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A Bootstrap Test for Conditional Symmetry

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This paper proposes a simple consistent nonparametric test of conditional symmetry based on the principle of characteristic functions. The test statistic is shown to be asymptotically normal under the null hypothesis of conditional symmetry and consistent against any conditional asymmetric distributions. We also study the power against local alternatives, propose a bootstrap version of the test, and conduct a small Monte Carlo simulation to evaluate the finite-sample performance of the test.

Key Words: Bootstrap; Conditional symmetry; Characteristic function; Test; U-statistics.

JEL Classification Numbers: C12, C14.

1. MOTIVATION

For the purpose of identification, three kinds of restrictions have been frequently imposed on the statistical relationship between observables and unobservables in many econometric models. They are conditional moment restrictions, independence and conditional symmetry. See Newey (1990) for a detailed description of how these types of restrictions are related. Conditional symmetry implies the distribution of one random variable, typically the unobservable error term, given another random vector, usually the observed independent variables, has a symmetric form. Even though it imposes a strong shape restriction, conditional symmetry allows for heteroskedasticity. There are a few semiparametric estimators proposed under conditional symmetry. Manski (1984) and Newey (1988) study a regression

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model under conditional symmetry. Powell's (1986a) and Newey's (1991) estimators for Tobit models under conditional symmetry also show its potential. The semiparametric efficiency bound for various types of Tobit models is also studied under conditional symmetry. See Newey (1991) and Chen (1995, 2000).

As noticed by Hyndman and Yao (2002), the symmetry of conditional density function is also of interest in modelling time series data in business and finance and in constructing predictive regions for nonlinear time series. For the former see Brännäs and De Gooijer (1992) and for the latter, see Hyndman (1995), De Gooijer and Grannoun (2000), and Polonik and Yao (2000).

Despite the wide use of the property of conditional symmetry, tests for conditional symmetry[1] have not been addressed very much in the literature. The first tests are proposed by Powell (1986b) for censored regression models and by Newey and Powell (1987) for linear regression models via asymmetric least squares estimation. As noticed by Zheng (1998), these tests are unlikely to be consistent against arbitrary conditional asymmetric distributions. Zheng proposes a test of conditional symmetry using a nonparametric kernel method but his test needs integral calculation and is hard to implement. Bai and Ng (2001) propose an alternative test for conditional symmetry for time series models via empirical distribution function approach. But their test relies on the correct specification of both conditional mean and conditional variance. In particular they assume multiplicative error and their test is essentially a test for symmetry. Hyndman and Yao (2002) develop a bootstrap test for the symmetry of conditional density functions based upon their improved methods for conditional density estimation.

In this paper we propose a simple test for conditional symmetry based on the principle that two conditional distributions are identical if and only if their respective conditional characteristic functions are. Our approach offers a convenient approach to testing for distributional hypotheses via an infinite number of conditional moment regressions. Also our test, unlike the aforementioned tests, allows both the dependent variable and the conditional variables to be multivariate.

The remainder of the paper is organized as follows. In section 2, we describe the hypothesis and test statistic. In section 3 we study the asymptotic null distribution of the test statistic, the consistency and local power properties of our test, and the validity of the simple bootstrap. We report some Monte Carlo evidence of our test in Section 4. All technical details are relegated to the Appendix. Throughout the paper, for a matrix

[1]In cases of test for unconditional symmetry, various tests have been proposed, including the most recent ones of Fan and Gencay (1995) Ahmad and Li (1997), Dicks and Tong (1999) and Osmoukhina (2001).

$A = (a_{ij})$, we denote its norm by $\|A\| = \{tr\,(A'A)\}^{1/2}$. Further, let $\sum_{i \neq j}$ and $\sum_{1 \leq i < j \leq n}$ denote $\sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n}$ and $\sum_{i=1}^{n} \sum_{j=2, j>i}^{n}$ respectively.

