)}{-x_t V''_{m+n}(x_t)}(\alpha - r\mathbf{1}_{m+n})\Sigma^{-1}, \tag{26}$$

for $t \geq 0$. Let S be the $(m+n)$ dimensional row vector the first m components of which are equal to those of $(\tilde\alpha - r\mathbf{1}_m)\Sigma_m^{-1}$ and the rest are equal to zero. With a $\hat{B} > 0$, whose determination is shown in Appendix A.1, we state the following theorem.

THEOREM 1. *Suppose that $U'(0) = \infty$. Then, the value function is given by*

$$V^*(x) = J(C(x; \hat{B}); \frac{\lambda_-}{\rho_-}\hat{B}) \tag{27}$$

for $0 < x \equiv x_0 < z$, and an optimal policy is given by

$$c_t = C(x_t; \hat{B}), \quad \pi_t = \frac{V^{*\prime}(x_t)}{-x_t V^{*\prime\prime}(x_t)}S \tag{28}$$

for $0 \leq t < T_z$, where $\hat{B} > 0$ is given in Appendix A.1, and

$$c_t = C_{m+n}(x_t), \quad \pi_t = \frac{V'_{m+n}(x_t)}{-x_t V''_{m+n}(x_t)}(\alpha - r\mathbf{1}_{m+n})\Sigma^{-1} \tag{29}$$

for $t \geq T_z$, where $V_{m+n}(\cdot)$ is given by (25).

Now we consider the case where $U'(0)$ is finite: Recall that $I : (0, U'(0)] \rightarrow$

$[0, \infty)$ is the inverse of $U' : [0, \infty) \rightarrow (0, U'(0)]$. We extend I by setting $I \equiv 0$ on $[U'(0), \infty)$. Define for $\hat{B} \geq 0, \hat{A} \geq 0$,

$$\mathcal{X}(y; \hat{B}) = \hat{B}y^{\lambda_-} + \mathcal{X}_0(y), \tag{30}$$

where

$$\mathcal{X}_0(y) = \frac{I(y)}{r} - \frac{1}{\kappa_1(\lambda_+ - \lambda_-)}[\frac{y^{\lambda_+}}{\lambda_+}\int_0^{I(y)}\frac{d\theta}{(U'(\theta))^{\lambda_+}} \\ + \frac{y^{\lambda_-}}{\lambda_-}\int_{I(y)}^{\infty}\frac{d\theta}{(U'(\theta))^{\lambda_-}}], \tag{31}$$

and

$$\mathcal{J}(y; \hat{A}) = \hat{A} y^{\rho_-} + \mathcal{J}_0(y), \tag{32}$$

where

$$\mathcal{J}_0(y) = \frac{U(I(y))}{\beta}$$

$$- \frac{1}{\kappa_1(\rho_+ - \rho_-)} \left[\frac{y^{\rho_+}}{\rho_+} \int_0^{I(y)} \frac{d\theta}{(U'(\theta))^{\lambda_+}} + \frac{y^{\rho_-}}{\rho_-} \int_{I(y)}^\infty \frac{d\theta}{(U'(\theta))^{\lambda_-}} \right], \tag{33}$$

for $y > 0$. Then $\mathcal{X}(\cdot; \hat{B})$ is strictly decreasing, maps $(0, \infty)$ onto $(0, \infty)$ and has an inverse $\mathcal{Y}(\cdot; \hat{B}) : (0, \infty) \to (0, \infty)$. Let $\mathcal{Y}_0(\cdot) \equiv \mathcal{Y}(\cdot; 0)$. That is, $\mathcal{Y}_0(\cdot)$ is the inverse function of $\mathcal{X}_0(\cdot)$. If the investment opportunity set consists constantly of the one riskless asset and the first m risky assets, then as is shown in Karatzas, Lehoczky, Sethi and Shreve [2], the optimal value at x becomes

$$V_m(x) = \mathcal{J}_0(\mathcal{Y}_0(x)), \tag{34}$$

for $x > 0$, and an optimal strategy is given by

$$c_t = I(V_m'(x_t)), \quad \tilde{\pi}_t = \frac{V_m'(x_t)}{-x_t V_m''(x_t)} (\tilde{\alpha} - r \mathbf{1}_m) \Sigma_m^{-1}, \tag{35}$$

for $t \geq 0$. Define

$$\mathcal{X}_{m+n}(y) \equiv \frac{I(y)}{r}$$

$$- \frac{1}{\kappa_2(\eta_+ - \eta_-)} \left[\frac{y^{\eta_+}}{\eta_+} \int_0^{I(y)} \frac{d\theta}{(U'(\theta))^{\eta_+}} + \frac{y^{\eta_-}}{\eta_-} \int_{I(y)}^\infty \frac{d\theta}{(U'(\theta))^{\eta_-}} \right]$$

Then, $\mathcal{X}_{m+n}(\cdot)$ is strictly decreasing, maps $(0, \infty)$ onto itself and has an inverse function $\mathcal{Y}_{m+n}(\cdot)$. We also define

$$\mathcal{J}_{m+n}(y) = \frac{U(I(y))}{\beta}$$

$$- \frac{1}{\kappa_2(\nu_+ - \nu_-)} \left[\frac{y^{\nu_+}}{\nu_+} \int_0^{I(y)} \frac{d\theta}{(U'(\theta))^{\eta_+}} + \frac{y^{\nu_-}}{\nu_-} \int_{I(y)}^\infty \frac{d\theta}{(U'(\theta))^{\eta_-}} \right], \tag{36}$$

for $y > 0$. Similarly to the above, if the investment opportunity set consists constantly of all the $m + n + 1$ assets, then the optimal value at x becomes

$$V_{m+n}(x) = \mathcal{J}_{m+n}(\mathcal{Y}_{m+n}(x)), \tag{37}$$

for $x > 0$ and an optimal strategy is given by

$$c_t = I(V'_{m+n}(x_t)), \quad \pi_t = \frac{V'_{m+n}(x_t)}{-x_t V''_{m+n}(x_t)}(\alpha - r\mathbf{1}_{m+n})\Sigma^{-1}, \qquad (38)$$

for $t \geq 0$. With a $\hat{B} > 0$, whose determination is shown in Appendix B, we state the following theorem.

THEOREM 2. *Suppose that $U'(0) < \infty$. Then, the value function is given by*

$$V^*(x) = \mathcal{J}(\mathcal{Y}(x; \hat{B}); \frac{\lambda_-}{\rho_-}\hat{B}) \qquad (39)$$

for $0 < x \equiv x_0 < z$, and an optimal policy is given by

$$c_t = I(V^{*\prime}(x_t)), \quad \pi_t = \frac{V^{*\prime}(x_t)}{-x_t V^{*\prime\prime}(x_t)}S \qquad (40)$$

for $0 \leq t < T_z$, where $\hat{B} > 0$ is given in Appendix B, and

$$c_t = I(V'_{m+n}(x_t)), \quad \pi_t = \frac{V'_{m+n}(x_t)}{-x_t V''_{m+n}(x_t)}(\alpha - r\mathbf{1}_{m+n})\Sigma^{-1}, \qquad (41)$$

for $t \geq T_z$, where $V_{m+n}(\cdot)$ is given by (37).

The following two propositions illustrate effects of the change of the investment opportunity set on optimal consumption and portfolio decisions. An interesting property is that the investor consumes less if the investor expects a better investment opportunity when she becomes rich than she does if she does not have such an expectation. Intuitively, she tries to accumulate her wealth fast enough to exploit a better investment opportunity and sacrifices her current consumption. Proposition 1 states this property.

PROPOSITION 1. *When $U'(0) = \infty$,*

$$C(x; \hat{B}) < C_0(x) \qquad (42)$$

for $0 < x < z$, where $C(x; \hat{B})$ is given in Theorem 1 and $C_0(x)$ in (22).

When $U'(0) < \infty$, if $\mathcal{X}_0(U'(0)) < z$ then

$$I(V^{*\prime}(x)) = I(V'_m(x)) = 0 \qquad (43)$$

for $x \leq \mathcal{X}_0(U'(0))$, and

$$I(V^{*'}(x)) < I(V'_m(x)) \tag{44}$$

for $\mathcal{X}_0(U'(0)) < x < z$, where $I(V^{*'}(x))$ is given in Theorem 2 and $I(V'_m(x))$ in (35).

The investor tends to take more risk and thereby tries to increase expected growth rate of her wealth when she expects a better investment opportunity at the critical wealth level. This is summarized in Proposition 2.

PROPOSITION 2. $\frac{V^{*'}(x)}{-V^{*''}(x)} > \frac{V'_m(x)}{-V''_m(x)}$ *for $0 < x < z$, in both cases where $U'(0) = \infty$ and where $U'(0) < \infty$.*

4. SOME EXAMPLES OF HARA UTILITY CLASS

In this section, we illustrate HARA utility class as examples. In calculations in this section, often used the relations are:

$$\lambda_+ + \lambda_- = \frac{r - \beta - \kappa_1}{\kappa_1}, \quad \lambda_+ \lambda_- = \frac{-r}{\kappa_1}, \quad \rho_+ \rho_- = \frac{-\beta}{\kappa_1}$$

and

$$\eta_+ + \eta_- = \frac{r - \beta - \kappa_2}{\kappa_2}, \quad \eta_+ \eta_- = \frac{-r}{\kappa_2}, \quad \nu_+ \nu_- = \frac{-\beta}{\kappa_2}.$$

4.1. CRRA utility case: $U'(0) = \infty$

When the utility function is given by

$$U(c) = \frac{c^{1-\gamma}}{1 - \gamma}$$

with $0 < \gamma \neq 1$, $U'(c) = c^{-\gamma}$. Hence $U'(0) = \infty$. As mentioned in section 2, (6) (or (14)) is equivalent to $K_1 > 0$, and (10) (or (15)) is equivalent to $K_2 > 0$.

