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The Distorted Theory of Rank-Dependent Expected Utility*

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This paper re-examines the rank-dependent expected utility theory. Firstly, we follow Quiggin's assumption (Quiggin 1982) to deduce the rank-dependent expected utility formula over lotteries and hence extend it to the case of general random variables. Secondly, we utilize the distortion function which reflects decision-makers' beliefs to propose a distorted independence axiom and then to prove the representation theorem of rank-dependent expected utility. Finally, we make direct use of the distorted independence axiom to explain the Allais paradox and the common ratio effect.

Key Words: Expected utility; Rank-dependent expected utility; Distortion function; Distorted independence axiom; The Allais paradox; The Common ratio effect.
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1. INTRODUCTION

It is well known that the independence axiom (IA), the key behavioral assumption of the expected utility (EU) theory, is often violated in practice in experimental studies. Amongst other theories, the anticipated utility (AU) theory, which is also known as rank-dependent expected utility —

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RDEU (Quiggin 1982, Segal 1989, and Quiggin and Wakker 1994), has successfully resolved this issue. There exist several axiomatic systems for this theory. The weak certainty equivalent substitution axiom in Quiggin (1982) and Quiggin and Wakker (1994) implies that the weights function maps $\frac{1}{2}$ to $\frac{1}{2}$.[1] However, such an assumption takes a lot of power out of this theory. Segal (1989) utilizes a measure approach to axiomatize RDEU theory. He proposes a projection IA to graphically compare two cumulative distribution functions (or lotteries), which lacks normative appeal. Here we propose a new axiomatic foundation to RDEU. In our opinion, the weights function in RDEU reflects decision-makers' beliefs and their attitude to risks, and therefore can be treated as exogenous. Based on the weights function, we propose a distorted independence axiom (DIA) and establish a new axiomatic system to build the distorted theory of RDEU. This paper also shows how DIA can be used *directly* to determine the specific forms of weights function under which some examples violating IA would no longer be paradoxical.

We first follow the assumption in Quiggin (1982) and Quiggin and Wakker (1994) that the transformation of cumulative distribution functions (CDFs) is continuous on the whole probability distribution. Applying the reduction of lottery dimensions we prove the RDEU formula over lotteries and show that it also holds for general (continuous) random variables. Quiggin's assumption is a breakthrough in successfully extending EU to RDEU. The essence of the von Neumann - Morgenstern EU theory is a set of restrictions imposed on the preference relations over lotteries that allows their representation by a mathematical expectation of a real function on the set of outcomes. One main aspect of this theory is the specific functional form of the representation, namely, the linearity in probabilities. The EU hypothesis is widely used in various disciplines. However, it sometimes fails to explain some counterexamples. Quiggin (1982) successfully resolves this issue by proposing that probability weights of every prospect are derived from the entire original probability distribution. He tries a special case of three outcomes and writes the general form of his RDEU formula. Segal (1987a) claims that this formula can be extended to any (one-stage) lottery. In this paper we prove Quiggin's RDEU formula over prospects by using the continuity of utility function (von Neumann - Morgenstern) and further generalize this formula for arbitrary random variables. For the case of general random variables we employ Helly theorem to prove the RDEU formula.

Then we re-axiomatize the RDEU theory along the line of Quiggin (1982), Chew (1985) and Segal (1989), using the methods in Fishburn (1982) and Yaari (1987). In the RDEU formula the transformation of CDFs or the deci-

[1]Later Chew (1985) removes this restriction.

sion weights function reflects decision-makers' beliefs. It 'distorts' prospects' probability distributions to reflect a decision maker's own evaluation of probabilities. Therefore we call it the distortion function. We can interpret the distortion function as the decision-maker's attitude to risk when choosing among lotteries. In this paper we assume that this distortion function is exogenous. We observe that the distortion function is also a CDF and can be represented by a random variable. Therefore, the distorted CDF also corresponds to a random variable, which is a compound of the inverse of the CDF of a risky prospect and the random variable corresponding to the distortion function. From this approach we provide our DIA, and hence prove the representation theorem of RDEU by modifying EU. The format of the distorted independence axiom (DIA) is analogous to IA, but in DIA we use a mixture operator instead of the conventional addition. The independence, instead of being hypothesized for convex combinations formed along the CDFs, is postulated for convex combinations formed along the distorted CDFs. Therefore, while IA is applied to the family of CDFs, DIA is applied to the distorted CDFs.

We can use DIA *directly* to rationalize the most famous "paradoxes" in uncertainty theory such as the Allais paradox (or the common consequences effect) and the common ratio effect (or the certainty effect) without resorting to the RDEU formula as in Segal (1987a). We use the distortion function to transform the unit triangle in Machina (1987) to obtain triangles under the framework of DIA. In the new unit triangles, indifference curves keep parallel but positions of prospects change such that lines of compared prospect pairs may not parallel. We show that the compared prospects form parallelograms in the transformed unit triangle if and only if DIA reduces to IA. In this case the inconsistency in these paradoxes would arise. Furthermore, we are able to show that in the transformed unit triangle under DIA, when the distortion function takes specific forms such that the lines of compared prospect pairs fan in, the behavioral patterns in these examples may be rational. Our approach here fundamentally departs from that of Machina (1987). In his diagrams, prospects are fixed and form parallelograms while indifference curves fan out.

The RDEU theory has received several axiomatizations. In Quiggin (1982) and Quiggin and Wakker (1994), a preference relation satisfies a set of axioms including the key weak certainty equivalent substitution axiom if and only if it has an expected utility with rank-dependent probabilities where the probability transformation function maps $\frac{1}{2}$ to $\frac{1}{2}$. As Chew, Karni and Safra (1987), Röell (1987) and Segal (1987a) suggest, risk aversion holds in this theory if and only if von Neumann-Morgendtern utility function is concave and the weights function is convex. Assuming that the weights function maps $\frac{1}{2}$ to $\frac{1}{2}$ takes much power out of the theory. Chew (1985) shows that the latter restriction is not necessary. Segal (1989) presents another set of axioms to prove RDEU by a measure representation approach. His projection IA is of a simple mathematical form which, in our opinion, is lack of interpretations in terms of behavioral foundations. Yaari (1987) also suggests an expected utility theory with rank-dependent probabilities, but with the roles of payment and probability reversed. He cites Fishburn's (1982) five axioms in EU and replaces IA with the dual IA. Our paper presents a more appealing axiomatic system for RDEU by replacing the dual IA in Yaari (1987) with our DIA. To some extent, Yaari's theory can be treated as a special case of our distorted theory of RDEU. The difference is that in RDEU the utility function is endogenous and the distortion function is exogenous while in Yaari the endogeneity of these two is reversed. Our paper further differs from Yaari (1987) in that all random variables in Yaari's model take values in the unit interval so that the inverse of a CDF is still a CDF, but in our model random variables take values from any (closed) interval. We use the distortion function which is also a CDF to "distort" the CDF of a risky prospect.

There are other papers on RDEU. Chew, Karni and Safra (1987) and Karni (1987) study the risk aversion in expected utility theory with rank-dependent probabilities and state-dependent preferences. The RDEU approach can be used not only to explain the examples with uncertainty such as the Allais paradox and the common ratio effect, but also to interpret the Ellsberg paradox (Segal 1987b). Furthermore, Karni and Schmeidler (1991) summarize the utility theory with uncertainty. On the other hand, RDEU can be used to explain ambiguity aversion, as Miao(2004, 2009) and Zou (2006).