2. THE HYPOTHESIS AND TEST STATISTIC

Let $(X, Y), (X_1, Y_1), \ldots, (X_n, Y_n)$ be independent observations with common joint probability $f_{XY}(\cdot, \cdot)$. Let $f_{Y|X}(\cdot|x)$ and $F_{Y|X}(\cdot|x)$ denote the conditional density and distribution of Y given $X = x \in \mathbb{R}^{d_1}$, respectively. Further denote the marginal density and distribution of X as $f_X(\cdot)$ and $F_X(\cdot)$, respectively. We are interested in testing whether the random variable Y is symmetric around zero conditional on X:

$$H_0 : \Pr[f_{Y|X}(y|X) = f_{Y|X}(-y|X)] = 1 \text{ for all } y \in \mathbb{R}^{d_2}. \tag{1}$$

The alternative hypothesis is

$$H_0 : \Pr[f_{Y|X}(y|X) = f_{Y|X}(-y|X)] < 1 \text{ for some } y \in \mathbb{R}^{d_2}. \tag{2}$$

The proposed test is based on the principle of characteristic functions. It is well known that two (conditional) distribution functions are equal almost everywhere (a.e.) if and only if their respective (conditional) characteristic functions are equal (a.e.). To state this precisely, let $\phi_{Y|X}(\cdot; x)$ be the conditional characteristic functions $\phi_{Y|X}$ of Y given $X = x$: $\phi_{Y|X}(u; x) = E[\exp(iu'Y)|X = x]$, where $i = \sqrt{-1}$ and $u \in \mathbb{R}^{d_2}$. Let $\psi(u; x) \equiv \phi_{Y|X}(u; x) - \phi_{Y|X}(-u; x)$. The conditional density of Y given X is symmetric if and only if $\psi(u; x) = 0$ a.e. $-x$ for every $u \in \mathbb{R}^{d_2}$. This motivates us to consider the following smooth functional

$$\Gamma \equiv \frac{1}{2} \iint \left|\phi_{Y|X}(u; x) - \phi_{Y|X}(-u; x)\right|^2 dG(u) dF_X(x), \tag{3}$$

where $dG(u) = g(u)du$ and we choose g to be a density function with full support on $\mathbb{R}^{d_2}$.

Under some regularity conditions (to allow the change of order of integration), one can write

$$\begin{aligned}
\Gamma &= \frac{1}{2} \iint \left|\phi_{Y|X}(u; x) - \phi_{Y|X}(-u; x)\right|^2 dG(u) dF_X(x) \\
&= \frac{1}{2} \iiiint \left[\exp(iu'y) - \exp(-iu'y))\right] \left[\exp(-iu'\widetilde{y}) - \exp(iu'\widetilde{y}))\right] dG(u) \\
&\quad dF_{Y|X}(y|x)\, dF_{Y|X}(\widetilde{y}|x)\, dF_X(x) \\
&= \frac{1}{2} \iiint \left[h(y - \widetilde{y}) + h(-y + \widetilde{y}) - h(y + \widetilde{y}) - h(-y - \widetilde{y})\right] dF_{Y|X}(y|x) \\
&\quad dF_{Y|X}(\widetilde{y}|x)\, dF_X(x).
\end{aligned}$$

where $h(y) \equiv \int e^{iu'y} dG(u)$, the characteristic function of the probability measure $dG(u)$. Without loss of generality, we assume h is symmetric so that

$$\Gamma = \iiint \left[h\left(y - \widetilde{y}\right) - h\left(y + \widetilde{y}\right)\right] dF_{Y|X}\left(y|x\right) dF_{Y|X}\left(\widetilde{y}|x\right) dF_X(x). \tag{4}$$

This integral facilitates the application of the convenient asymptotic distribution theory for U-statistics.

To introduce the test statistic of interest, let K be a kernel function on $\mathbb{R}^{d_1}$ and $B \equiv B(n)$ be the bandwidth $d_1 \times d_1$ matrix. Define $K_B(u) \equiv |B|^{-1} K(B^{-1}u)$, where $|B|$ is the determinant of B. We propose the following test statistic

$$\Gamma_n = \frac{2}{n\left(n-1\right)|B|^{1/2}} \sum_{1 \le i < j \le n} H_n\left(Z_i, Z_j\right), \tag{5}$$

where $Z_i = (X_i, Y_i)$, and
$H_n\left(Z_i, Z_j\right) = |B|^{1/2}\left[h\left(Y_i - Y_j\right) - h\left(Y_i + Y_j\right)\right] K_B\left(X_i - X_j\right)$.

The test statistic Γ_n has the advantage that it has zero mean under H_0 and hence it does not have a finite sample bias term. We will show that after being appropriately scaled, Γ_n is asymptotically normally distributed under H_0.

3. THE ASYMPTOTIC DISTRIBUTION AND BOOTSTRAP TEST

In this section we establish the asymptotic property of Γ_n under H_0, H_1, and a sequence of local alternatives. We also prove the validity of a bootstrap test for the null of conditional symmetry.