For $\hat{B} \geq 0$, by calculation, (17) and (18) become

$$X(c; \hat{B}) = \hat{B}c^{-\gamma\lambda_-} + X_0(c) \tag{45}$$

for $c > 0$, where $X_0(c) = \frac{1}{K_1}c$. Hence $C_0(x) = K_1 x$, for $x > 0$. For $\hat{A} \geq 0$, by calculating, (19) and (20) become

$$J(c; \hat{A}) = \hat{A}c^{-\gamma\rho_-} + J_0(c) \tag{46}$$

for $c > 0$, where

$$J_0(c) = \frac{1}{(1-\gamma)K_1}c^{1-\gamma}. \tag{47}$$

Hence, if the investment opportunity set consists constantly of the one riskless asset and the first m risky assets, the value function becomes

$$V_m(x) = J_0(C_0(x)) = \frac{K_1^{-\gamma}}{1-\gamma}x^{1-\gamma} \tag{48}$$

for $x > 0$, and an optimal strategy given by (22) becomes

$$c_t = K_1 x_t, \quad \tilde{\pi}_t = \frac{1}{\gamma}(\tilde{\alpha} - r\mathbf{1}_m)\Sigma_m^{-1}, \tag{49}$$

for $t \geq 0$. By calculation, (23) becomes

$$X_{m+n}(c) = \frac{1}{K_2}c$$

for $c > 0$, hence $C_{m+n}(x) = K_2 x$ for $x > 0$, and (24) becomes

$$J_{m+n}(c) = \frac{1}{(1-\gamma)K_2}c^{1-\gamma}.$$

Hence, when the investment opportunity set consists constantly of all the $m + n + 1$ assets the value function becomes

$$V_{m+n}(x) = J_{m+n}(C_{m+n}(x)) = \frac{K_2^{-\gamma}}{1-\gamma}x^{1-\gamma} \tag{50}$$

for $x > 0$, and an optimal strategy given by (26) becomes

$$c_t = K_2 x_t, \quad \pi_t = \frac{1}{\gamma}(\alpha - r\mathbf{1}_{m+n})\Sigma^{-1}, \tag{51}$$

for $t \geq 0$. By (48) and (50), The inequality (16) is equivalent to the condition

$$\kappa_1 < \kappa_2. \tag{52}$$

Recall that $C(\cdot, \hat{B})$ is the inverse function of $X(\cdot, \hat{B})$. With a $\hat{B} > 0$, whose determination is shown generally in Appendix A.1, the value function (27) in Theorem 1 becomes

$$V^*(x) = J(C(x; \hat{B}); \frac{\lambda_-}{\rho_-}\hat{B})$$

$$= \frac{\lambda_-}{\rho_-}\hat{B}(C(x; \hat{B}))^{-\gamma\rho_-} + \frac{1}{(1-\gamma)K_1}(C(x; \hat{B}))^{1-\gamma}, \quad (53)$$

and the optimal policy in Theorem 1 is given by

$$c_t = C(x_t, \hat{B}), \quad \pi_t = \frac{V^{*\prime}(x_t)}{-x_t V^{*\prime\prime}(x_t)}S$$

$$= \frac{1}{x_t}(-\lambda_-\hat{B}c_t^{-\gamma\lambda_-} + \frac{1}{\gamma K_1}c_t)S \quad (54)$$

for $0 \le t < T_z$, and

$$c_t = K_2 x_t, \quad \pi_t = \frac{V'_{m+n}(x_t)}{-x_t V''_{m+n}(x_t)}(\alpha - r\mathbf{1}_{m+n})\Sigma^{-1}$$

$$= \frac{1}{\gamma}(\alpha - r\mathbf{1}_{m+n})\Sigma^{-1} \quad (55)$$

for $t \ge T_z$.

Figure 1 compares consumption rate in the case where the investment opportunity set consists constantly of one riskless asset and one risky asset with one in the case where the investor can invest in only these assets before her wealth level touching $z = 1$, however, she can invest in another risky asset once her wealth level touches $z = 1$. As is shown in the figure, the investor consume less before touching the critical wealth level in the latter case than in the former case, illustrating the inequality (42) in Proposition 1.

Figure 2 compares total investments in the risky assets in the two cases. As is shown in the figure, the investor invests more in the risky assets before touching the critical wealth level in the latter case than in the former case, illustrating the inequality in Proposition 2.

Consider the case when utility function is given by

$$U(c) = \log c,$$

for $c > 0$. In this case, $U'(c) = \frac{1}{c}$ so that $U'(0) = \infty$. As mentioned in section 2, (6) (or (14)) and (10) (or (15)) are automatically satisfied. We

FIG. 1.   Comparison of consumption rates with $U(c) = \frac{c^{1-\gamma}}{1-\gamma}$.
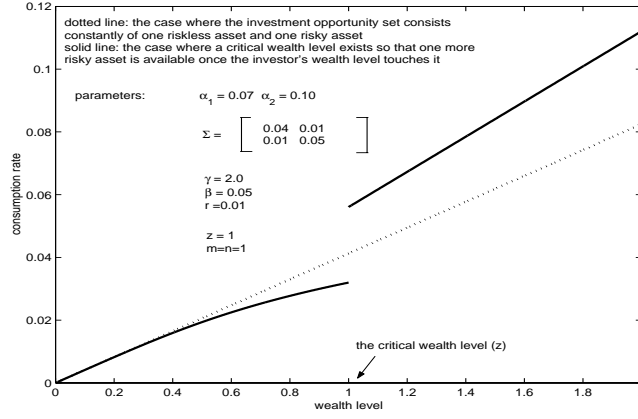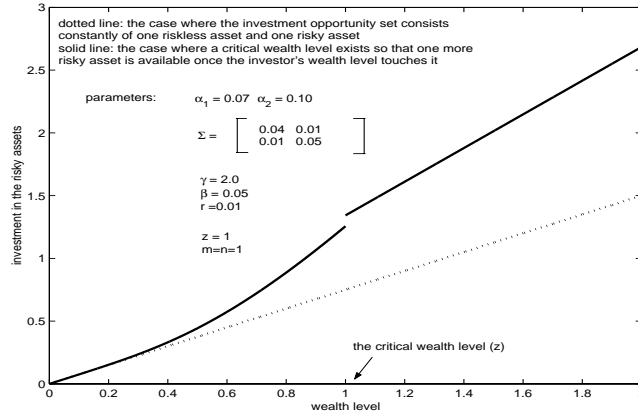

FIG. 2.   Comparison of investments in the risky assets with $U(c) = \frac{c^{1-\gamma}}{1-\gamma}$.

have for $\hat{B} \geq 0$,

$$X(c; \hat{B}) = \hat{B}c^{-\lambda_-} + X_0(c) \qquad (56)$$

for $c > 0$, where

$$X_0(c) = \frac{1}{\beta}c.$$

Hence

$$C_0(x) = \beta x,$$

for $x > 0$. For $\hat{A} \geq 0$, we have

$$J(c; \hat{A}) = \hat{A}c^{-\rho_-} + J_0(c) \qquad (57)$$

for $c > 0$, where

$$J_0(c) = \frac{\beta \log c + \kappa_1 + r - \beta}{\beta^2}.$$ (58)

Hence, if the investment opportunity set consists constantly of the one riskless asset and the first m risky assets, the value function becomes

$$V_m(x) = J_0(C_0(x)) = \frac{\beta \log \beta x + \kappa_1 + r - \beta}{\beta^2}$$ (59)

for $x > 0$, and an optimal strategy given by (22) becomes

$$c_t = \beta x_t, \quad \tilde{\pi}_t = (\tilde{\alpha} - r\mathbf{1}_m)\Sigma_m^{-1},$$ (60)

for $t \geq 0$. We get $X_{m+n}(c) = \frac{1}{\beta}c$ for $c > 0$, and $C_{m+n}(x) = \beta x$ for $x > 0$. We have

$$J_{m+n}(c) = \frac{\beta \log c + \kappa_2 + r - \beta}{\beta^2}.$$

Thus, when the investment opportunity set consists constantly of all the $m + n + 1$ assets the value function becomes

$$V_{m+n}(x) = J_{m+n}(C_{m+n}(x)) = \frac{\beta \log \beta x + \kappa_2 + r - \beta}{\beta^2}.$$ (61)

for $x > 0$, and an optimal strategy given by (26) becomes

$$c_t = \beta x_t, \quad \pi_t = (\alpha - r\mathbf{1}_{m+n})\Sigma^{-1},$$ (62)

for $t \geq 0$. By (59) and (61), The inequality (16) is equivalent to the condition (52) in this case too.

Recall that $C(\cdot, \hat{B})$ is the inverse function of $X(\cdot, \hat{B})$. With a $\hat{B} > 0$, whose determination is shown generally in Appendix A.1, the value function (27) in Theorem 1 becomes

$$\begin{aligned} V^*(x) &= J(C(x; \hat{B}); \frac{\lambda_-}{\rho_-}\hat{B}) \\ &= \frac{\lambda_-}{\rho_-}\hat{B}(C(x; \hat{B}))^{-\rho_-} + \frac{\beta \log C(x; \hat{B}) + \kappa_1 + r - \beta}{\beta^2}, \end{aligned}$$ (63)

and the optimal policy in Theorem 1 is given by

$$\begin{aligned} c_t &= C(x_t, \hat{B}), \\ \pi_t &= \frac{V^{*\prime}(x_t)}{-x_t V^{*\prime\prime}(x_t)}S = (1 + \frac{(-1 - \lambda_-)\hat{B}c_t^{-\lambda_-}}{x_t})S \end{aligned}$$ (64)

for $0 \le t < T_z$, and

$$c_t = \beta x_t,$$
$$\pi_t = \frac{V'_{m+n}(x_t)}{-x_t V''_{m+n}(x_t)}(\alpha - r\mathbf{1}_{m+n})\Sigma^{-1} = (\alpha - r\mathbf{1}_{m+n})\Sigma^{-1} \qquad (65)$$

for $t \ge T_z$.

Figure 3 compares consumption rate in the case where the investment opportunity set consists constantly of one riskless asset and one risky asset with one in the case where the investor can invest in only these assets before her wealth level touching $z = 1$, however, she can invest in another risky asset once her wealth level touches $z = 1$. As is shown in the figure, the investor consume less before touching the critical wealth level in the latter case than in the former case, illustrating the inequality (42) in Proposition 1.

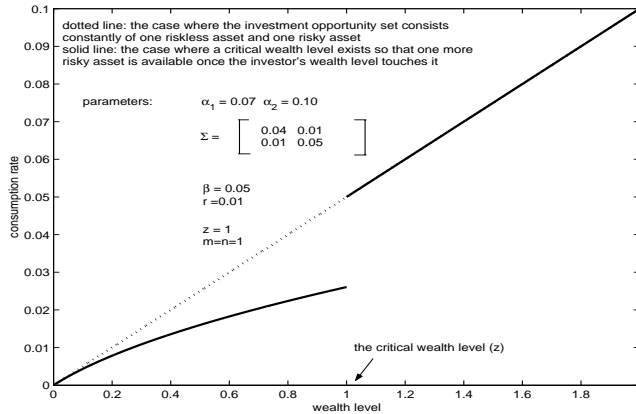**FIG. 3.** Comparison of consumption rates with $U(c) = \log c$.

Figure 4 compares total investments in the risky assets in the two cases. As is shown in the figure, the investor invests more in the risky assets before touching the critical wealth level in the latter case than in the former case, illustrating the inequality in Proposition 2.

In Figure 1 and Figure 3, we can observe that the consumption rates have jumps at time T_z in the case when the critical wealth level exists. We state the following lemma whose proof is given in Appendix B.1.

LEMMA 1. *When the utility function is given by* $U(c) = \frac{c^{1-\gamma}}{1-\gamma}, \quad c > 0,$ *or* $U(c) = \log c, \quad c > 0,$ *the optimal consumption process has a jump of positive size at time* T_z.