This paper is composed of five sections. In section 2 we formally generalize the RDEU formula from Quiggin (1982) and Quiggin and Wakker (1994). In section 3 we propose an axiomatic system with DIA and prove the representation theorem of RDEU. Section 4 explains the Allais paradox and the common ratio effect by directly using DIA, in addition to using the RDEU formula. Section 5 concludes this paper.

2. RANK-DEPENDENT EXPECTED UTILITY FORMULA

This section outlines the RDEU theory, which represents decision makers' preferences using mathematical expectations of a utility function with respect to a transformation of probabilities on a set of outcomes. The transformation function can be found by induction, and generally is not a linear function of the CDF. Each component of the transformation is a function of the whole probability distribution of the prospect and does not depend upon the winning probability of this prize only. For the case of discrete random variables we show that, for any natural number $N = 1, 2, \cdots$, the N-th component is an increment of the transformation of the sum of the winning probabilities for the smallest $N-1$ outcomes. Further, we can extend the EDRU formula to a more general one by using a convergence theorem of CDFs.

Consider a compact interval $[m, M]$ of monetary prizes. Let $\mathscr{L}$ be the set of lotteries (probability measures) over $[m, M]$ and $\mathscr{L}_0$ in $\mathscr{L}$ be the set of lotteries with finite support. For each $X \in \mathscr{L}$, its CDF of X is defined by $F_X(x) = P\{X \leq x\}$ for $x \in [m, M]$. [2]

Now we build the RDEU theory on $\mathscr{L}_0$. For any natural number $N = 1, 2, \cdots$, denote $X^N = (x_1^N, p_1^N; \cdots; x_N^N, p_N^N)$ as a prospect, which yields x_n^N dollars with probability p_n^N for $n = 1, \cdots, N$, where $x_1^N \leq x_2^N \leq \cdots \leq x_{N-1}^N \leq x_N^N$ in $[m, M]$, $p_n^N \geq 0$ for $n = 1, \cdots, N$, and $\sum_{n=1}^N p_n^N = 1$. The RDEU function is defined to be

$$RDEU(X^N) = \sum_{n=1}^{N} H_n^N(p_1^N, \cdots, p_N^N) U(x_n^N) \tag{1}$$

where $H_n^N : [0,1]^N \to [0,1]$ is a continuous function for $n = 1, \cdots, N$ with $\sum_{n=1}^N H_n^N(p_1^N, \cdots, p_N^N) = 1$, and U is a continuous and increasing von Neumann - Morgenstern utility function. $(H_1^N, \cdots, H_N^N)$ is a transformation of the probability distribution and produces a new probability distribution.

Quiggin (1982) assumes that, for $n = 1, \cdots, N$, $H_n^N(p_1^N, \cdots, p_N^N)$ is a function of $(p_1^N, \cdots, p_N^N)$. Under the environment of the RDEU theory, $H_n^N(p_1^N, \cdots, p_N^N)$ is a function of $(p_1^N, \cdots, p_n^N)$ for $n = 1, \cdots, N$, which is proved in this section by induction. For any $N = 1, 2, \cdots$, $X^N = (x_1^N, p_1^N; \cdots; x_N^N, p_N^N)$ with $\sum_{n=1}^N p_n^N = 1$, we have

$$\begin{aligned} H_1^N(p_1^N, \cdots, p_N^N) &= g(p_1^N) & (2) \\ H_n^N(p_1^N, \cdots, p_N^N) &= g\left(\sum_{n'=1}^{n} p_{n'}^N\right) - g\left(\sum_{n'=1}^{n-1} p_{n'}^N\right), \text{ for } n = 2, \cdots, N & (3) \end{aligned}$$

where $g(p) = H_1^2(p, 1-p)$ for $p \in [0, 1]$. Thus we have the RDEU formula in $\mathscr{L}_0$, which is an assertion in Quggin (1982).

THEOREM 1 (Quiggin 1982). *In the rank-dependent expected utility function (1), the behavior of $H_n^N(p_1^N, \cdots, p_N^N)$ on arbitrary probability distributions is fully determined by the values of g as in (2) and (3).*

[2] The CDF F_X satisfies the following three conditions: [1] F_X is non-decreasing; [2] $F_X(m-) = 0$ and $F_X(M) = 1$; and [3] F_X is right-continuous. If function F satisfies the three conditions, then there exists a random variable X such that its CDF F_X is equal to F.

From this theorem, the RDEU function in $\mathscr{L}_0$ is, for $X^N = (x_1^N, p_1^N; \cdots; x_N^N, p_N^N)$,

$$RDEU(X^N) = g(p_1^N)U(x_1^N) + \sum_{n=2}^{N} \left[g\left(\sum_{n'=1}^{n} p_{n'}^N\right) - g\left(\sum_{n'=1}^{n-1} p_{n'}^N\right)\right] U(x_n^N). \quad (4)$$

From (2) and (3), we have $H_n^N(p_1^N, \cdots, p_N^N)$ is the increment of the transformation of the sum of the first $n-1$ winning probabilities. Expression (4) can be re-written in a general form as

$$RDEU(X^N) = \int_{[m,M]} U(x)dg(F_{X^N}(x)).$$

From the proof of Theorem 1 in the Appendix, we summarize the property of function g as follows.

Proposition 1. *The function g is a continuous and increasing function with $g(0) = 0$ and $g(1) = 1$.*

Furthermore, on $\mathscr{L}$ we can also prove the rank-dependent expected utility function by using a convergence theorem of CDFs.

Theorem 2. *The rank-dependent expected utility function in $\mathscr{L}$ is*

$$RDEU(X) = \int_{[m,M]} U(x)dg(F_X(x)). \quad (5)$$

The RDEU formula describes a class of models of decision making under risk in which risks are represented by CDFs, and preference relations on risks are represented by mathematical expectations of a utility function with respect to a transformation of probabilities on a set of outcomes. The distinguishing characteristic of these models is that the transformed probability of an outcome depends on the rank of the outcomes in the induced preference ordering on the set of outcomes. When the function g reduces to the identity, the RDEU formula reduces to the EU one. However, EU does not depend on the rank of the outcomes.

3. A REPRESENTATION THEOREM OF RANK-DEPENDENT EXPECTED UTILITY

In this section we axiomatize the RDEU theory, following the axiomatic systems of Fishburn (1982) and Yaari (1987). We present our five axioms and then prove the representation theorem.

As we know, the existence of von Neumann and Morgenstern EU is equivalent to three axioms: the preference relation axiom, the independence (substitution) axiom, and the Archimedean axiom. The representation of linearity in probabilities in EU is a direct consequence of the restriction on preference relations known as IA. Fishburn (1982) proves the representation theorem of EU from an axiomatic system with five axioms which has been widely used. The five axioms are the neutrality axiom, the complete weak order axiom, the continuity (with respect to L_1- convergence) axiom, the monotonicity (with respect to first-order stochastic dominance) axiom, and the independence axiom. IA is the key behavioral assumption of EU, which is often violated in experimental studies. Yaari (1987) cites Fishburn's (1982) five axioms and replaces IA with the dual IA, and hence establishes the dual theory of choice under risk. At the core of the dual theory is the dual IA. Yaari (1987) develops an expected utility theory with rank-dependent probabilities (EURDP) with the roles of payments and probabilities reversed. In this paper, we replace the dual IA in Yaari (1987) with our DIA. We then use this system of five axioms to prove the representation theorem of RDEU.