3.1. Asymptotic distributions

To study the asymptotic distribution of Γ_n, we make the following assumptions.

A1 $f_{XY}\left(x, y\right)$ is continuous and has uniformly bounded second order derivatives with respect to x.

A2 The density function g is uniformly bounded such that its characteristic function h is uniformly bounded and symmetric.

A3 The kernel function $K\left(\cdot\right)$ is a symmetric, bounded and continuous density on $\mathbb{R}^{d_1}$ satisfying $\int \|u\|^2 K\left(u\right) du < \infty$.

A4 As $n \to \infty$, $\|B\| \to 0$, and $n\,|B| \to \infty$.

Assumption A1 imposes the smoothness condition on f_{XY} and it can be weakened to a Lipschitz continuity with some modifications on the proof.

The uniform boundedness of h in A2 comes free as one important property of characteristic functions. From the derivation of our results, it is easy to see that the symmetry of h can be easily relaxed. However, many of the commonly used g functions are symmetric. For example, g can be either a normal density function or a double exponential density function. Both A3 and A4 are standard in the literature. In practice, one frequently chooses B to be a diagonal matrix: $B = diag(b_1, \ldots, b_{d_1})$.

We can state our main results.

THEOREM 1. *Under Assumptions A1-A4 and under* $H_0, T_n \equiv n\,|B|^{1/2}\,\Gamma_n/\widehat{\sigma} \xrightarrow{d} N(0,1)$, *where* $\widehat{\sigma}^2 = 2\,(n\,(n-1))^{-1} \sum_{i \neq j} H_n^2\,(Z_i, Z_j)$ *is a consistent estimator for*

$$\sigma^2 = 2 \int K^2\,(u)\,du \left\{ \iiint \left[h\,(y_1 - y_2) - h\,(y_1 + y_2)\right]^2 f_{XY}\,(x, y_1)\,f_{XY}\,(x, y_2)\,dx dy_1 dy_2 \right\}. \tag{6}$$

The proof of the above theorem is relegated to the Appendix. To implement the test, we compare T_n with the one-sided critical value z_α from the standard normal distribution, and reject the null when $T_n > z_\alpha$.

The following result provides the asymptotic behavior of T_n under H_1.

THEOREM 2. *Under Assumptions A1-A4 and under* H_1,

$$T_n/(n\,|B|^{1/2}) = \Gamma_n/\widehat{\sigma} \xrightarrow{p} \frac{1}{2\sigma} \int \int |\psi(u;x)|^2\,dG\,(u)\,f_X^2\,(x)\,dx > 0. \tag{7}$$

Thus $T_n \to \infty$ under H_1 and the test is consistent.

We next study the local power of the test. For simplicity, we specify the local alternative in terms of conditional characteristic functions:

$$H_1(\alpha_n) : \phi_{Y|X}(u;x) = \phi_{Y|X}(-u;x) + \alpha_n \Delta(u,x), \tag{8}$$

where $\Delta(u,x)$ satisfies $\gamma \equiv \frac{1}{2} \int \int |\Delta(u,x)|^2\,dG(u) f_X^2\,(x)\,dx < \infty$, and $\alpha_n \to 0$ as $n \to \infty$.

The following theorem shows that our test can distinguish local alternatives $H_1(\alpha_n)$ at rate $\alpha_n = n^{-1/2}\,|B|^{-1/4}$.

THEOREM 3. *Suppose that* $\alpha_n = n^{-1/2}\,|B|^{-1/4}$ *in* $H_1(\alpha_n)$. *Under Assumptions A1-A4, the local power of the test satisfies* $\Pr(T_n \geq z_\alpha | H_1(\alpha_n)) \to 1 - \Phi(z_\alpha - \gamma/\sigma)$.

Theorem 3 implies that with a proper choice of B, the test is consistent against any local alternatives approaching the null at rates arbitrarily close to (but slower than) the parametric rate $n^{-1/2}$.

3.2. The bootstrap test

Although T_n has asymptotic normal distribution under H_0, its convergence rate is only of the order $|B|^{1/2}$. Thus we next study the bootstrap method as an alternative approximation to the null distribution of T_n.