FIG. 4. Comparison of investments in the risky assets with $U(c) = \log c$.
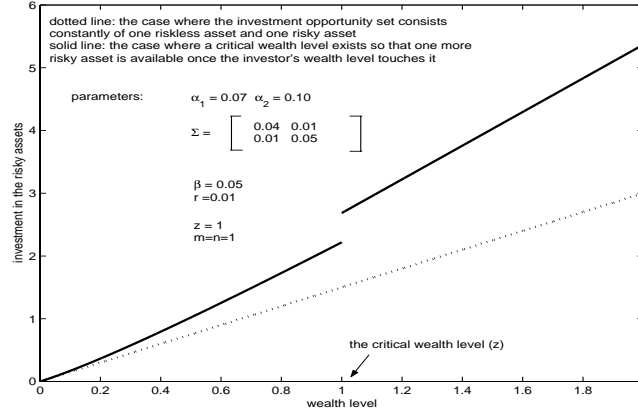
4.2. CARA utility class: $U'(0) < \infty$

When the utility function is given by

$$U(c) = -\exp(-ac),$$

where $a > 0$ is the absolute risk aversion coefficient, $U'(c) = a \exp(-ac)$ so that $U'(0) = a < \infty$. As mentioned in section 2, (6) (or (14)) and (10) (or (15)) are automatically satisfied. $I(\cdot) : (0, \infty) \to [0, \infty)$ becomes

$$I(y) = \begin{cases} \frac{1}{a} \log \frac{a}{y}, & 0 < y \leq a = U'(0), \\ 0, & y \geq a. \end{cases} \qquad (66)$$

(30) and (31) by calculation become

$$\mathcal{X}(y; \hat{B}) = \hat{B} y^{\lambda_-} + \mathcal{X}_0(y), \qquad (67)$$

where

$$\mathcal{X}_0(y) = \frac{I(y)}{r} - \frac{y^{\lambda_+}}{\kappa_1 \lambda_+^2 (\lambda_+ - \lambda_-) a^{\rho_+}} [\exp(a\lambda_+ I(y)) - 1]$$

$$+ \frac{y^{\lambda_-}}{\kappa_1 \lambda_-^2 (\lambda_+ - \lambda_-) a^{\rho_-}} \exp(a\lambda_- I(y))$$

and (32) and (33) become

$$\mathcal{J}(y; \hat{A}) = \hat{A} y^{\rho_-} + \mathcal{J}_0(y), \qquad (68)$$

where

$$\mathcal{J}_0(y) = \frac{U(I(y))}{\beta} - \frac{y^{\rho_+}}{\kappa_1 \lambda_+ \rho_+ (\rho_+ - \rho_-) a^{\rho_+}} [\exp(a\lambda_+ I(y)) - 1]$$
$$+ \frac{y^{\rho_-}}{\kappa_1 \lambda_- \rho_- (\rho_+ - \rho_-) a^{\rho_-}} \exp(a\lambda_- I(y)).$$

Hence, if the investment opportunity set consists constantly of the one riskless asset and the first m risky assets the value function becomes

$$V_m(x) = \mathcal{J}_0(\mathcal{Y}_0(x))$$
$$= \frac{U(I(\mathcal{Y}_0(x)))}{\beta} - \frac{(\mathcal{Y}_0(x))^{\rho_+}}{\kappa_1 \lambda_+ \rho_+ (\rho_+ - \rho_-) a^{\rho_+}} [\exp(a\lambda_+ I(\mathcal{Y}_0(x))) - 1]$$
$$+ \frac{(\mathcal{Y}_0(x))^{\rho_-}}{\kappa_1 \lambda_- \rho_- (\rho_+ - \rho_-) a^{\rho_-}} \exp(a\lambda_- I(\mathcal{Y}_0(x))) \tag{69}$$

for $x > 0$. Similarly to (B.9) in Appendix B, $V_m'(x) = \mathcal{Y}_0(x)$, hence an optimal strategy given by (35) becomes

$$c_t = I(\mathcal{Y}_0(x_t))$$
$$= \begin{cases} \frac{1}{a} \log \frac{a}{\mathcal{Y}_0(x_t)}, & x_t \geq \mathcal{X}_0(a), \\ 0, & x_t \leq \mathcal{X}_0(a). \end{cases} \tag{70}$$

$$\tilde{\pi}_t = \frac{\mathcal{Y}_0(x_t) \mathcal{X}_0'(\mathcal{Y}_0(x_t))}{-x_t} (\tilde{\alpha} - r\mathbf{1}_m) \Sigma_m^{-1}, \tag{71}$$

for $t \geq 0$.

Similarly, we get

$$\mathcal{X}_{m+n}(y) = \frac{I(y)}{r} - \frac{y^{\eta_+}}{\kappa_2 \eta_+^2 (\eta_+ - \eta_-) a^{\nu_+}} [\exp(a\eta_+ I(y)) - 1]$$
$$+ \frac{y^{\eta_-}}{\kappa_2 \eta_-^2 (\eta_+ - \eta_-) a^{\nu_-}} \exp(a\eta_- I(y)),$$

and we have

$$\mathcal{J}_{m+n}(y) = \frac{U(I(y))}{\beta} - \frac{y^{\nu_+}}{\kappa_2 \eta_+ \nu_+ (\nu_+ - \nu_-) a^{\nu_+}} [\exp(a\eta_+ I(y)) - 1]$$
$$+ \frac{y^{\nu_-}}{\kappa_2 \eta_- \nu_- (\nu_+ - \nu_-) a^{\nu_-}} \exp(a\eta_- I(y)).$$

Hence, if the investment opportunity set consists constantly of all the $m + n + 1$ assets, then the value function becomes

$$
\begin{aligned}
V_{m+n}(x) &= \mathcal{J}_{m+n}(\mathcal{Y}_{m+n}(x)) \\
&= \frac{U(I(\mathcal{Y}_{m+n}(x)))}{\beta} \\
&\quad - \frac{(\mathcal{Y}_{m+n}(x))^{\nu_+}}{\kappa_2 \eta_+ \nu_+ (\nu_+ - \nu_-) a^{\nu_+}} [\exp(a\eta_+ I(\mathcal{Y}_{m+n}(x))) - 1] \\
&\quad + \frac{(\mathcal{Y}_{m+n}(x))^{\nu_-}}{\kappa_2 \eta_- \nu_- (\nu_+ - \nu_-) a^{\nu_-}} \exp(a\eta_- I(\mathcal{Y}_{m+n}(x))), \qquad (72)
\end{aligned}
$$

for $x > 0$. Similarly to (B.9) in Appendix B, $V'_{m+n}(x) = \mathcal{Y}_{m+n}(x)$, hence an optimal strategy given by (38) becomes

$$
\begin{aligned}
c_t &= I(\mathcal{Y}_{m+n}(x_t)) \\
&= \begin{cases} \frac{1}{a} \log \frac{a}{\mathcal{Y}_{m+n}(x_t)}, & x_t \geq \mathcal{X}_{m+n}(a), \\ 0, & x_t \leq \mathcal{X}_{m+n}(a). \end{cases} \qquad (73)
\end{aligned}
$$

$$
\pi_t = \frac{\mathcal{Y}_{m+n}(x_t)\mathcal{X}'_{m+n}(\mathcal{Y}_{m+n}(x_t))}{-x_t}(\alpha - r\mathbf{1}_{m+n})\Sigma^{-1}, \qquad (74)
$$

for $t \geq 0$.

Remark 4.1. The value functions (69), (72), and the strategies (70), (71), (73), (74) are different from those in Merton [7] since Merton [7] does not consider the nonnegative wealth constraint (2). In particular, the dollar amount of optimal investment in risky stocks is not constant for an investor with a CARA utility function if the nonnegative wealth constraint (2) is imposed. (See Karatzas, Lehoczky, Sethi and Shreve [2].)

With a $\hat{B} > 0$, whose determination is shown generally in Appendix B, the value function (39) in Theorem 2 becomes

$$
\begin{aligned}
V^*(x) &= \mathcal{J}(\mathcal{Y}(x; \hat{B}); \frac{\lambda_-}{\rho_-}\hat{B}) \\
&= \frac{\lambda_-}{\rho_-}\hat{B}(\mathcal{Y}(x; \hat{B}))^{\rho_-} + \\
&\quad + \frac{U(I(\mathcal{Y}(x; \hat{B})))}{\beta} - \frac{(\mathcal{Y}(x; \hat{B}))^{\rho_+}}{\kappa_1 \lambda_+ \rho_+ (\rho_+ - \rho_-) a^{\rho_+}} [\exp(a\lambda_+ I(\mathcal{Y}(x; \hat{B}))) - 1] \\
&\quad + \frac{(\mathcal{Y}(x; \hat{B}))^{\rho_-}}{\kappa_1 \lambda_- \rho_- (\rho_+ - \rho_-) a^{\rho_-}} \exp(a\lambda_- I(\mathcal{Y}(x; \hat{B}))).
\end{aligned}
$$

By (B.9) in Appendix B, the optimal policy in Theorem 2 is given by following: For $0 < t < T_z$,

$$
\begin{aligned}
c_t &= I(\mathcal{Y}(x_t; \hat{B})) \\
&= \begin{cases} \frac{1}{a} \log \frac{a}{\mathcal{Y}(x_t; \hat{B})}, & x_t \geq \mathcal{X}(a; \hat{B})), \\ 0, & x_t \leq \mathcal{X}(a; \hat{B}). \end{cases}
\end{aligned}
\tag{75}
$$

$$
\tilde{\pi}_t = \frac{\mathcal{Y}(x_t, \hat{B})\mathcal{X}'(\mathcal{Y}(x_t; \hat{B}); \hat{B})}{-x_t}(\tilde{\alpha} - r\mathbf{1}_m)\Sigma_m^{-1},
\tag{76}
$$

and for $t \geq T_z$,

$$
\begin{aligned}
c_t &= I(\mathcal{Y}_{m+n}(x_t)) \\
&= \begin{cases} \frac{1}{a} \log \frac{a}{\mathcal{Y}_{m+n}(x_t)}, & x_t \geq \mathcal{X}_{m+n}(a), \\ 0, & x_t \leq \mathcal{X}_{m+n}(a). \end{cases}
\end{aligned}
\tag{77}
$$

$$
\pi_t = \frac{\mathcal{Y}_{m+n}(x_t)\mathcal{X}'_{m+n}(\mathcal{Y}_{m+n}(x_t))}{-x_t}(\alpha - r\mathbf{1}_{m+n})\Sigma^{-1}.
\tag{78}
$$

Figure 5 compares consumption rate in the case where the investment opportunity set consists constantly of one riskless asset and one risky asset with one in the case where the investor can invest in only these assets before her wealth level touching $z = 3$, however, she can invest in another risky asset once her wealth level touches $z = 3$. As is shown in the figure, during $x_t \leq \mathcal{X}_0(U'(0)) \approx 1.3$, the consumption rates are equal to zero in both cases, however, the investor's consumption rates are still equal to zero until $x_t = z = 3$ in the latter case while far away from zero for $\mathcal{X}_0(U'(0)) \approx 1.3 < x_t < z = 3$ in the former case, illustrating the equality (43) and the inequality (44) in Proposition 1.

Figure 6 compares investments in the risky assets in the two cases. As is shown in the figure, the investor invests more in the risky assets before touching the critical wealth level in the latter case than in the former case, illustrating the inequality in Proposition 2.