We first describe the five axioms in Yaari (1987) and the representation theorem of EU. A strict preference relation $\succ$ is assumed to be defined on $\mathscr{L}$. Let the symbols $\succsim$ and $\sim$ stand for preference relation and indifference relation, respectively. The following axiom suggests itself:

[**Axiom A1 - Neutrality**]: Let X_1 and X_2 belong to $\mathscr{L}$ with respective CDFs F_{X_1} and F_{X_2}. If $F_{X_1} = F_{X_2}$ then $X_1 \sim X_2$.

Denote $\mathscr{F}$ to be a family of CDFs by $\mathscr{F} = \{F : [m, M] \to [0, 1] \mid F \text{ is a CDF}\}$. Define $\succ$ on $\mathscr{F}$ by $F_1 \succ F_2$ if and only if $X_1 \succ X_2$ for which $F_1 = F_{X_1}$ and $F_2 = F_{X_2}$.

[**Axiom A2 - Weak Order**]: $\succ$ is asymmetric and negatively transitive.

[**Axiom A3 - Continuity with respect to L_1-Convergence**]: Let F_1, F_2, F_1', F_2' belong to $\mathscr{F}$; assume that $F_1 \succ F_2$. Then there exists an $\varepsilon > 0$ such that $\|F_1 - F_1'\| < \varepsilon$ and $\|F_2 - F_2'\| < \varepsilon$ imply $F_1' \succ F_2'$, where $\|\cdot\|$ is the L_1-norm $\|F\| = \int_{[m,M]} |F(x)|dx$.

[**Axiom A4 - Monotonicity with respect to First-Order Stochastic Dominance**]: If $F_{X_1}(x) \le F_{X_2}(x)$ for all $x \in [m, M]$, then $F_{X_1} \succsim F_{X_2}$.

[**Axiom A5EU - Independence (Substitution)**]: If F_1, F_2 and F belong to $\mathscr{F}$ and α is a real number satisfying $0 < \alpha < 1$, then $F_1 \succ F_2$ implies $\alpha F_1 + (1 - \alpha)F \succ \alpha F_2 + (1 - \alpha)F$.

By using the five axioms, Yaari (1987) proves the following representation theorem of EU, which is a modification of Fishburn (1982).

THEOREM 3. *A preference relation $\succsim$ satisfies Axioms A1 - A4 and A5EU if and only if there exists a continuous and non-decreasing real function u, defined on $[m, M]$, such that, for all X_1 and X_2 belonging to $\mathscr{L}$,*

$$X_1 \succ X_2 \qquad \Longleftrightarrow \qquad E[u(X_1)] > E[u(X_2)].$$

Moreover, the function u, which is unique up to a positive transformation, can be selected in such a way that, for all $x \in [m, M]$, $u(x)$ solves the preference equation

$$(m, 1 - u(x); M, u(x)) \sim (x, 1).$$

From Theorem 3, the expected utility is given by

$$EU(X) = E[u(X)] = \int_{[m,M]} u(x) dF_X(x).$$

We now present our DIA and prove the representation theorem of RDEU. The distorted theory of choice under risk is obtained when IA (Axiom A5EU) of EU is replaced. We postulate independence for convex combinations that are formed along the distorted CDFs instead of for convex combinations formed along the CDFs. The best way to achieve that is to consider an appropriately defined distortion of CDFs.

In RDEU (Quiggin 1982 and Segal 1989), the rank-dependent expected utility value is

$$RDEU(X) = \int_{[m,M]} u(x) dg(F_X(x)) = \int_{[m,M]} u(x) d[g \circ F_X](x)$$

which is in Section 2. The function $g(p) = H_1^2(p, 1-p)$ for each $p \in [0, 1]$ defines the behavior of (H_1^2, H_2^2) on the pair $(p, 1-p)$ in Theorem 1. Then $g : [0, 1] \to [0, 1]$ is a transformation to change probability distributions. As we have explained in above section, we can treat the function g as exogenous; and we call it the distortion function. Suppose that g satisfies some conditions such that $g \circ F_X$ is a CDF, then we can represent the RDEU value in the form of mathematical expectations. Thus the representation theorem of RDEU can be checked by using Theorem 3 of EU. We now consider the corresponding axiom for independence.

We assume that the distortion function g satisfies Proposition 1. Then the function g is onto and invertible. The inverse of the function g, $g^{-1} : [0, 1] \to [0, 1]$, also satisfies Proposition 1. In addition, We assume that the function $g : [0, 1] \to [0, 1]$ and its inverse $g^{-1} : [0, 1] \to [0, 1]$ satisfy Lipschitz conditions.

From Proposition 1, the function $g : [0, 1] \to [0, 1]$ satisfies all the conditions of CDFs. Then we can consider it as a CDF. Therefore there exists a random variable ξ on $[0, 1]$ such that $g(p) = P\{\xi \le p\}$. From now on we always use ξ as the random variable with the CDF g.

The *inverse* $F^{-1} : [0,1] \to [m,M]$ of a CDF $F : [m,M] \to [0,1]$ is given by:

$$F^{-1}(p) = \sup\{x \in [m,M] | F(x) \leq p\}.$$

Proposition 2. *Let $X \in \mathscr{L}$ be a random variable, then $F_X(X)$ follows a uniform distribution on $[0,1]$. If the random variable θ follows a uniform distribution on $[0,1]$, for any CDF F, $F^{-1}(\theta)$ follows the CDF F.*

If the random variable θ follows the uniform distribution on $[0,1]$, the random variable ξ on $[0,1]$ can be chosen from Proposition 2 as $\xi = g^{-1}(\theta)$. In fact,

$$P\{\xi \leq p\} = P\{g^{-1}(\theta) \leq p\} = P\{\theta \leq g(p)\} = g(p).$$

For any CDF F, we consider the compound function $g \circ F$ of the two CDFs g and F, which is defined by

$$[g \circ F](x) = g(F(x)).$$

It is clear that $g \circ F$ satisfies all the conditions of CDFs. Then $g \circ F$ is a CDF, and we call it a distorted CDF of CDF F.

For any distorted CDF $g \circ F$, we have that

$$[g \circ F](x) = g(F(x)) = P\{\xi \leq F(x)\} = P\{F^{-1}(\xi) \leq x\} = F_{F^{-1}(\xi)}(x)$$

which is the CDF of random variable $F^{-1}(\xi) = F^{-1}(g^{-1}(\theta)) = [F^{-1} \circ g^{-1}](\theta)$ $= [g \circ F]^{-1}(\theta)$. Thus the compound function $g \circ F : [m,M] \to [0,1]$ is also a CDF. Hence

$$g \circ F = F_{F^{-1}(\xi)}.$$

We denote the set of such distorted CDFs as

$$\mathscr{F}^\circ = \{g \circ F \in \mathscr{F} | F \in \mathscr{F}\} = \{F_{F^{-1}(\xi)} \in \mathscr{F} | F \in \mathscr{F}\}.$$

We can simply write $\mathscr{F}^\circ = g(\mathscr{F})$. From the property of Proposition 1 we have $\mathscr{F}^\circ = \mathscr{F}$.