Noticing that under $H_0, (X, -Y)$ has the same distribution as (X, Y), so we resample from the sample $D_n \equiv \{(X_1, Y_1), \ldots, (X_n, Y_n), (X_1, -Y_1), \ldots, (X_n, -Y_n)\}$. Let $\mathcal{Z}_n = \{X_i, Y_i\}_{i=1}^n$, and let $\{Z_i^* \equiv (X_i^*, Y_i^*)\}_{i=1}^n$ denote the bootstrap sample obtained by sampling with replacement from D_n. The bootstrap version of T_n is given by

$$T_n^* = \frac{2}{(n-1)} \sum_{1 \le i < j \le n} H_n\left(Z_i^*, Z_j^*\right) / \widehat{\sigma}^*, \tag{9}$$

where

$$\widehat{\sigma}^{*2} = \frac{2}{n(n-1)} \sum_{i \ne j} H_n^2\left(Z_i^*, Z_j^*\right). \tag{10}$$

It is easy to establish the following theorem.

Theorem 4. *Under Assumptions A1-A4 and H_0, we have*

$$T_n^* | \mathcal{Z}_n \xrightarrow{d} N(0, 1). \tag{11}$$

Theorem 4 only says that the bootstrap method works asymptotically. We could follow the proof of Theorem 3 of Li and Wang (1998), or Theorem 2.3 in Li (1999) and show theoretically that the bootstrap method provides a better null approximation than the asymptotic normal distribution. This involves further complication, and we leave it for future research.

4. MONTE CARLO RESULTS

In this section we conduct a small set of Monte Carlo simulations to evaluate the finite-sample performance of the our bootstrap test for conditional symmetry. Like Zheng (1998), we consider the following data generating process (DGP):

$$W = 1 + X + X\varepsilon,$$

where X and ε are two independent random variables, and X is drawn from the standard normal distribution. Let $Y = W - 1 - X$. The null hypothesis

H_0 we are interested in is that Y is symmetric about zero conditioning on X. In practice, we observe the data on (W, X) so that Y is estimated by the least-squares residual $\widehat{Y}$ in the regression of W on the constant and X.

To yield conditionally symmetric and asymmetric distributions for Y, we draw ε from symmetric and asymmetric distributions following Zheng (1998). That is, we first generate ε from the standard normal distribution and the t-distribution with 10 degrees of freedom and label the resultant DGPs as DGP1 and DGP2, respectively. We then draw ε from another six distributions, two symmetric and four asymmetric, from the generalized lambda family (GLF) and label the resultant DGPs as DGP3-DGP8. The distributions in the GLF family are defined in terms of the inverses of the cumulative distribution functions: $F^{-1}(u) = \lambda_1 + \left[u^{\lambda_3} - (1-u)^{\lambda_4}\right]/\lambda_2$ for $u \in (0, 1)$. The parameters defining the six distributions are listed in Table 1 of Zheng (1998). We transform all distributions from this family to have mean 0 and variance 1.

To implement our test, we choose both the kernel function $K(\cdot)$ and the weight function $g(\cdot)$ to be the standard normal density. The characteristic function of $g(\cdot)$ is then given by $h(u) = \exp\left(-u^2/2\right)$. Since our bootstrap version of the test is not sensitive to the choice of bandwidth B, we follow the Silverman's rule of thumb by setting $B = 1.06\widehat{\sigma}_X n^{-1/5}$ where $\widehat{\sigma}_X$ is the sample standard deviation of X.

TABLE 1.

Proportion of rejections for distributions 1-8

	$n = 100$			$n = 200$			$n = 400$		
	1%	5%	10%	1%	5%	10%	1%	5%	10%
Symmetric									
DGP1	0.018	0.076	0.154	0.016	0.072	0.150	0.016	0.064	0.124
DGP2	0.020	0.072	0.130	0.016	0.066	0.130	0.014	0.064	0.126
DGP3	0.022	0.060	0.112	0.014	0.060	0.118	0.014	0.056	0.114
DGP4	0.024	0.084	0.160	0.014	0.076	0.152	0.012	0.072	0.146
Asymmetric									
DGP5	0.112	0.348	0.526	0.322	0.740	0.872	0.848	0.992	1.000
DGP6	0.154	0.368	0.550	0.298	0.682	0.836	0.848	0.992	0.996
DGP7	0.102	0.340	0.582	0.290	0.700	0.866	0.804	0.980	1.000
DGP8	0.130	0.378	0.564	0.306	0.694	0.874	0.782	1.000	1.000

We conduct the simulation with sample sizes $n = 100, 200$, and 400, and the number of bootstrap resamples is 1000. Each experiment is based on 500 replications. The results are reported in Table 1. For small sizes, the test is oversized and the power is fairly good. As the sample increases,

the sizes become reasonably closer to their nominal level and the power dominates that of Zheng (1998) significantly.