5. CONCLUSION

We have studied a consumption and investment problem of an investor who expects a better investment opportunity when she becomes rich enough, that is, when her wealth level touches a critical level. We provide a closed form solution to the problem when she has a general time-separable von

FIG. 5. Comparison of consumption rates with $U(c) = -\exp(-ac)$.
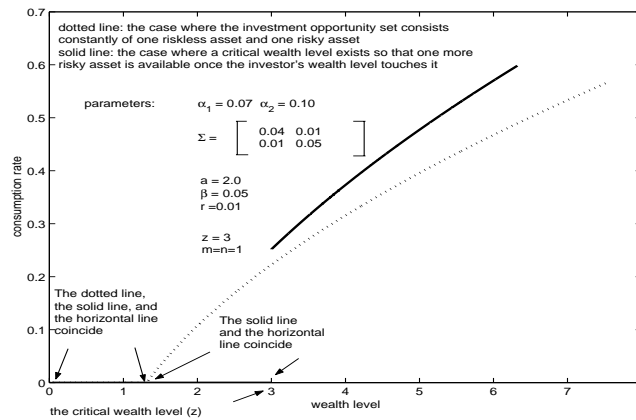
FIG. 6. Comparison of investment in the risky assets with $U(c) = -\exp(-ac)$.
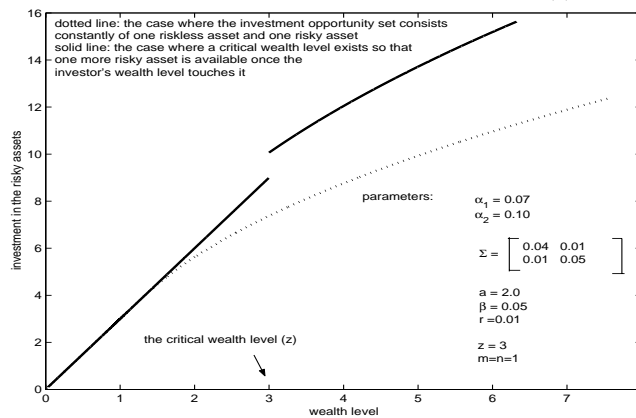
Neumann-Morgenstern utility function, and show that she consumes less and takes more risk in risky assets in order to reach the critical wealth level early enough. We illustrate the optimal policies by using a constant relative risk aversion as well as a constant absolute risk aversion utility functions. We also show that optimal consumption jumps to a higher level when the wealth touches the critical level under the constant relative risk aversion utility functions.

APPENDIX A

We prove all the assertions and statements in Section 3 and Lemma 1 in Section 4.1. The HJB equation for $0 < t < T_z$ is given by

$$\beta V(x) = \max_{c \geq 0, \tilde{\pi}} \left\{ (\tilde{\alpha} - r\mathbf{1}_m)\tilde{\pi}^T x V'(x) + (rx - c)V'(x) \right.$$
$$\left. + \frac{1}{2}\tilde{\pi}\Sigma_m\tilde{\pi}^T x^2 V''(x) + U(c) \right\}, \tag{A.1}$$

for $0 < x < z$.

A.1. SOLVING THE PROBLEM WHEN $U'(0) = \infty$

We first consider the case when $U'(0) = \infty$ in Section 3. Fix $\hat{B} \geq 0$ and let

$$X(c) \equiv X(c; \hat{B}),$$

where $X(\cdot; \hat{B})$ is defined as in (17) in section 3. We have the following lemma.

LEMMA 2.   If $U'(0) = \infty$, then

$$\lim_{c \downarrow 0} \frac{U(c)}{U'(c)} = 0, \tag{A.2}$$

$$\lim_{c \downarrow 0}(U'(c))^{\lambda_+} \int_0^c \frac{d\theta}{(U'(\theta))^{\lambda_+}} = 0 \tag{A.3}$$

and

$$\lim_{c \downarrow 0}(U'(c))^{\lambda_-} \int_c^\infty \frac{d\theta}{(U'(\theta))^{\lambda_-}} = 0. \tag{A.4}$$

Proof. When $U(0)$ is finite, (A.2) trivially holds. When $U(0) = -\infty$,

$\limsup_{c \downarrow 0} \frac{U(c)}{U'(c)} \leq 0$. For every $\epsilon > 0$ and $0 < c < \epsilon$, $U(c) \geq U(\epsilon) - U'(c)(\epsilon - c)$. Therefore,

$$\liminf_{c \downarrow 0} \frac{U(c)}{U'(c)} \geq \liminf_{c \downarrow 0} \left(\frac{U(\epsilon)}{U'(c)} - \epsilon + c \right) = -\epsilon.$$

Since $\epsilon > 0$ is arbitrary,

$$\liminf_{c \downarrow 0} \frac{U(c)}{U'(c)} \geq 0.$$

Hence (A.2) holds.
Since

$$
\begin{aligned}
0 &\le \liminf_{c\downarrow 0}(U'(c))^{\lambda_+}\int_0^c \frac{d\theta}{(U'(\theta))^{\lambda_+}} \\
&\le \limsup_{c\downarrow 0}(U'(c))^{\lambda_+}\int_0^c \frac{d\theta}{(U'(\theta))^{\lambda_+}} \\
&\le \limsup_{c\downarrow 0}(U'(c))^{\lambda_+}\int_0^c \frac{d\theta}{(U'(c))^{\lambda_+}} \\
&= \limsup_{c\downarrow 0} c \\
&= 0,
\end{aligned}
$$

(A.3) holds. Finally, since, for every $\epsilon > 0$,

$$
\begin{aligned}
0 &\le \liminf_{c\downarrow 0}(U'(c))^{\lambda_-}\int_c^\infty \frac{d\theta}{(U'(\theta))^{\lambda_-}} \\
&\le \limsup_{c\downarrow 0}(U'(c))^{\lambda_-}\int_c^\infty \frac{d\theta}{(U'(\theta))^{\lambda_-}} \\
&\le \limsup_{c\downarrow 0}\int_c^\epsilon (\frac{U'(c)}{U'(\theta)})^{\lambda_-}d\theta + \limsup_{c\downarrow 0}(U'(c))^{\lambda_-}\int_\epsilon^\infty \frac{d\theta}{(U'(\theta))^{\lambda_-}} \\
&\le \limsup_{c\downarrow 0}(\epsilon - c) \\
&= \epsilon,
\end{aligned}
$$

(A.4) holds. ∎
By (A.3) and (A.4), we have

$$
X(0) \equiv \lim_{c\downarrow 0} X(c) = 0
$$

if $U'(0) = \infty$. Similarly to (6.11) of Karatzas, Lehoczky, Sethi, and Shreve [2], $\lim_{c\uparrow\infty} X(c) = \infty$. Using the relation $\lambda_+\lambda_- = -\frac{r}{\kappa_1}$, we have

$$
\begin{aligned}
X'(c) &= \lambda_-\hat{B}(U'(c))^{\lambda_- - 1}U''(c) \\
&- \frac{U''(c)}{\kappa_1(\lambda_+ - \lambda_-)}\Big\{(U'(c))^{\lambda_+ - 1}\int_0^c \frac{d\theta}{(U'(\theta))^{\lambda_+}} + (U'(c))^{\lambda_- - 1}\int_c^\infty \frac{d\theta}{(U'(\theta))^{\lambda_-}}\Big\}.
\end{aligned}
$$

Since $U(\cdot)$ is strictly concave, $X'(c) > 0$ for all $c > 0$. Hence $X(\cdot)$ is strictly increasing and maps $[0, \infty)$ onto itself so that the inverse function of it, $C(\cdot) \equiv C(\cdot, \hat{B})$, exists and is also strictly increasing and maps $[0, \infty)$ onto itself as stated in Section 3.

Put

$$F(c) = \frac{\lambda_- c}{r} U'(c) - \frac{\rho_-}{\beta} U(c)$$
$$+ \frac{1}{\kappa_1 \lambda_+ \rho_+} (U'(c))^{\rho_+} \int_0^c \frac{d\theta}{(U'(\theta))^{\lambda_+}} - \lambda_- z U'(c)$$
$$+ \rho_- V_{m+n}(z). \tag{A.5}$$

Then, by Lemma 2,

$$\lim_{c \downarrow 0} F(c) = \infty.$$

Using the inequality (16) in Section 2, and (21) in Section 3,

$$F(C_0(z)) = \frac{\lambda_- C_0(z)}{r} U'(C_0(z)) - \frac{\rho_-}{\beta} U(C_0(z))$$
$$+ \frac{1}{\kappa_1 \lambda_+ \rho_+} (U'(C_0(z)))^{\rho_+} \int_0^{C_0(z)} \frac{d\theta}{(U'(\theta))^{\lambda_+}}$$
$$- \lambda_- z U'(C_0(z)) + \rho_- V_{m+n}(z)$$
$$< \frac{\lambda_- C_0(z)}{r} U'(C_0(z)) - \frac{\rho_-}{\beta} U(C_0(z))$$
$$+ \frac{1}{\kappa_1 \lambda_+ \rho_+} (U'(C_0(z)))^{\rho_+} \int_0^{C_0(z)} \frac{d\theta}{(U'(\theta))^{\lambda_+}}$$
$$- \lambda_- z U'(C_0(z)) + \rho_- V_m(z)$$
$$= \frac{\lambda_- C_0(z)}{r} U'(C_0(z)) - \frac{\rho_-}{\beta} U(C_0(z))$$
$$+ \frac{1}{\kappa_1 \lambda_+ \rho_+} (U'(C_0(z)))^{\rho_+} \int_0^{C_0(z)} \frac{d\theta}{(U'(\theta))^{\lambda_+}}$$
$$- \lambda_- z U'(C_0(z)) + \rho_- J_0(C_0(z))$$
$$= 0,$$

where $C_0(\cdot) \equiv C(\cdot; 0)$ is the inverse function of $X_0(\cdot) \equiv X(\cdot; 0)$ as defined in Section 3. Hence, by the intermediate value theorem, there exists $0 < d < C_0(z)$ such that

$$F(d) = 0. \tag{A.6}$$

Choose $\hat{B}$ so that

$$X(d) \equiv X(d; \hat{B}) = z, \tag{A.7}$$

i.e.

$$\hat{B} = \frac{z - X_0(d)}{(U'(d))^{\lambda_-}}. \tag{A.8}$$

Then, since $X_0(\cdot)$ is an increasing function,

$$X_0(d) < X_0(C_0(z)) = z$$

so that $\hat{B} > 0$ as required.

From now on, we proceed with this $\hat{B}$ in this section.

Define

$$V(x) \equiv J(C(x); \frac{\lambda_-}{\rho_-}\hat{B}) \qquad (A.9)$$

for $x > 0$, where $J(\cdot; \frac{\lambda_-}{\rho_-}\hat{B})$ is defined as in (19) in Section 3 and $C(\cdot) \equiv C(\cdot; \hat{B})$, then we have the following lemma.

LEMMA 3. *V(x) defined by (A.9) is strictly increasing, strictly concave and satisfies the HJB equation (A.0) for $0 = X(0) < x < X(d) = z$.*

Proof. By calculation, we have

$$V'(x) \;=\; \frac{J'(C(x); \frac{\lambda_-}{\rho_-}\hat{B})}{X'(C(x))} \qquad (A.10)$$

$$=\; U'(C(x)) \qquad (A.11)$$

$$>\; 0, \quad x > 0, \qquad (A.12)$$

and

$$V''(x) \;=\; U''(C(x))C'(x) \qquad (A.13)$$

$$<\; 0, \quad x > 0. \qquad (A.14)$$

Thus, $V(\cdot)$ is strictly increasing and strictly concave.