A mixture operation for distorted CDFs in $\mathscr{F}^\circ$ may now be defined as follows: if $g \circ F_1$ and $g \circ F_2$ belong to $\mathscr{F}^\circ$ and if $0 \leq \alpha \leq 1$, then $\alpha[g \circ F_1] \oplus (1-\alpha)[g \circ F_2] \in \mathscr{F}^\circ$ is given by

$$\alpha[g \circ F_1] \oplus (1-\alpha)[g \circ F_2] \equiv g \circ [\alpha F_1 + (1-\alpha)F_2].$$

Equivalently, if $F_{F_1^{-1}(\xi)}$ and $F_{F_2^{-1}(\xi)}$ belong to $\mathscr{F}^\circ$ and if $0 \le \alpha \le 1$, then $\alpha F_{F_1^{-1}(\xi)} \oplus (1-\alpha)F_{F_2^{-1}(\xi)} \in \mathscr{F}^\circ$ is given by

$$\alpha F_{F_1^{-1}(\xi)} \oplus (1-\alpha)F_{F_2^{-1}(\xi)} \equiv F_{[\alpha F_1+(1-\alpha)F_2]^{-1}(\xi)}.$$

For some $0 \le \alpha \le 1$, $\alpha[g\circ F_1] \oplus (1-\alpha)[g\circ F_2] = \alpha F_{F_1^{-1}(\xi)} \oplus (1-\alpha)F_{F_2^{-1}(\xi)}$ is called a harmonic convex combination of F_1 and F_2. With the operation $\oplus$, the set $\mathscr{F}^\circ$ of all distorted CDFs becomes a mixture space.

Returning to the preference relation $\succsim$, we are now in a position to state the axiom that gives rise to the distorted theory of choice under risk:
[**Axiom A5 - Distorted Independence**]: If $g\circ F_1$, $g\circ F_2$ and $g\circ F$ belong to $\mathscr{F}^\circ$ and α is a real number satisfying $0 < \alpha < 1$, then $g\circ F_1 \succ g\circ F_2$ implies

$$\alpha[g\circ F_1] \oplus (1-\alpha)[g\circ F] \succ \alpha[g\circ F_2] \oplus (1-\alpha)[g\circ F].$$

Equivalently, this axiom can be written as
[**Axiom A5 - Distorted Independence**]: If $F_{F_1^{-1}(\xi)}$, $F_{F_2^{-1}(\xi)}$ and $F_{F^{-1}(\xi)}$ belong to $\mathscr{F}^\circ$ and α is a real number satisfying $0 < \alpha < 1$, then $F_{F_1^{-1}(\xi)} \succ F_{F_2^{-1}(\xi)}$ implies

$$\alpha F_{F_1^{-1}(\xi)} \oplus (1-\alpha)F_{F^{-1}(\xi)} \succ \alpha F_{F_2^{-1}(\xi)} \oplus (1-\alpha)F_{F^{-1}(\xi)}.$$

For any distorted CDF F° in $\mathscr{F}^\circ$, there exists a CDF F in $\mathscr{F}$ such that $F^\circ = g\circ F$, then $F^\circ(x) = [g\circ F](x) = g(F(x))$ and $F(x) = g^{-1}(F^\circ(x)) = [g^{-1}\circ F^\circ](x)$ for $x \in [m, M]$, hence $F = g^{-1}\circ F^\circ$. A mixture operation for CDFs in $\mathscr{F}^\circ$ can be defined as follows: if F_1° and F_2° belong to $\mathscr{F}^\circ$ and if $0 \le \alpha \le 1$, then $\alpha F_1^\circ \oplus (1-\alpha)F_2^\circ \in \mathscr{F}^\circ$ is given by

$$\alpha F_1^\circ \oplus (1-\alpha)F_2^\circ \equiv g\circ\{\alpha[g^{-1}\circ F_1^\circ] + (1-\alpha)[g^{-1}\circ F_2^\circ]\}.$$

Then we can write DIA in a simple form.
[**Axiom A5 - Distorted Independence**]: If F_1 , F_2 and F belong to $\mathscr{F}^\circ$ and α is a real number satisfying $0 < \alpha < 1$, then $F_1 \succ F_2$ implies $\alpha F_1 \oplus (1-\alpha)F \succ \alpha F_2 \oplus (1-\alpha)F$.

From the above axioms, we have the following representation theorem of RDEU by using Theorem 3.

THEOREM 4. *Assume that the distortion function g and its inverse g^{-1} satisfy Lipschitz conditions. A preference relation $\succsim$ satisfies Axioms A1 - A5 if and only if there exists a continuous and non-decreasing real function u, defined on $[m, M]$, such that, for all X_1 and X_2 belonging to $\mathscr{L}$,*

$$X_1 \succ X_2 \Longleftrightarrow \int_{[m,M]} u(x)dg(F_{X_1}(x)) > \int_{[m,M]} u(x)dg(F_{X_2}(x)). \tag{6}$$

Moreover, the function u, which is unique up to a positive transformation, can be selected in such a way that, for all $x \in [m, M]$, $u(x)$ solves the preference equation

$$g \circ F_{(m,1-u(x);M,u(x))} \sim g \circ F_{(x,1)}. \tag{7}$$

We first note that g^{-1} satisfying Lipschitz condition is not required for the proof of the sufficient condition for (6). We also note that from Theorem 4, the rank-dependent expected utility is given by

$$\begin{aligned} RDEU(X) &= \int_{[m,M]} u(x) dg(F_X(x)) = \int_{[m,M]} u(x) d[g \circ F_X](x) \\ &= \int_{[m,M]} u(x) dF_{F_X^{-1}(\xi)}(x) = E[u(F_X^{-1}(\xi))] \end{aligned} \tag{8}$$

which is presented in Quiggin (1982) and Segal (1987a and 1989). Chew, Karni and Safra (1987) state that the distortion function g is Lipschitz continuous on $[0, 1]$ if and only if the RDEU functional in (8) is weakly Gateaux differentiable on $\mathscr{F}$.

If all random variables take values in the unit interval $[0, 1]$, Yaari (1987) proposes the dual IA as follows: If F_{X_1} , F_{X_2} and F_X in $\mathscr{F}$ are pairwise comonotonic and α is a real number satisfying $0 < \alpha < 1$, then $F_{X_1} \succ F_{X_2}$ implies

$$\alpha F_{X_1} \boxplus (1-\alpha) F_X \succ \alpha F_{X_2} \boxplus (1-\alpha) F_X$$

where $\alpha F_{X'} \boxplus (1-\alpha) F_X \equiv F_{\alpha X' + (1-\alpha) X}$. Using Axioms A1 - A4 and his dual IA, he then proves his EURDP (Yaari's Theorem of Dual Theory) in which the utility function in payments is linear. The dual utility is given by

$$DU(X) = \int_{[0,1]} f_0(1 - F_X(x)) dx = -\int_{[0,1]} x df_0(1 - F_X(x)).$$

Define $g_0 : [0, 1] \to [0, 1]$ by $g_0(p) = 1 - f_0(1 - p)$, then we have

$$DU(X) = \int_{[0,1]} x dg_0(F_X(x)).$$

If we take the utility function u in Theorem 4 to be linear, then RDEU degenerates into Yaari's dual utility. Yaari's dual utility can be considered as a special case of Quiggin's AU/RDEU. In Theorem 4 of RDEU, the utility function u is endogenous and the distortion function g is exogenous, while, in Yaari's Theorem of Dual Theory, the weights function f_0 (and hence g_0) is endogenous but the utility function u is exogenous and linear.