APPENDIX A

Let $A \approx C$ denote $A = C(1+o(1))$ componentwise for any matrices A, C of the same dimension. Let $Z = (X, Y)$, and $z_i = (x_i, y_i)$, $i = 1, 2$. Let E^* and var^* denote the expectation and variance conditional on the data $\mathcal{Z}_n$.

Proof of Theorem 1

The proof of the first part is straightforward by applying Theorem 1 of Hall (1984). So we only sketch the proof. By construction and Assumptions A2-A3, $H_n(z_1, z_2) = H_n(z_2, z_1)$.

$$\begin{aligned}
& E\left[H_n(z_1, Z_2)\right] \\
= \; & |B|^{1/2} \int \left\{ \int \left[h(y_1 - y) - h(y_1 + y)\right] dF_{Y|X}(y|x) \right\} K_B(x_1 - x)\, dF_X(x) \\
= \; & 0
\end{aligned}$$

under H_0.

$$\begin{aligned}
& E\left[H_n^2(Z_1, Z_2)\right] \\
= \; & |B| \int \int \left[h(y_1 - y_2) - h(y_1 + y_2)\right]^2 K_B^2(x_1 - x_2)\, dF_{XY}(x_1, y_1)\, dF_{XY}(x_2, y_1) \\
= \; & \int K^2(u)\, du \left\{ \int \int \int \left[h(y_1 - y_2) - h(y_1 + y_2)\right]^2 f_{XY}(x, y_1) f_{XY}(x, y_2)\, dx dy_1 dy_2 \right\} \\
& + O\left(\|B\|^2\right) \\
= \; & \sigma^2/2 + O\left(\|B\|^2\right).
\end{aligned}$$

Let $G_n(z_1, z_2) = E\left[H_n(Z, z_1) H_n(Z, z_2)\right]$. Then

$$\begin{aligned}
& E\left[G_n^2(Z_1, Z_2)\right] \\
= \; & |B|^2 E\left[\int \left[h(y - Y_1) - h(y + Y_1)\right] \left[h(y - Y_2) - h(y + Y_2)\right] K_B(x - X_1) \right. \\
& \left. K_B\left(x - X_2\right) dF_{XY}(x, y) \right]^2 \\
= \; & O(|B|).
\end{aligned}$$

$$\begin{aligned}
& E\left[H_n^4(Z_1, Z_2)\right] \\
= \; & |B|^2 \int \int \left[h(y_1 - y_2) - h(y_1 + y_2)\right]^4 K_B^4(x_1 - x_2)\, dF_{XY}(x_1, y_1)\, dF_{XY}(x_2, y_2) \\
= \; & O\left(|B|^{-1}\right).
\end{aligned}$$

So as $n \to \infty$,

$$\frac{E\left[G_n^2\left(Z_1,Z_2\right)\right]+n^{-1}E\left[H_n^4\left(Z_1,Z_2\right)\right]}{\left\{E\left[H_n^2\left(Z_1,Z_2\right)\right]\right\}^2}=\frac{O\left(|B|\right)+O\left(n^{-1}\left|B\right|^{-1}\right)}{\left\{\sigma^2/2+O(\|B\|^2)\right\}^2}\to 0.$$

The conditions of Theorem 1 of Hall (1984) are satisfied and the result follows.

Next, we show that $\widehat{\sigma}^2 = 2\left(n\left(n-1\right)\right)^{-1}\sum_{i\neq j}H_n^2\left(Z_i,Z_j\right)$ is a consistent estimator for σ^2. Noticing that $E\left(\widehat{\sigma}^2\right) = 2E\left[H_n^2\left(Z_1,Z_2\right)\right] = \sigma^2+O\left(\|B\|^2\right)$ and $E\left(\widehat{\sigma}^2\right)^2 = \sigma^4+O\left(n^{-1}\right)+O\left(n^{-2}\left|B\right|^{-1}\right)+O\left(\|B\|^2\right)$, $\mathrm{var}\left(\widehat{\sigma}^2\right) = o(1)$ and $\widehat{\sigma}^2 \xrightarrow{p} \sigma^2$ by the Cauchy-schwartz inequality. ▮