Hence applying this $V(\cdot)$ in the HJB equation (A.0) and maximizing over investment ratios in risky assets gives

$$\tilde{\pi} = \frac{V'(x)}{-xV''(x)}(\tilde{\alpha} - r\mathbf{1}_m)\Sigma_m^{-1}.$$

Hence the HJB equation (A.0) becomes

$$\beta V(x) = -\kappa_1 \frac{(V'(x))^2}{V''(x)} + \max_{c \geq 0}\{(rx - c)V'(x) + U(c)\}. \qquad (A.15)$$

By (A.11)and (A.13), (A.15) takes the form

$$\beta V(x) = -\kappa_1 \frac{(U'(C(x)))^2 X'(C(x))}{U''(C(x))} + (rx - C(x))V'(x) + U(C(x)) \quad (A.16)$$

for $0 < x < z$, which is equivalent to

$$\beta J(c; \frac{\lambda_-}{\rho_-}\hat{B}) = -\kappa_1 \frac{(U'(c))^2 X'(c)}{U''(c)} + (rX(c) - c)U'(c) + U(c) \qquad \text{(A.17)}$$

for $0 < c < d$. By calculation and using the relation $\rho_+\rho_- = -\frac{\beta}{\kappa_1}$, (A.17) can be shown to hold. Hence $V(\cdot)$ satisfies the HJB equation (A.0). ∎
  By (A.7), we have

$$\lim_{x\uparrow z} C(x) = C(z) = d, \qquad \text{(A.18)}$$

so that

$$\lim_{x\uparrow z} V(x) = J(d; \frac{\lambda_-}{\rho_-}\hat{B}). \qquad \text{(A.19)}$$

By (A.6), (A.7) and definition of $X(\cdot)$, we have

$$\begin{aligned}
V_{m+n}(z) &= -\frac{\lambda_- d}{r\rho_-}U'(d) + \frac{\rho_-}{\beta\rho_-}U(d) \\
&\quad - \frac{1}{\kappa_1\lambda_+\rho_+\rho_-}(U'(d))^{\rho_+}\int_0^d \frac{d\theta}{(U'(\theta))^{\lambda_+}} + \frac{\lambda_- z}{\rho_-}U'(d) \\
&= -\frac{\lambda_- d}{r\rho_-}U'(d) + \frac{\rho_-}{\beta\rho_-}U(d) \\
&\quad - \frac{1}{\kappa_1\lambda_+\rho_+\rho_-}(U'(d))^{\rho_+}\int_0^d \frac{d\theta}{(U'(\theta))^{\lambda_+}} + \frac{\lambda_- X(d)}{\rho_-}U'(d) \\
&= J(d; \frac{\lambda_-}{\rho_-}\hat{B}).
\end{aligned}$$

 Hence by (A.19), we get

$$\lim_{x\uparrow z} V(x) = V_{m+n}(z). \qquad \text{(A.20)}$$

Similarly to Lemma 8.7 of [2], we have

$$\lim_{x\downarrow 0} V(x) = \frac{U(0)}{\beta}. \qquad \text{(A.21)}$$

Let x_0 be a given initial wealth with $0 < x_0 < z$ and consider the strategy

$$c_t = C(x_t), \quad \pi_t = \frac{V'(x_t)}{-x_t V''(x_t)}S \qquad \text{(A.22)}$$

for $0 \le t \le T_0 \wedge T_z$.

Using (A.11) and (A.13), similarly to the equation (7.4) in Karatzas, Lehoczky, Sethi, and Shreve [2], the stochastic differential equation for $\{c_t \equiv C(x_t), 0 \le t \le T_0 \wedge T_z\}$ becomes

$$dy_t = -(r - \beta)y_t dt - y_t SD d\mathbf{w}^T(t),$$

where $y_t = U'(c_t)$. Hence

$$U'(c_t) = y_t = U'(c_0) \exp[-(r - \beta + \kappa_1)t - SD\mathbf{w}^T(t)], \quad 0 \le t \le T_0 \wedge T_z,$$

so that we get

$$c_t = I(U'(c_0) \exp[-(r - \beta + \kappa_1)t - SD\mathbf{w}^T(t)]), \quad 0 \le t \le T_0 \wedge T_z.$$

Therefore, if $U'(0) = \infty$, then

$$T_0 = \inf\{t \ge 0 : x_t = 0\} = \inf\{t \ge 0 : c_t = 0\} = \inf\{t \ge 0 : y_t = \infty\} = \infty, a.s. \tag{A.23}$$

Hence, if an investor use control (A.22) with initial wealth $0 < x_0 = x < z$, then

$$T_0 \wedge T_z = T_z. \tag{A.24}$$

That is, bankruptcy does not occur before his wealth level touches z.

Now consider the strategy $(\mathbf{c}, \pi)$ in Theorem 1 in Section 3 with V replacing V^*.

For $0 < c_0 < d$ (or equivalently $0 < x < z$), let

$$H(c_0) \equiv V_{(\mathbf{c},\pi)}(x_0) = E_{x_0}\Big[\int_0^{T_z} \exp(-\beta t)U(c_t)dt + \exp(-\beta T_z)V_{m+n}(z)\Big], \tag{A.25}$$

where the equality comes from strong Markov property and (A.24). Using $H(c_0) \le V_{m+n}(x_0)$, then similarly to Karatzas, Lehoczky, Sethi, and Shreve [2] $H(c)$ is well defined and finite for all c with $0 < c < d$. Define

$$G(y_0) = H(I(y_0)) = E_{x_0}\Big[\int_0^{T_z} \exp(-\beta t)U(I(y_t))dt + \exp(-\beta T_z)V_{m+n}(z)\Big]$$

for $U'(d) < y_0 < U'(0) = \infty$ where $\{y_t, 0 \le t \le T_z\}$ is given by above with $y_0 = U'(c_0)$. By Theorem 13.16 of Dynkin [1](Feynman-Kac formula), G is C^2 on $(U'(d), \infty)$ and satisfies

$$\beta G(y) = -(r - \beta)yG'(y) + \kappa_1 y^2 G''(y) + U(I(y)) \tag{A.26}$$

for $U'(d) < y_0 < \infty$ with $\lim_{y\downarrow U'(d)} G(y) = V_{m+n}(z)$. Hence H is C^2 on $(0, d)$ and satisfies

$$\beta H(c) = -\frac{U'(c)}{U''(c)}[r - \beta + \kappa_1 \frac{U'(c)U'''(c)}{(U''(c))^2}]H'(c) + \kappa_1(\frac{U'(c)}{U''(c)})^2 H''(c) + U(c) \tag{A.27}$$

for $0 < c < d$ with $\lim_{c\uparrow d} H(c) = V_{m+n}(z)$. The general solution to the above equation (A.27) is

$$A(U'(c))^{\rho+} + J(c; \hat{A})$$

for $0 < c < d$. Hence for $0 < c < d$,

$$H(c) = A(U'(c))^{\rho+} + J(c; \hat{A})$$

for some A and $\hat{A}$ such that

$$\lim_{c\uparrow d} H(c) = A(U'(d))^{\rho+} + J(d; \hat{A}) = V_{m+n}(z). \tag{A.28}$$

Similarly to Theorem 8.8 of Karatzas, Lehoczky, Sethi, and Shreve [2], it is shown that $A = 0$ when $U'(0) = \infty$ so that for $0 < c < d$,

$$H(c) = J(c; \hat{A})$$

for some $\hat{A}$ such that

$$\lim_{c\uparrow d} H(c) = J(d; \hat{A}) = V_{m+n}(z). \tag{A.29}$$

Using (A.6), (A.7) and (A.29), we get $\hat{A} = \frac{\lambda_-}{\rho_-}\hat{B}$ so that

$$V(x) = H(C(x)). \tag{A.30}$$

Now we prove Theorem 1 in Section 3.

Proof of Theorem 1

We first consider the case in which $U(0)$ is finite. Given any admissible strategy $(\mathbf{c}, \pi)$, $V_{(\mathbf{c},\pi)}(x)$ is greater than or equal to $\frac{U(0)}{\beta}$ for all $x > 0$. However it is less than or equal to the optimal value at x when the investment opportunity set consists constantly of all the $m + n + 1$ assets, which is finite (see Karatzas, Lehoczky, Sethi, and Shreve [2]). Hence $V_{(\mathbf{c},\pi)}(x)$ is

finite for all $x > 0$. Now fix $0 < x < z$. By the strong Markovian property,

$$
\begin{aligned}
E_x \int_0^\infty \exp(-\beta t)U(c_t)dt \;=\; & E_x[\int_0^{T_0 \wedge T_z} \exp(-\beta t)U(c_t)dt \\
& + \mathbf{1}_{\{T_0 < T_z\}}\exp(-\beta T_0)\frac{U(0)}{\beta} \\
& + \mathbf{1}_{\{T_0 > T_z\}}\exp(-\beta T_z)V_{(\mathbf{c},\pi)}(z)].
\end{aligned}
$$

Choose $0 < \xi_1 < x < \xi_2 < z$ and define $S_n = \inf\{t \geq 0 : \int_0^t \pi_s \Sigma \pi_s^T ds = n\}$. From the fact that $V(x)$ defined by (A.9) satisfies the HJB equation (A.0) and using the Itô's rule, we get

$$
\begin{aligned}
& E_x \int_0^{T_{\xi_1} \wedge T_{\xi_2} \wedge S_n} \exp(-\beta t)U(c_t)dt \\
\leq \;& E_x[\int_0^{T_{\xi_1} \wedge T_{\xi_2} \wedge S_n} \exp(-\beta t)[\beta V(x_t) \\
& - (\tilde{\alpha} - r\mathbf{1}_m)\tilde{\pi}_t^T x_t V'(x_t) - (rx_t - c_t)V'(x_t) \\
& - \frac{1}{2}\tilde{\pi}_t \Sigma_1 \tilde{\pi}_t^T x_t^2 V''(x_t)]dt] \\
= \;& E_x[\int_0^{T_{\xi_1} \wedge T_{\xi_2} \wedge S_n} [-d(\exp(-\beta t)V(x_t)) \\
& + \exp(-\beta t)x_t V'(x_t)\pi_t D d\mathbf{w}^T(t)]] \\
= \;& -E_x \exp(-\beta(T_{\xi_1} \wedge T_{\xi_2} \wedge S_n))V(x(T_{\xi_1} \wedge T_{\xi_2} \wedge S_n)) + V(x).
\end{aligned}
$$

Hence

$$
\begin{aligned}
V(x) \;\geq\; & E_x \int_0^{T_{\xi_1} \wedge T_{\xi_2} \wedge S_n} \exp(-\beta t)U(c_t)dt \\
& + E_x \exp(-\beta(T_{\xi_1} \wedge T_{\xi_2} \wedge S_n))V(x(T_{\xi_1} \wedge T_{\xi_2} \wedge S_n)).
\end{aligned}
$$

Letting $\xi_1 \downarrow 0, \xi_2 \uparrow z, n \to \infty$, we have $T_{\xi_1} \wedge T_{\xi_2} \wedge S_n \to T_0 \wedge T_z$. By the monotone convergence theorem and Fatou's lemma, we get

$$
\begin{aligned}
V(x) \;\geq\; & E_x[\int_0^{T_0 \wedge T_z} \exp(-\beta t)U(c_t)dt \\
& + \mathbf{1}_{\{T_0 < T_z\}}\exp(-\beta T_0)V(0+) + \mathbf{1}_{\{T_0 > T_z\}}\exp(-\beta T_z)V(z-)].
\end{aligned}
$$

By (A.20) and (A.21), we have

$$
\begin{aligned}
V(x) \;\geq\; & E_x[\int_0^{T_0 \wedge T_z} \exp(-\beta t)U(c_t)dt \\
& + \mathbf{1}_{\{T_0 < T_z\}}\exp(-\beta T_0)\frac{U(0)}{\beta} + \mathbf{1}_{\{T_0 > T_z\}}\exp(-\beta T_z)V_{m+n}(z)].
\end{aligned}
$$

Since $V_{m+n}(\cdot)$ is the value function when the investment opportunity set consists constantly of all the $m+n+1$ assets, we have

$$V_{m+n}(z) \geq V_{(\mathbf{c},\pi)}(z).$$

Therefore we have

$$
\begin{aligned}
V(x) \;\geq\; & E_x\big[\int_0^{T_0 \wedge T_z} \exp\left(-\beta t\right)U(c_t)dt + \mathbf{1}_{\{T_0 < T_z\}} \exp\left(-\beta T_0\right)\frac{U(0)}{\beta} \\
& +\; \mathbf{1}_{\{T_0 > T_z\}} \exp\left(-\beta T_z\right)V_{(\mathbf{c},\pi)}(z)\big] \\
=\; & E_x \int_0^\infty \exp\left(-\beta t\right)U(c_t)dt.
\end{aligned}
$$

Since $(\mathbf{c},\pi)$ is an arbitrary admissible strategy, we have

$$V(x) \geq V^*(x),$$

where $V^*(\cdot)$ is the optimal value function as defined in (4) in Section 2.