Yaari (1987) considers random variables assuming values in the unit interval $[0, 1]$ (that is, $[m, M] = [0, 1]$); then the inverse of a CDF is also a CDF. However, for a general interval $[m, M] \neq [0, 1]$, the inverse of a CDF is not a CDF. Therefore we introduce the distortion function such that the distorted CDF is a CDF. As we have seen earlier, the distorted function is a CDF and there exists a random variable following this distribution; hence we can find the random variable associated with the distorted CDF. This is what leads us to obtain the distorted theory of rank-dependent expected utility.

To unravel the paradoxes in next section, we need to use the specific forms of the RDEU formula. From Theorems 2 and 4, the rank-dependent expected utility value of random variable $X \in \mathscr{L}$ is

$$RDEU(X) = \int_{[m,M]} u(x) dg(F_X(x)) = u(M) - \int_m^M g(F_X(x)) du(x).$$

We define a decision-weights function $f : [0, 1] \to [0, 1]$ by $f(p) = 1 - g(1-p)$; then it also satisfies $f(0) = 0$ and $f(1) = 1$. Therefore the RDEU functional is given by

$$RDEU(X) = -\int_{[m,M]} u(x) df(1-F_X(x)) = u(m) + \int_m^M f(1-F_X(x)) du(x).$$

When we consider a simple lottery $X = (x_1, p_1; \cdots; x_N, p_N)$, the RDEU functional is

$$\begin{aligned} RDEU(X) &= g(p_1)u(x_1) + \sum_{n=2}^{N} \left[g\left(\sum_{n'=1}^{n} p_{n'} \right) - g\left(\sum_{n'=1}^{n-1} p_{n'} \right) \right] u(x_n) \\ &= u(x_N) - \sum_{n=2}^{N} g\left(\sum_{n'=1}^{n-1} p_{n'} \right) [u(x_n) - u(x_{n-1})]. \end{aligned} \tag{9}$$

and

$$\begin{aligned} RDEU(X) &= \sum_{n=1}^{N-1} \left[f\left(\sum_{n'=n}^{N} p_{n'} \right) - f\left(\sum_{n'=n+1}^{N} p_{n'} \right) \right] u(x_n) + f(p_N)u(x_N) \\ &= u(x_1) + \sum_{n=2}^{N} f\left(\sum_{n'=n}^{N} p_{n'} \right) [u(x_n) - u(x_{n-1})]. \end{aligned} \tag{10}$$

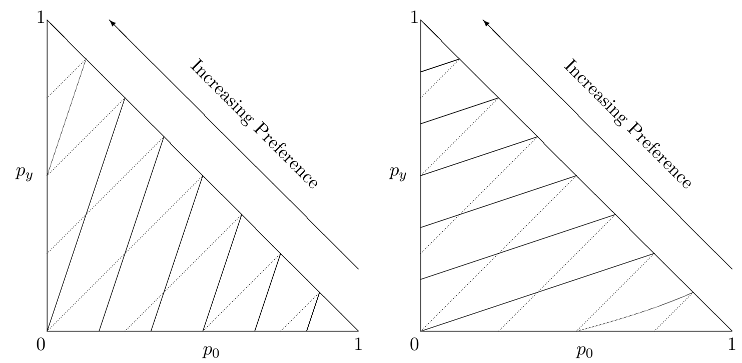
The expressions (9) – (10) are more concise formulas of the RDEU theory for discrete random variables.

4. DIRECT APPLICATIONS OF DISTORTED INDEPENDENCE AXIOM

In this section, we show the significance of DIA. We use DIA directly to rationalize two famous examples — the Allais paradox and the common ratio effect. The two examples are inconsistent with IA and EU, but may agree with RDEU, which can be checked by applying RDEU formulas (9)—(10) (Segal 1987a and 1989). Using DIA directly, we can determine the functional forms of the distortion function such that the two examples are paradoxical. Furthermore, we look beyond these functional forms and are able to obtain conditions under which the two examples accord with DIA.

In order to explain the roles played by IA and DIA for the two examples, we will use isosceles right triangles in Machina (1987). As he demonstrates in his well-known article, the set of all prospects over the fixed outcome levels $0 < x < y$ can be represented by the set of all probability triples of the form (p_0, p_x, p_y) where $p_0 = P\{X = 0\}$, $p_x = P\{X = x\}$, $p_y = P\{X = y\}$, and $p_0 + p_x + p_y = 1$. Graphically, this set of gambles can be represented in two dimensions, in (p_0, p_y) plane (Figure 1), since the third dimension, p_x, is implicit in the graph by $p_x = 1 - p_0 - p_y$. Then the indifference curves under EU in the triangle diagram are straight lines with the same slope, which illustrates the property of linearity in probabilities. Attitude to risk can also be illustrated in the unit triangle where relatively steep utility indifference curves represent risk aversion and relatively flat utility indifference curves represent risk loving.

FIG. 1. Indifference curves under EU in the triangle diagram. Solid lines are indifference curves and dotted lines are iso-expected value lines.



4.1. The Allais Paradox

[**The Allais Paradox**]: Consider the following two problems:
Problem 1: Choose between

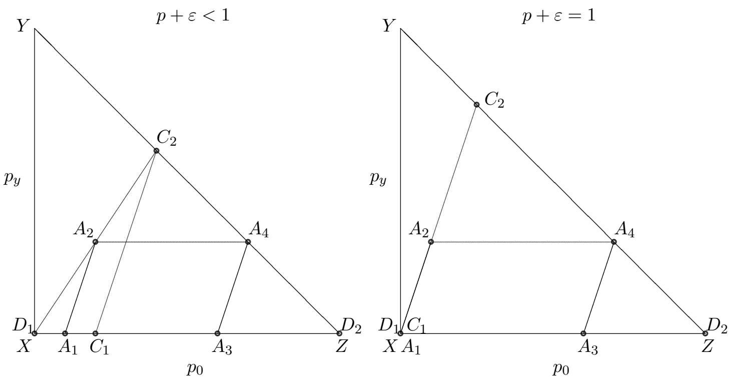
$$A_1 = (0, 1-p-\varepsilon;\ x, p+\varepsilon) \qquad \text{and} \qquad A_2 = (0, 1-q-\varepsilon;\ x, \varepsilon;\ y, q);$$

Problem 2: Choose between

$$A_3 = (0, 1-p;\ x, p) \qquad \text{and} \qquad A_4 = (0, 1-q;\ y, q)$$

where $0 < x < y$, $0 < q < p < 1$, and $0 < \varepsilon \leq 1-p$. Most people prefer A_1 to A_2 and A_4 to A_3 (Allais 1953). However, they are not consistent with IA or EU. Allais (1953) takes the parameter values as $x = \$1m$, $y = \$5m$, $p = 0.11$, $q = 0.10$, and $\varepsilon = 0.89$. In Kahneman and Tversky (1979), $x = 2400$, $y = 2500$, $p = 0.34$, $q = 0.33$, and $\varepsilon = 0.66$.