Proof of Theorem 2

Under H_1,

$$\begin{aligned}
& E\left(\Gamma_n\right)\\
= \;& E\left[\left|B\right|^{-1/2}H_n\left(Z_1,Z_2\right)\right]\\
\approx \;& \int\int\int\left[h\left(y_1-y_2\right)-h\left(y_1+y_2\right)\right]dF_{Y|X}\left(y_1|x\right)dF_{Y|X}\left(y_2|x\right)f_X^2\left(x\right)dx\\
= \;& \frac{1}{2}\int\int\int\left[h\left(y_1-y_2\right)+h\left(y_1-y_2\right)-h\left(y_1+y_2\right)-h\left(-y_1-y_2\right)\right]dF_{Y|X}\left(y_1|x\right)\\
& dF_{Y|X}\left(y_2|x\right)f_X^2\left(x\right)dx\\
= \;& \frac{1}{2}\int\int\int\int\left[\exp\left(iu'y_1\right)-\exp\left(-iu'y_1\right)\right]\left[\exp\left(-iu'y_2\right)-\exp\left(iu'y_2\right)\right]dG\left(u\right)\\
& dF_{Y|X}\left(y_1|x\right)dF_{Y|X}\left(y_2|x\right)f_X^2\left(x\right)dx\\
= \;& \frac{1}{2}\int\int\left|\int\left[\exp\left(iu'y\right)-\exp\left(-iu'y\right)\right]f_{Y|X}\left(y|x\right)dy\right|^2dG\left(u\right)f_X^2\left(x\right)dx\\
= \;& \frac{1}{2}\int\int\left|\psi(u;x)\right|^2dG\left(u\right)f_X^2\left(x\right)dx
\end{aligned}$$

Simple but tedious calculations show $\mathrm{var}(\Gamma_n) = o(1)$. So

$$\Gamma_n \xrightarrow{p} \frac{1}{2}\int\int\left|\psi(u;x)\right|^2dG\left(u\right)f_X^2\left(x\right)dx$$

by the Chebyshev's inequality. Noting that $\widehat{\sigma}^2 = \sigma^2+o_p\left(1\right)$ also holds under H_1, the result follows. ▮

Proof of Theorem 3

The proof is similar to that of 2 except that now under $H_1\left(\alpha_n\right)$, $E\left[n\left|B\right|^{1/2}\Gamma_n\right] = n\left|B\right|^{1/2}\frac{1}{2}\int\int\left|\psi(u;x)\right|^2dG\left(u\right)f_X^2\left(x\right)dx = n\left|B\right|^{1/2}\alpha_n^2\gamma = \gamma$. ▮

Proof of Theorem 4

The proof follows closely to that of 1 (see also Lemma 2.2 of Li (1999)). First,

$$\begin{aligned}
& E^*[H_n(Z_1^*, Z_2^*)|Z_1^*] \\
= \; & |B|^{1/2} E^*[\{[h(Y_1^* - Y_2^*) - h(Y_1^* + Y_2^*)] K_B(X_1^* - X_2^*) | Z_1^*\} \\
= \; & \frac{1}{2n} \sum_{i=1}^{n} [h(Y_1^* - Y_i) - h(Y_1^* + Y_i)] K_B(X_1^* - X_i) \\
& + \frac{1}{2n} \sum_{i=1}^{n} [h(Y_1^* + Y_i) - h(Y_1^* - Y_i)] K_B(X_1^* - X_i) \\
= \; & 0.
\end{aligned}$$

So conditional on $\mathcal{Z}_n$ $H_n(Z_i^*, Z_j^*)$ is a degenerate U-statisitc and $E^*[T_n^*] = 0$. Noting that conditional on $\mathcal{Z}_n$, $\{Z_i^*\}$ are i.i.d., we have

$$\begin{aligned}
\mathrm{var}^*(T_n^*) &= E^*\left[H_n^2(Z_1^*, Z_2^*)\right] \\
&= \frac{1}{4n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left\{ H_n^2((X_i, Y_i), (X_j, Y_j)) + H_n^2((X_i, Y_i), (X_j, -Y_j)) \right. \\
&\quad \left. + H_n^2((X_i, -Y_i), (X_j, Y_j)) + H_n^2((X_i, -Y_i), (X_j, -Y_j)) \right\} \\
&= \frac{n-1}{n} \widehat{\sigma}^2 + o_p(1) = \sigma^2 + o_p(1).
\end{aligned}$$

Finally, we can show that the other conditions of Theorem 1 of Hall (1984) also holds. The result in 4 follows. ▮

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