However, by (A.30) we have

$$V(x) = H(C(x)) \leq V^*(x)$$

Hence $V(x) = V^*(x)$, that is, $V(x)$ is the value function for $0 < x < z$ and an optimal policy is given as in Theorem 1 in Section 3.

When $U(0) = -\infty$, by the corollary 10.3 of Karatzas, Lehoczky, Sethi, and Shreve [2] we can replace the admissible set by the set whose elements are contained in the original admissible set and the the expected total reward of them are finite. Thus we may assume that the expected total rewards of controls of the admissible set are finite and bankruptcy does not occur under admissible policies. Hence for any admissible strategy $(\mathbf{c},\pi)$,

$$E_x \int_0^\infty \exp\left(-\beta t\right)U(c_t)dt = E_x\big[\int_0^{T_z} \exp\left(-\beta t\right)U(c_t)dt + \exp\left(-\beta T_z\right)V_{\mathbf{c},\pi}(z)\big]$$

Similarly to the case that $U(0)$ is finite, the conclusion holds. ∎

Similarly to (A.11),

$$V'_{m+n}(x) = U'(C_{m+n}(x)). \tag{A.31}$$

By (A.11) and (A.13), and (A.31), the optimal vector of ratios of wealth invested in the risky assets can be rewritten as

$$\pi_t = \frac{U'(C(x_t))X'(C(x_t))}{-x_t U''(C(x_t))}S \tag{A.32}$$

for $0 \le t < T_z$, and

$$\pi_t = \frac{U'(C_{m+n}(x_t))X'_{m+n}(C_{m+n}(x_t))}{-x_t U''(C_{m+n}(x_t))}(\alpha - r\mathbf{1}_{m+n})\Sigma^{-1} \qquad (\text{A.33})$$

for $t \ge T_z$.

Now the inequality (42) in Proposition 1 and Proposition 2 in Section 3 when $U'(0) = \infty$ can be proved.

Proof of Proposition 1 when $U'(0) = \infty$

When the investment opportunity set consists constantly of the one riskless asset and the first m risky assets for all levels of wealth, the function from consumption to wealth in her optimal behavior is $X_0(\cdot)$ as in (22) in Section 3. Since $\hat{B}$ is larger than zero, $X(c) > X_0(c)$ for all $c > 0$. Hence their inverse functions(from wealth to consumption) have the relation $C(x) < C_0(x)$ for all $x > 0$ since $X(\cdot)$ and $X_0(\cdot)$ are increasing functions. ∎

Proof of Proposition 2 when $U'(0) = \infty$

By (A.32), the investor's optimal vector of investments in the risky assets at time $t < T_z$ is

$$x_t \pi_t = -\frac{U'(C(x_t))X'(C(x_t))}{U''(C(x_t))}S.$$

Similarly to (A.11),

$$V'_m(x) = U'(C_0(x)).$$

Hence when the investment opportunity set consists constantly of the one riskless asset and the first m risky assets for all levels of wealth, the investor's optimal vector of investments in the risky assets at time t is

$$-\frac{U'(C_0(x_t))X'_0(C_0(x_t))}{U''(C_0(x_t))}(\tilde{\alpha} - r\mathbf{1}_m)\Sigma_m^{-1},$$

as mentioned in (22) in Section 3 using $V'_m(x) = U'(C_0(x))$. Some calculation gives

$$-\frac{U'(C(x))X'(C(x))}{U''(C(x))} = -\lambda_-\{x - \frac{C(x)}{r} + \frac{1}{r}(U'(C(x)))^{\lambda_+}\int_0^{C(x)} \frac{d\theta}{(U'(\theta))^{\lambda_+}}\}$$

and

$$-\frac{U'(C_0(x))X'_0(C_0(x))}{U''(C_0(x))} = -\lambda_-\{x - \frac{C_0(x)}{r} + \frac{1}{r}(U'(C_0(x)))^{\lambda_+}\int_0^{C_0(x)} \frac{d\theta}{(U'(\theta))^{\lambda_+}}\}.$$

By differentiation it is easily checked that $-\lambda_-\{x-\frac{c}{r}+\frac{1}{r}(U'(c))^{\lambda_+}\int_0^c \frac{d\theta}{(U'(\theta))^{\lambda_+}}\}$ is a decreasing function of c. Since $C(x) < C_0(x)$,

$$-\frac{U'(C(x))X'(C(x))}{U''(C(x))} > -\frac{U'(C_0(x))X_0'(C_0(x))}{U''(C_0(x))} > 0.$$

Hence the investor invests more in the risky assets when there is such a critical wealth level compared to the case when the investment opportunity set consists constantly of the one riskless asset and the first m risky assets for all levels of wealth. ∎

APPENDIX B

We now consider the case when $U'(0)$ is finite so that $U(0)$ is also finite. Recall

$$I : (0, U'(0)] \to [0, \infty)$$

which is the inverse of U'. We extended I by setting $I \equiv 0$ on $[U'(0), \infty)$. If V is C^2, strictly increasing, and strictly concave, then the HJB equation (A.0) becomes

$$\beta V(x) = -\kappa_1 \frac{(V'(x))^2}{V''(x)} + [rx - I(V'(x))]V'(x) + U(I(V'(x))), \quad \text{(B.1)}$$

for $z > x > 0$. For $c \geq 0$, we have $c = I(U'(c))$, hence for $\hat{B} \geq 0, \hat{A} \geq 0$,

$$\mathcal{X}(U'(c); \hat{B}) = X(c; \hat{B}),$$

and

$$\mathcal{J}(U'(c); \hat{A}) = J(c; \hat{A}),$$

where $\mathcal{X}(\cdot; \hat{B})$, $\mathcal{J}(\cdot; \hat{A})$, $X(\cdot; \hat{B})$ and $J(\cdot; \hat{A})$ are defined as in (30), (32), (17) and (19), respectively in Section 3. For $y > 0$ and $y \neq U'(0)$, using the relation $\lambda_+\lambda_- = -\frac{r}{\kappa_1}$, we get

$$
\begin{aligned}
\mathcal{X}'(y) &= \hat{B}\lambda_- y^{\lambda_- - 1} \\
&\quad - \frac{1}{\kappa_1(\lambda_+ - \lambda_-)}\left[y^{\lambda_+}\int_0^{I(y)} \frac{d\theta}{(U'(\theta))^{\lambda_+}} + y^{\lambda_-}\int_{I(y)}^\infty \frac{d\theta}{(U'(\theta))^{\lambda_-}}\right] \\
&< 0.
\end{aligned}
$$

Hence $\mathcal{X}(\cdot; \hat{B})$ is strictly decreasing. Furthermore,

$$
\begin{aligned}
\lim_{y \downarrow 0} \mathcal{X}(y; \hat{B}) &= \lim_{c \uparrow \infty} \mathcal{X}(U'(c); \hat{B}) \\
&= \lim_{c \uparrow \infty} X(c; \hat{B}) \\
&= \infty
\end{aligned}
$$

and

$$
\begin{aligned}
\lim_{y \uparrow \infty} \mathcal{X}(y; \hat{B}) &= \lim_{y \uparrow \infty} [\hat{B} y^{\lambda_-} - \frac{1}{\kappa_1 (\lambda_+ - \lambda_-)} \frac{y^{\lambda_-}}{\lambda_-} \int_0^\infty \frac{d\theta}{(U'(\theta))^{\lambda_-}}] \\
&= 0.
\end{aligned}
$$

Therefore $\mathcal{X}(\cdot; \hat{B})$ maps $(0, \infty)$ onto $(0, \infty)$ and has the inverse function $\mathcal{Y}(\cdot; \hat{B}) : (0, \infty) \to (0, \infty)$ as stated in Section 3. Put

$$
\begin{aligned}
\Phi(y) &= \frac{\lambda_- y}{r} I(y) - \frac{\rho_-}{\beta} U(I(y)) + \frac{1}{\kappa_1 \lambda_+ \rho_+} y^{\rho_+} \int_0^{I(y)} \frac{d\theta}{(U'(\theta))^{\lambda_+}} - \lambda_- yz \\
&\quad + \rho_- V_{m+n}(z).
\end{aligned} \tag{B.2}
$$