FIG. 2. The Allais Paradox and the Independence Axiom



EU implies that $A_1 \succ A_2$ and $A_3 \succ A_4$ are equivalent. Under RDEU, $A_1 \succ A_2$ and $A_4 \succ A_3$ are compatible if and only if the distortion function g is concave (Segal 1987a). Under IA, $A_1 \succ A_2$ is only compatible with $A_3 \succ A_4$, which we use the unit triangle to illustrate. Later we will apply DIA to the unit triangle to resolve the paradox. Figure 2 shows us the four prospects A_1, A_2, A_3 and A_4 in the plane (p_0, p_y), where $\overline{A_1A_2} \parallel \overline{A_3A_4}$ and $\overline{A_1A_3} \parallel \overline{A_2A_4}$. Slope of $\overline{A_1A_2}$ = Slope of $\overline{A_3A_4} = \dfrac{q}{p-q}$. We can find two pairs of prospects to represent the original four prospects A_1, A_2, A_3, and A_4. Figure 2 reports how to take the two pairs of new prospects. First, we take point C_2 to be the intersection of line $\overline{XA_2}$ and line $\overline{ZY}$ and point C_1 to be the point on the line $\overline{XZ}$ such that $\overline{C_1C_2} \parallel \overline{A_1A_2}$. Then we take D_1 to be X (the origin) and D_2 to be Z. The two pairs of prospects are defined as $C_1 = \left(0, \frac{1-p-\varepsilon}{1-\varepsilon}; x, \frac{p}{1-\varepsilon}\right)$ and $C_2 = \left(0, \frac{1-q-\varepsilon}{1-\varepsilon}; y, \frac{q}{1-\varepsilon}\right)$, $D_1 = (x, 1)$ and $D_2 = (0, 1)$. Then $A_1 = (1-\varepsilon)C_1 + \varepsilon D_1$ and $A_2 = (1-\varepsilon)C_2 + \varepsilon D_1$, $A_3 = (1-\varepsilon)C_1 + \varepsilon D_2$ and $A_4 = (1-\varepsilon)C_2 + \varepsilon D_2$. In addition, $F_{A_1} = (1-\varepsilon)F_{C_1} + \varepsilon F_{D_1}$ and $F_{A_2} = (1-\varepsilon)F_{C_2} + \varepsilon F_{D_1}$, $F_{A_3} = (1-\varepsilon)F_{C_1} + \varepsilon F_{D_2}$ and $F_{A_4} = (1-\varepsilon)F_{C_2} + \varepsilon F_{D_2}$. From IA, $F_{C_1} \succ F_{C_2}$ implies $F_{A_1} \succ F_{A_2}$ and $F_{A_3} \succ F_{A_4}$. Then $A_1 \succ A_2$ is only compatible with $A_3 \succ A_4$.

4.1.1. Under DIA, the distortion function g is the identity if and only if $\overline{A_1'A_2'} \parallel \overline{A_3'A_4'}$, and in this case $A_1 \succ A_2$ and $A_3 \succ A_4$ hold simultaneously

Now we explain this paradox by directly using DIA. In Figure 3, we define four CDFs $F_{A_1'}$, $F_{A_2'}$, $F_{A_3'}$ and $F_{A_4'}$ in $\mathscr{F}$ as $F_{A_1'} = g^{-1}\circ F_{A_1}$, $F_{A_2'} = g^{-1}\circ F_{A_2}$, $F_{A_3'} = g^{-1}\circ F_{A_3}$ and $F_{A_4'} = g^{-1}\circ F_{A_4}$. That is,

$$\begin{aligned}
A_1' &= (0, g^{-1}(1-p-\varepsilon);\ x, 1 - g^{-1}(1-p-\varepsilon)),\\
A_2' &= (0, g^{-1}(1-q-\varepsilon);\ x, g^{-1}(1-q) - g^{-1}(1-q-\varepsilon);\ y, 1 - g^{-1}(1-q)),\\
A_3' &= (0, g^{-1}(1-p);\ x, 1 - g^{-1}(1-p)),\\
A_4' &= (0, g^{-1}(1-q);\ y, 1 - g^{-1}(1-q)).
\end{aligned}$$

We take point C_2' to be the intersection of line $\overline{X'A_2'}$ and line $\overline{Z'Y'}$ and point C_1' to be the point on the line $\overline{X'Z'}$ such that $\overline{C_1'C_2'} \parallel \overline{A_1'A_2'}$. Then the two prospects can be expressed as

$$\begin{aligned}
C_1' &= \left(0, \frac{g^{-1}(1-p-\varepsilon)}{1-g^{-1}(1-q)+g^{-1}(1-q-\varepsilon)};\ x, 1-\frac{g^{-1}(1-p-\varepsilon)}{1-g^{-1}(1-q)+g^{-1}(1-q-\varepsilon)}\right);\\
C_2' &= \left(0, \frac{g^{-1}(1-q-\varepsilon)}{1-g^{-1}(1-q)+g^{-1}(1-q-\varepsilon)};\ y, \frac{1-g^{-1}(1-q)}{1-g^{-1}(1-q)+g^{-1}(1-q-\varepsilon)}\right).
\end{aligned}$$

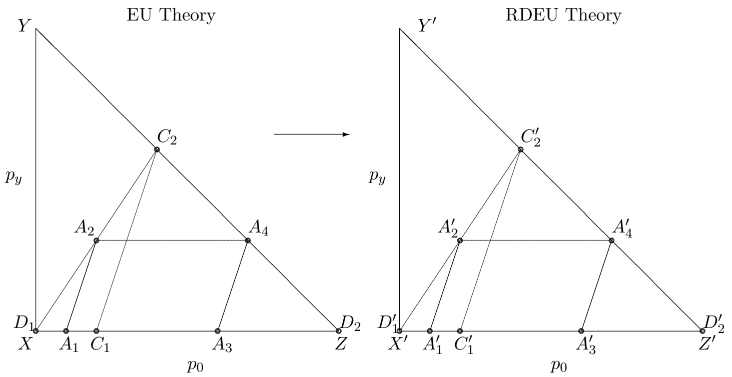
Then $A_1' = (1-\varepsilon')C_1' + \varepsilon' D_1'$ and $A_2' = (1-\varepsilon')C_2' + \varepsilon' D_1'$ where $\varepsilon' = g^{-1}(1-q) - g^{-1}(1-q-\varepsilon)$. In addition, $F_{A_1'} = (1-\varepsilon')F_{C_1'} + \varepsilon' F_{D_1'}$ and $F_{A_2'} = (1-\varepsilon')F_{C_2'} + \varepsilon' F_{D_1'}$. Then

$$\begin{aligned}
F_{A_1} &= g\circ F_{A_1'} = g\circ[(1-\varepsilon')F_{C_1'} + \varepsilon' F_{D_1'}] = (1-\varepsilon')[g\circ F_{C_1'}] \oplus \varepsilon'[g\circ F_{D_1'}];\\
F_{A_2} &= g\circ F_{A_2'} = g\circ[(1-\varepsilon')F_{C_2'} + \varepsilon' F_{D_1'}] = (1-\varepsilon')[g\circ F_{C_2'}] \oplus \varepsilon'[g\circ F_{D_1'}].
\end{aligned}$$

From DIA, $g\circ F_{C_1'} \succ g\circ F_{C_2'}$ implies $F_{A_1} \succ F_{A_2}$.

As we know $\overline{A_1'A_3'} \parallel \overline{A_2'A_4'}$, then $\frac{|\overline{D_2'A_4'}|}{|\overline{D_2'C_2'}|} = \frac{|\overline{D_1'A_2'}|}{|\overline{D_1'C_2'}|} = \frac{|\overline{D_1'A_1'}|}{|\overline{D_1'C_1'}|} = 1-\varepsilon'$. If $\frac{|\overline{D_2'A_3'}|}{|\overline{D_2'C_1'}|} = 1-\varepsilon'$, then $\overline{C_1'C_2'} \parallel \overline{A_3'A_4'}$, and hence $A_3' = (1-\varepsilon')C_1' + \varepsilon' D_2'$

FIG. 3. The Allais Paradox and the Distorted Independence Axiom



and $A'_4 = (1-\varepsilon')C'_2 + \varepsilon' D'_2$. In addition, $F_{A'_3} = (1-\varepsilon')F_{C'_1} + \varepsilon' F_{D'_2}$ and $F_{A'_4} = (1-\varepsilon')F_{C'_2} + \varepsilon' F_{D'_2}$. Then

$$F_{A_3} = g \circ F_{A'_3} = g \circ [(1-\varepsilon')F_{C'_1} + \varepsilon' F_{D'_2}] = (1-\varepsilon')[g \circ F_{C'_1}] \oplus \varepsilon'[g \circ F_{D'_2}];$$
$$F_{A_4} = g \circ F_{A'_4} = g \circ [(1-\varepsilon')F_{C'_2} + \varepsilon' F_{D'_2}] = (1-\varepsilon')[g \circ F_{C'_2}] \oplus \varepsilon'[g \circ F_{D'_2}].$$

From DIA, $g \circ F_{C'_1} \succ g \circ F_{C'_2}$ implies $F_{A_3} \succ F_{A_4}$.

The condition $\dfrac{|\overline{D'_2 A'_3}|}{|\overline{D'_2 C'_1}|} = 1-\varepsilon'$ implies

$$\frac{g^{-1}(1-p)}{1-\dfrac{g^{-1}(1-p-\varepsilon)}{1-g^{-1}(1-q)+g^{-1}(1-q-\varepsilon)}} = 1-g^{-1}(1-q)+g^{-1}(1-q-\varepsilon).$$

Therefore, we have, for any $0 < q < p < 1$ and $0 < \varepsilon \leq 1-p$,

$$g^{-1}(1-q) - g^{-1}(1-q-\varepsilon) = g^{-1}(1-p) - g^{-1}(1-p-\varepsilon). \tag{11}$$

Then $g^{-1}(1-p) - g^{-1}(1-p-\varepsilon)$ is independent of p. For any $0 < p < 1$ and $0 < \varepsilon \leq 1-p$,

$$g^{-1}(1-p) - g^{-1}(1-p-\varepsilon) = R(\varepsilon). \tag{12}$$

Differentiating (12) with respect to p, we then have $[g^{-1}]'(1-p) = [g^{-1}]'(1-p-\varepsilon)$. That is, $[g^{-1}]'(1-p)$ is a constant and thus g^{-1} is linear. Since $g^{-1}(0) = 0$ and $g^{-1}(1) = 1$, then $g^{-1}(p) = p$ and $g(p) = p$. Therefore the distortion function g is the identity. In this case $A_1 \succ A_2$ is only compatible with $A_3 \succ A_4$.

In summary, we find the condition for the distortion function such that $\overline{A_1'A_2'} \parallel \overline{C_1'C_2'} \parallel \overline{A_3'A_4'}$ and hence

$$\begin{aligned} \frac{1-g^{-1}(1-q)}{g^{-1}(1-q-\varepsilon)-g^{-1}(1-p-\varepsilon)} &= \text{Slope of } \overline{A_1'A_2'} \\ &= \text{Slope of } \overline{A_3'A_4'} = \frac{1-g^{-1}(1-q)}{g^{-1}(1-q)-g^{-1}(1-p)}. \end{aligned}$$

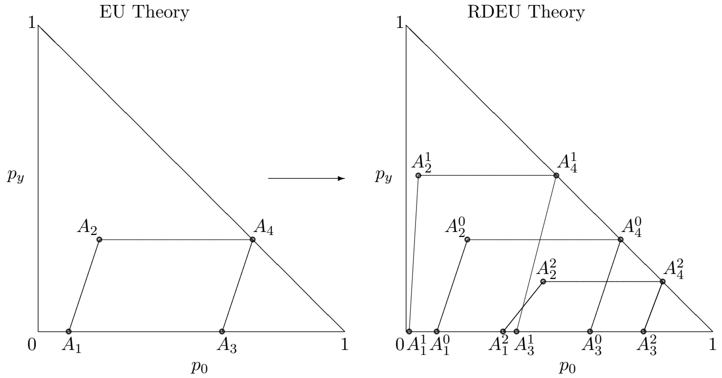
Then we have (11) and the distortion function g is the identity. In this case DIA reduces to IA and the RDEU formula reduces to the EU one, and we have $A_1 \succ A_2$ and $A_3 \succ A_4$ hold simultaneously.

4.1.2. A Closer Look

Now we take a closer look at the Allais paradox by using DIA in the unit triangle. In the left diagram of Figure 3, the indifference curves under EU are linear; and in the right diagram of Figure 3 which is the transformed triangle, the indifference curves are also linear. Based on Figure 3, we can use DIA directly to explain the Allais paradox. As we have shown above,

$$\begin{aligned} \text{Slope of } \overline{A_1'A_2'} &= \frac{1-g^{-1}(1-q)}{g^{-1}(1-q-\varepsilon)-g^{-1}(1-p-\varepsilon)}; \\ \text{Slope of } \overline{A_3'A_4'} &= \frac{1-g^{-1}(1-q)}{g^{-1}(1-q)-g^{-1}(1-p)}. \end{aligned}$$

FIG. 4. Rationalization of the Allais Paradox by DIA. A^0, A^1, and A^2 correspond to the cases that g is the identity, g is concave, and g is convex, respectively.



The distortion function g is the identity if and only if Slope of $\overline{A_1'A_2'} =$ Slope of $\overline{A_3'A_4'}$ if and only if $\overline{A_1'A_2'} \parallel \overline{A_3'A_4'}$ if and only if $A_1 \succ A_2$ and

$A_3 \succ A_4$. In the right-hand-side of Figure 4 we use superscript ^{0} to replace $'$ in A' to denote for this case. As we know, under RDEU, $A_1 \succ A_2$ and $A_4 \succ A_3$ are compatible if and only if g is concave. Does this result hold under DIA? We discuss the two cases where g is not the identity as follows. Note $[g^{-1}]''(g(p)) = -\dfrac{g''(p)}{[g'(p)]^3}$.

[1] The distortion function g is concave if and only if the weights function g^{-1} is convex if and only if $g^{-1}(1-p) - g^{-1}(1-p-\varepsilon) < g^{-1}(1-q) - g^{-1}(1-q-\varepsilon)$ if and only if $g^{-1}(1-q-\varepsilon) - g^{-1}(1-p-\varepsilon) < g^{-1}(1-q) - g^{-1}(1-p)$ if and only if Slope of $\overline{A_1'A_2'}$ > Slope of $\overline{A_3'A_4'}$. In the right-hand-side of Figure 4 we use superscript ^{1} to replace $'$ in A'. In this case, it is possible that $A_1 \succ A_2$ and $A_4 \succ A_3$ are compatible.