Then since, for sufficiently large y,

$$
\Phi(y) = -\frac{\rho_-}{\beta} U(0) - \lambda_- yz + \rho_- V_{m+n}(z),
$$

we have

$$
\lim_{y \to \infty} \Phi(y) = \infty.
$$

By the inequality (16) in Section 2 and (34) in Section 3, we get

$$
\begin{aligned}
\Phi(\mathcal{Y}_0(z)) &= \frac{\lambda_- \mathcal{Y}_0(z)}{r} I(\mathcal{Y}_0(z)) - \frac{\rho_-}{\beta} U(I(\mathcal{Y}_0(z))) \\
&\quad + \frac{1}{\kappa_1 \lambda_+ \rho_+} (\mathcal{Y}_0(z))^{\rho_+} \int_0^{I(\mathcal{Y}_0(z))} \frac{d\theta}{(U'(\theta))^{\lambda_+}} - \lambda_- \mathcal{Y}_0(z) z + \rho_- V_{m+n}(z) \\
&< \frac{\lambda_- \mathcal{Y}_0(z)}{r} I(\mathcal{Y}_0(z)) - \frac{\rho_-}{\beta} U(I(\mathcal{Y}_0(z))) \\
&\quad + \frac{1}{\kappa_1 \lambda_+ \rho_+} (\mathcal{Y}_0(z))^{\rho_+} \int_0^{I(\mathcal{Y}(z))} \frac{d\theta}{(U'(\theta))^{\lambda_+}} - \lambda_- \mathcal{Y}_0(z) \mathcal{X}_0(\mathcal{Y}_0(z)) + \rho_- V_m(z) \\
&= \frac{\lambda_- \mathcal{Y}_0(z)}{r} I(\mathcal{Y}_0(z)) - \frac{\rho_-}{\beta} U(I(\mathcal{Y}_0(z))) \\
&\quad + \frac{1}{\kappa_1 \lambda_+ \rho_+} (\mathcal{Y}_0(z))^{\rho_+} \int_0^{I(\mathcal{Y}(z))} \frac{d\theta}{(U'(\theta))^{\lambda_+}} - \lambda_- \mathcal{Y}_0(z) \mathcal{X}_0(\mathcal{Y}_0(z)) + \rho_- \mathcal{J}_0(\mathcal{Y}_0(z)) \\
&= 0,
\end{aligned}
$$

where $\mathcal{Y}_0(\cdot) \equiv \mathcal{Y}(\cdot; 0)$ is the inverse function of $\mathcal{X}_0(\cdot) \equiv \mathcal{X}(\cdot; 0)$ as defined in Section 3. Hence, by the intermediate value theorem, there exists $\hat{y} > \mathcal{Y}_0(z)$ such that

$$\Phi(\hat{y}) = 0. \tag{B.3}$$

Choose $\hat{B}$ so that

$$\mathcal{X}(\hat{y}; \hat{B}) = z \tag{B.4}$$

holds with this $\hat{y}$, that is, $\hat{B} = \frac{z - \mathcal{X}_0(\hat{y})}{\hat{y}^{\lambda_-}}$. Then, since $\mathcal{X}_0(\cdot)$ is a strictly decreasing function,

$$\mathcal{X}_0(\hat{y}) < \mathcal{X}_0(\mathcal{Y}_0(z)) = z,$$

so that $\hat{B} > 0$ as required.

From now on, we proceed with this $\hat{B}$ in this section.

Define

$$\mathcal{V}(x) = \mathcal{J}(\mathcal{Y}(x; \hat{B}); \frac{\lambda_-}{\rho_-} \hat{B}), \tag{B.5}$$

for $x > 0$.

LEMMA 4.   *The function $\mathcal{V}$ defined by (B.5) is strictly increasing, strictly concave, satisfies the HJB equation (A.0) for $0 < x < z$,*

$$\lim_{x \downarrow 0} \mathcal{V}(x) = \frac{U(0)}{\beta} \tag{B.6}$$

and

$$\lim_{x \uparrow z} \mathcal{V}(x) = \mathcal{V}_{m+n}(z). \tag{B.7}$$

Proof. By calculation, we have

$$\mathcal{V}'(x) = \frac{\mathcal{J}'(\mathcal{Y}(x; \hat{B}); \frac{\lambda_-}{\rho_-} \hat{B})}{\mathcal{X}'(\mathcal{Y}(x; \hat{B}); \hat{B})} \tag{B.8}$$

$$= \mathcal{Y}(x; \hat{B}) \tag{B.9}$$

$$> 0, \quad x > 0, \tag{B.10}$$

and

$$\mathcal{V}''(x) = \mathcal{Y}'(x; \hat{B}) \tag{B.11}$$

$$= \frac{1}{\mathcal{X}'(\mathcal{Y}(x; \hat{B}); \hat{B})} \tag{B.12}$$

$$< 0, \quad x > 0. \tag{B.13}$$

Thus, $\mathcal{V}(\cdot)$ is strictly increasing and strictly concave. By (B.9) and (B.12), the right side of (B.1) applying this $\mathcal{V}(\cdot)$ becomes

$$
\begin{aligned}
&- \kappa_1(\mathcal{Y}(x;\hat{B}))^2 \mathcal{X}'(\mathcal{Y}(x,\hat{B}),\hat{B}) \\
&+ [r\mathcal{X}(\mathcal{Y}(x;\hat{B});\hat{B}) - I(\mathcal{Y}(x;\hat{B}))]\mathcal{Y}(x;\hat{B}) + U(I(\mathcal{Y}(x;\hat{B}))),
\end{aligned}
$$

which, by calculation and using the relation $\rho_+\rho_- = -\frac{\beta}{\kappa_1}$, equals to $\beta\mathcal{J}(\mathcal{Y}(x;\hat{B});\frac{\lambda_-}{\rho_-}\hat{B}) = \beta\mathcal{V}(x)$, the left side of (B.1) applying $\mathcal{V}(\cdot)$. Hence $\mathcal{V}(\cdot)$ satisfies the HJB equation (B.1) (or equivalently (A.0)).

$$
\begin{aligned}
\lim_{x\downarrow 0}\mathcal{V}(x) &= \lim_{y\uparrow\infty}\mathcal{J}(y;\frac{\lambda_-}{\rho_-}\hat{B}) \\
&= \lim_{y\uparrow\infty}[\frac{\lambda_-}{\rho_-}\hat{B}y^{\rho_-} + \frac{U(0)}{\beta} - \frac{1}{\kappa_1(\rho_+ - \rho_-)}\frac{y^{\rho_-}}{\rho_-}\int_0^\infty \frac{d\theta}{(U'(\theta))^{\lambda_-}}] \\
&= \frac{U(0)}{\beta}.
\end{aligned}
$$

Thus (B.6) holds. By (B.3), (B.4), and by calculation, we have

$$
\begin{aligned}
V_{m+n}(z) &= -\frac{\lambda_-\hat{y}}{\rho_- r}I(\hat{y}) + \frac{U(I(\hat{y}))}{\beta} \\
&\quad - \frac{1}{\kappa_1\lambda_+\rho_+\rho_-}\hat{y}^{\rho_+}\int_0^{I(\hat{y})}\frac{d\theta}{(U'(\theta))^{\lambda_+}} + \frac{\lambda_-\hat{y}}{\rho_-}z \\
&= -\frac{\lambda_-\hat{y}}{\rho_- r}I(\hat{y}) + \frac{U(I(\hat{y}))}{\beta} \\
&\quad - \frac{1}{\kappa_1\lambda_+\rho_+\rho_-}\hat{y}^{\rho_+}\int_0^{I(\hat{y})}\frac{d\theta}{(U'(\theta))^{\lambda_+}} + \frac{\lambda_-\hat{y}}{\rho_-}\mathcal{X}(\hat{y};\hat{B}) \\
&= \mathcal{J}(\hat{y};\frac{\lambda_-}{\rho_-}\hat{B}) \\
&= \mathcal{J}(\mathcal{Y}(z;\hat{B});\frac{\lambda_-}{\rho_-}\hat{B}) \\
&= \lim_{x\uparrow z}\mathcal{V}(x)
\end{aligned}
$$

Hence (B.7) holds. ∎

Let x_0 be a given initial wealth with $0 < x_0 < z$ and let

$$
c_t = I(\mathcal{V}'(x_t)), \pi_t = \frac{\mathcal{V}'(x_t)}{-x_t\mathcal{V}''(x_t)}S \tag{B.14}
$$

for $0 \le t \le T_0 \wedge T_z$, where $\mathcal{V}(\cdot)$ is given by (B.5). We set

$$
Y_t \equiv \mathcal{V}'(x_t),
$$

hence $Y_0 = \mathcal{V}'(x_0)$.

Similarly to Lemma 13.2 in Karatzas, Lehoczky, Sethi, and Shreve [2], the stochastic process $\{Y_t, 0 \leq t \leq T_0 \wedge T_z\}$ satisfies

$$dY_t = -(r - \beta)Y_t dt - Y_t SDd\mathbf{w}^T(t),$$

so that

$$Y_t = Y_0 \exp[-(r - \beta + \kappa_1)t - SDw^T(t)].$$

Since $Y_t = \mathcal{Y}(x_t; \hat{B})$ by (B.9), we have

$$
\begin{aligned}
T_0 &= \inf\{t \geq 0 : x_t = 0\} \\
&= \inf\{t \geq 0 : \mathcal{Y}(x_t; \hat{B} = \infty\} \\
&= \inf\{t \geq 0 : Y_t = \infty\} \\
&= \infty.
\end{aligned}
$$

Hence

$$T_0 \wedge T_z = T_z. \tag{B.15}$$

Therefore, bankruptcy does not occur before the investor's wealth level touching z if she use the strategy (B.14).

Now consider the strategy $(\mathbf{c}, \pi)$ in Theorem 2 in Section 3 with $\mathcal{V}$ replacing V^*.

For $0 < x_0 < z$ (or equivalently $Y_0 > \hat{y}$), let

$$
\begin{aligned}
\mathcal{H}(Y_0) &\equiv V_{(\mathbf{c}, \pi)}(x_0) \\
&= E_{x_0}\left[\int_0^{T_z} \exp\left(-\beta t\right)U(I(Y_t))dt + \exp\left(-\beta T_z\right)V_{m+n}(z)\right], \tag{B.16}
\end{aligned}
$$

where the equality comes from the strong Markov property and (B.15). Since $U(0)$ is finite and $\mathcal{H}(Y_0) \leq V_{m+n}(x_0)$, $\mathcal{H}(y)$ is well defined and finite for all $y > \hat{y}$. According to Theorem(13.16) of Dynkin [1], $\mathcal{H}$ given by (B.16) is C^2 on $(\hat{y}, \infty)$, satisfies (A.26) on this interval and

$$\lim_{y \downarrow \hat{y}} \mathcal{H}(y) = V_{m+n}(z).$$

The general solution to (A.26) is

$$\mathcal{J}(y; \hat{A}) + Ay^{\rho_+}.$$

Hence for $y > \hat{y}$,

$$\mathcal{H}(y) = \mathcal{J}(y; \hat{A}) + Ay^{\rho+}$$

for some A and $\hat{A}$ such that

$$\lim_{y \downarrow \hat{y}} \mathcal{H}(y) = \mathcal{J}(\hat{y}; \hat{A}) + A\hat{y}^{\rho+} = V_{m+n}(z)$$

By (B.16), $\mathcal{H}$ is bounded above on $[\hat{y}, \infty)$. However, since by (B.6)

$$\lim_{y \uparrow \infty} \mathcal{J}(y; \hat{A}) = \frac{U(0)}{\beta}$$

which is finite, we must have

$$A \leq 0,$$

for otherwise $\lim_{y \downarrow \hat{y}} \mathcal{H}(y) = \infty$. If $y^* > Y_0 > \hat{y}$ is given, we can select A_0 so that $\mathcal{J}(\hat{y}; A_0) = V_{m+n}(z)$. Using Itô's rule and the fact that $\mathcal{J}(\cdot; A_0)$ satisfies the equation (A.26), we get

$$d[\exp(-\beta t)\mathcal{J}(Y_t; A_0)] = -\exp(-\beta t)[U(I(Y_t))dt + Y_t \mathcal{J}'(Y_t; A_0)SDd\mathbf{w}^T(t)].$$

Letting $\mathcal{X}(y^*) = x^*$, then we have

$$
\begin{aligned}
\mathcal{J}(Y_0; A_0) \;=\; & E_{x_0}\Big[\int_0^{T_z \wedge T_{x^*}} \exp(-\beta t)U(I(Y_t))dt \\
& + 1_{\{T_z < T_{x^*}\}} \exp(-\beta T_z)V_{m+n}(z) \\
& + 1_{\{T_z > T_{x^*}\}} \exp(-\beta T_{x^*})\mathcal{J}(y^*; A_0)\Big]. \qquad \text{(B.17)}
\end{aligned}
$$

Since

$$\lim_{y \uparrow \infty} \mathcal{J}(y; A_0) = \frac{U(0)}{\beta},$$

$\mathcal{J}(y^*; A_0)$ is bounded above for y^* in a neighborhood of ∞. We have

$$U(I(Y_t)) \leq U(I(\hat{y}))$$

for $t \in [0, T_z \wedge T_{x^*}]$. Hence letting $y^* \to \infty$, Fatou's lemma can be applied to (B.17) to obtain

$$
\begin{aligned}
\mathcal{J}(Y_0; A_0) \;\leq\; & E_{x_0}\Big[\int_0^{T_z} \exp(-\beta t)U(I(Y_t))dt + \exp(-\beta T_z)V_{m+n}(z)\Big] \\
\;=\; & \mathcal{H}(Y_0) = \mathcal{J}(Y_0; \hat{A}) + AY_0^{\rho+}.
\end{aligned}
$$

This inequality will fail for large Y_0 if $A < 0$ since $\lim_{y\uparrow\infty} \mathcal{J}(y; A_0) = \lim_{y\uparrow\infty} \mathcal{J}(y; \hat{A}) = \frac{U(0)}{\beta}$ and $\lim_{y\uparrow\infty} Ay^{\rho+} = -\infty$ if $A < 0$. Hence $A \geq 0$. As a result, $A = 0$. Hence

$$\mathcal{H}(y) = \mathcal{J}(y; \hat{A})$$

for some $\hat{A}$ such that $\mathcal{J}(\hat{y}; \hat{A}) = V_{m+n}(z)$. Since

$$\mathcal{J}(\hat{y}; \hat{A}) = V_{m+n}(z),$$

$$\lim_{y\downarrow\hat{y}} \mathcal{X}(y; \hat{B}) = z$$

and

$$\Phi(\hat{y}) = 0,$$

we have

$$\hat{A} = \frac{\lambda_-}{\rho_-}\hat{B}.$$

Hence we can prove Theorem 2 in Section 3 with $V^* = \mathcal{V}$ replacing V^* by the same way as the proof of Theorem 1.

The equality (43) and the inequality (44) in Proposition 1 and Proposition 2 in Section 3 can be proved when $U'(0)$ is finite too.

Proof of Proposition 1 when $U'(0)$ is finite   When the investment opportunity set consists constantly of the one riskless asset and the first m risky assets for all levels of wealth, the optimal consumption rate at time t equals to $I(V'_m(x_t))$, as mentioned in (35) in Section 3. Since $\hat{B}$ is larger than zero, $\mathcal{X}(y; \hat{B}) > \mathcal{X}_0(y)$ for all $y > 0$. Hence their inverse functions have the relation $\mathcal{Y}(x; \hat{B}) > \mathcal{Y}_0(x)$ for all $x > 0$ since $\mathcal{X}(\cdot; \hat{B})$ and $\mathcal{X}_0(\cdot)$ are decreasing functions. By (B.9), $\mathcal{V}'(x) = \mathcal{Y}(x; \hat{B})$ and similarly $V'_m(x) = \mathcal{Y}_0(x)$. If $x \leq \mathcal{X}_0(U'(0))$, then $\mathcal{Y}(x; \hat{B} > \mathcal{Y}_0(x) \geq U'(0)$. Therefore $I(\mathcal{V}'(x)) = I(V'_m(x)) = 0$ for $x \leq \mathcal{X}(U'(0))$ since $I \equiv 0$ on $[U'(0), \infty)$. If $\mathcal{X}(U'(0)) < x \leq \mathcal{X}(U'(0); \hat{B})$, then $\mathcal{Y}_0(x) < U'(0)$ and $\mathcal{Y}(x; \hat{B}) \geq U'(0)$. Hence $I(V'_m(x)) > 0$ and $I(\mathcal{V}'(x)) = 0$ for $\mathcal{X}(U'(0)) < x \leq \mathcal{X}(U'(0); \hat{B})$ since $I(y) > 0$ for $0 < y < U'(0)$ and $I \equiv 0$ on $[U'(0), \infty)$. If $x > \mathcal{X}(U'(0); \hat{B})$, then $0 < \mathcal{Y}_0(x) < \mathcal{Y}(x; \hat{B}) < U'(0)$. Hence $I(\mathcal{V}'(x)) < I(V'_m(x))$ for $x > \mathcal{X}(U'(0); \hat{B})$ since $I(\cdot)$ is strictly decreasing for $0 < y < U'(0)$. ∎

Proof of Proposition 2 when $U'(0)$ is finite As is shown in this section, the investor's optimal vector of investments in the risky assets at time t is

$$\frac{\mathcal{V}'(x_t)}{-\mathcal{V}''(x_t)}S.$$

By calculation, it can be shown that

$$\frac{\mathcal{V}'(x_t)}{-\mathcal{V}''(x_t)} = -\lambda_- x_t + \frac{\lambda_-}{r}I(\mathcal{Y}(x_t; \hat{B}))$$

$$+ \frac{1}{\kappa_1 \lambda_+}(\mathcal{Y}(x_t; \hat{B}))^{\lambda_+} \int_0^{I(\mathcal{Y}(x_t;\hat{B}))} \frac{d\theta}{(U'(\theta))^{\lambda_+}}.$$

When the investment opportunity set consists constantly of the one riskless asset and the first m risky assets for all levels of wealth, the investor's optimal vector of investments in the risky assets at time t is

$$\frac{V'_m(x_t)}{-V''_m(x_t)}(\tilde{\alpha} - r\mathbf{1}_m)\Sigma_m^{-1}$$

as mentioned in (35) in Section 3. Similarly to above, it can be shown that

$$\frac{V'_m(x_t)}{-V''_m(x_t)} = -\lambda_- x_t + \frac{\lambda_-}{r}I(\mathcal{Y}_0(x_t)) + \frac{1}{\kappa_1 \lambda_+}(\mathcal{Y}_0(x_t))^{\lambda_+}\int_0^{I(\mathcal{Y}_0(x_t))} \frac{d\theta}{(U'(\theta))^{\lambda_+}}.$$

By differentiation with respect to y it is easily checked that

$$-\lambda_- x_t + \frac{\lambda_-}{r}I(y) + \frac{1}{\kappa_1 \lambda_+}y^{\lambda_+}\int_0^{I(y)} \frac{d\theta}{(U'(\theta))^{\lambda_+}}$$

is a strictly increasing function of y. Hence, since $\mathcal{Y}(x; \hat{B}) > \mathcal{Y}_0(x)$, we have

$$\frac{\mathcal{V}'(x_t)}{-\mathcal{V}''(x_t)} > \frac{V'_m(x_t)}{-V''_m(x_t)}.$$

∎

B.1. CONSUMPTION JUMPS UNDER CRRA UTILITY FUNCTION CLASS

In this section we prove Lemma 1 in Section 4.1.

We first consider the case when the utility function is given by

$$U(c) = \frac{c^{1-\gamma}}{1-\gamma}$$

with $0 < \gamma \neq 1$, $U'(c) = c^{-\gamma}$. Equation (A.5) becomes

$$F(c) = \frac{-1-\gamma\lambda_-}{(1-\gamma)K_1}c^{1-\gamma} - \lambda_- zc^{-\gamma} + \rho_-\frac{K_2^{-\gamma}}{1-\gamma}z^{1-\gamma}. \tag{B.1}$$

We can calculate

$$\begin{aligned}
\lim_{c\downarrow 0} F(c) &= \lim_{c\downarrow 0} c^{-\gamma}\left(\frac{-1-\gamma\lambda_-}{1-\gamma}c - \lambda_- z + \rho_-\frac{K_2^{-\gamma}}{1-\gamma}z^{1-\gamma}c^{\gamma}\right) \\
&= \infty,
\end{aligned}$$

and

$$F'(c) = c^{-1-\gamma}\left(\frac{-1-\gamma\lambda_-}{K_1}c + \gamma\lambda_- z\right). \tag{B.2}$$

Since $1 + \gamma\lambda_- < 0$ as mentioned in Section 2, $F(\cdot)$ is strictly decreasing for $0 < c < \frac{\gamma\lambda_-}{1+\gamma\lambda_-}K_1 z$ and strictly increasing for $c > \frac{\gamma\lambda_-}{1+\gamma\lambda_-}K_1 z$. Calculation gives

$$F(K_1 z) = \frac{-\rho_-}{1-\gamma}K_2^{-\gamma}\left[\left(\frac{K_2}{K_1}\right)^{\gamma} - 1\right]z^{1-\gamma}. \tag{B.3}$$

By (52) in Section 4.1, (8), and (12) in Section 2, if $0 < \gamma < 1$ then $K_1 > K_2 > 0$ and if $\gamma > 1$ then $0 < K_1 < K_2$. Therefore we have

$$F(K_1 z) < 0. \tag{B.4}$$

Hence there exists a unique constant d such that

$$0 < d < K_1 z \tag{B.5}$$

and

$$F(d) = 0.$$

As is shown in Appendix A.1, the $\hat{B}$ is chosen so that $X(d; \hat{B}) = z$. Similarly to (B.4), we get

$$F(K_2 z) = \frac{-1-\gamma\lambda_-}{(1-\gamma)K_1}(K_2 - K_1)K_2^{-\gamma}z^{1-\gamma} < 0. \tag{B.6}$$

Hence d is less than $K_2 z$ so that

$$C(z; \hat{B}) = d < K_2 z = C_{m+n}(z),$$

where the last equality comes from (51) in Section 4.1. Therefore consumption jump occurs at time T_z.

Consider the case when utility function is given by

$$U(c) = \log c,$$

for $c > 0$. (A.5) becomes

$$F(c) = \frac{-\beta - \kappa_2 \rho_+ - r\rho_+}{\kappa_1 \rho_+^2 \beta} - \frac{\rho_-}{\beta} \log c - \frac{\lambda_- z}{c} + \frac{\rho_-}{\beta} \log \beta z. \qquad (B.7)$$

We can calculate

$$\lim_{c \downarrow 0} F(c) = \infty,$$

and

$$F'(c) = \frac{1}{c^2} \left(\frac{-\rho}{\beta} c + \lambda_- z \right). \qquad (B.8)$$

Since $\frac{-\rho}{\beta} > 0$, $F(\cdot)$ is strictly decreasing for $0 < c < \frac{\lambda_-}{\rho_-} \beta z$ and strictly increasing for $c > \frac{\lambda_-}{\rho_-} \beta z$. Calculation gives

$$F(\beta z) = -\frac{\rho_+ (\kappa_2 - \kappa_1)}{\kappa_1 \rho_+^2 \beta}. \qquad (B.9)$$

Since $\kappa_1 < \kappa_2$ as mentioned in Section 4.1, we have

$$F(\beta z) < 0. \qquad (B.10)$$

Hence there exists a unique constant d such that

$$0 < d < \beta z$$

and

$$F(d) = 0.$$

As is shown in Appendix A.1, the $\hat{B}$ is chosen so that $X(d; \hat{B}) = z$. Hence

$$C(z; \hat{B}) = d < \beta z = C_{m+n}(z),$$

where the last equality comes from (62) in Section 4.1. Therefore consumption jump occurs at time T_z. ▊

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