[2] The distortion function g is convex if and only if the weights function g^{-1} is concave if and only if $g^{-1}(1-p) - g^{-1}(1-p-\varepsilon) > g^{-1}(1-q) - g^{-1}(1-q-\varepsilon)$ if and only if $g^{-1}(1-q-\varepsilon) - g^{-1}(1-p-\varepsilon) > g^{-1}(1-q) - g^{-1}(1-p)$ if and only if Slope of $\overline{A_1'A_2'}$ < Slope of $\overline{A_3'A_4'}$. In the right-hand-side of Figure 4 we use superscript ^{2} to replace $'$ in A'. In this case, $A_1 \succ A_2$ and $A_3 \succ A_4$ are compatible.

We summarize the above results from the view of DIA as follows.

1. The distortion function g is the identity if and only if $\overline{A_1'A_2'} \parallel \overline{A_3'A_4'}$, then $A_1 \succ A_2$ and $A_3 \succ A_4$ hold simultaneously.
2. The distortion function g is concave if and only if Slope of $\overline{A_1'A_2'}$ > Slope of $\overline{A_3'A_4'}$, and it is possible that $A_1 \succ A_2$ and $A_4 \succ A_3$ are compatible.
3. The distortion function g is convex if and only if Slope of $\overline{A_1'A_2'}$ < Slope of $\overline{A_3'A_4'}$, then $A_1 \succ A_2$ and $A_3 \succ A_4$ are compatible.

4.2. The Common Ratio Effect

[The Common Ratio Effect]: Consider the following two problems:

Problem 1: Choose between

$$A_1 = (0, 1-p;\ x, p) \qquad \text{and} \qquad A_2 = (0, 1-q;\ y, q);$$

Problem 2: Choose between

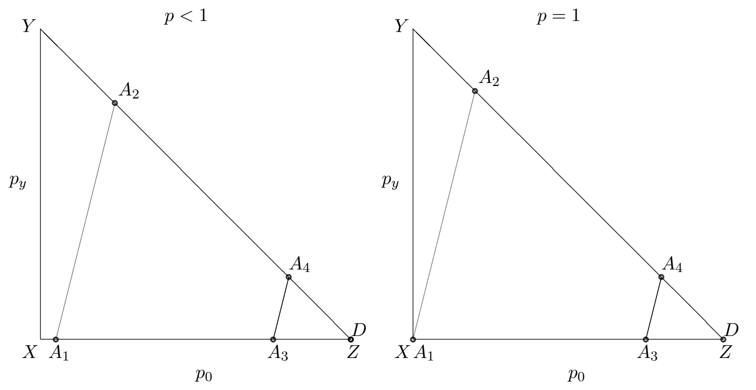
$$A_3 = (0, 1-\lambda p;\ x, \lambda p) \qquad \text{and} \qquad A_4 = (0, 1-\lambda q;\ y, \lambda q)$$

where $0 < x < y$, $0 < q < p \leq 1$, and $0 < \lambda < 1$. Most people prefer A_1 to A_2 and A_4 to A_3 (MacCrimmon and Larsson 1979). However, they are not consistent with IA or EU. MacCrimmon and Larsson (1979) take the parameter values as $x = \$1m$, $y = \$5m$, $p = 1.00$, $q = 0.80$, and $\lambda = 0.05$.

In Kahneman and Tversky (1979), $x = 3000$, $y = 4000$, $p = 1.00$, $q = 0.80$, and $\lambda = 0.25$.

EU implies that $A_1 \succ A_2$ and $A_3 \succ A_4$ are equivalent. Under RDEU $A_1 \succ A_2$ and $A_4 \succ A_3$ are compatible if and only if the elasticity of the weights function f is increasing (Segal 1987a). Under IA, $F_{A_1} \succ F_{A_2}$ implies $F_{A_3} \succ F_{A_4}$, which we illustrate in the unit triangle. Later we will apply DIA to the unit triangle to resolve the common ratio effect. Figure 5 shows us the four prospects A_1, A_2, A_3 and A_4 in the plane (p_0, p_y), where $\overline{A_1A_2} \parallel \overline{A_3A_4}$ (Slope of $\overline{A_1A_2}$ = Slope of $\overline{A_3A_4} = \dfrac{q}{p-q}$). Define $D = (0, 1)$. then $A_3 = \lambda A_1 + (1-\lambda)D$ and $A_4 = \lambda A_2 + (1-\lambda)D$, $F_{A_3} = \lambda F_{A_1} + (1-\lambda)F_D$ and $F_{A_4} = \lambda F_{A_2} + (1-\lambda)F_D$. From IA, $F_{A_1} \succ F_{A_2}$ implies $F_{A_3} \succ F_{A_4}$.

FIG. 5. The Common Ratio Effect and the Independence Axiom

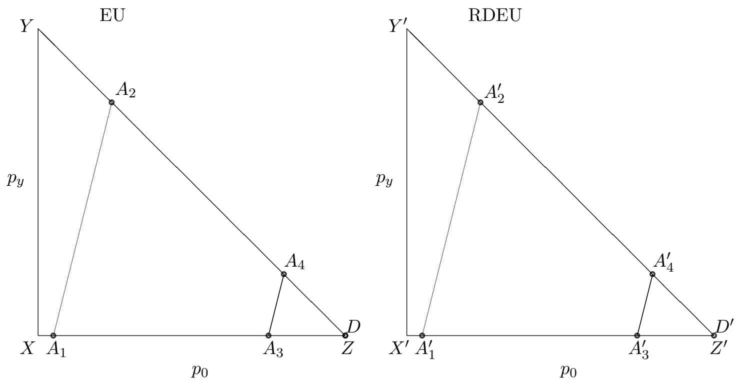


4.2.1. Under DIA, the weights function f is of constant elasticity if and only if $\overline{A'_1A'_2} \parallel \overline{A'_3A'_4}$, and in this case $A_1 \succ A_2$ and $A_3 \succ A_4$ hold simultaneously

Now we explain this example by using DIA. Define two new prospects to be $A'_1 = (0, g^{-1}(1-p);\ x, 1 - g^{-1}(1-p))$ and $A'_2 = (0, g^{-1}(1-q);\ y, 1 - g^{-1}(1-q))$ such that their CDFs $F_{A'_1}$ and $F_{A'_2}$ in $\mathscr{F}$ satisfy $F_{A_1} = g \circ F_{A'_1}$ and $F_{A_2} = g \circ F_{A'_2}$, as illustrated in Figure 6. By DIA, $F_{A_1} \succ F_{A_2}$ if and only if $g \circ F_{A'_1} \succ g \circ F_{A'_2}$ implies, for any $\alpha \in [0, 1]$ and $0 \le z \le x$, $\alpha[g \circ F_{A'_1}] \oplus (1-\alpha)[g \circ G_z] \succ \alpha[g \circ F_{A'_2}] \oplus (1-\alpha)[g \circ G_z]$. That is, $g \circ [\alpha F_{A'_1} + (1-\alpha)G_z] \succ g \circ [\alpha F_{A'_2} + (1-\alpha)G_z]$.

We take D' such that $F_{D'} = g^{-1} \circ F_D$, then $D' = D = (0, 1)$. For $0 \le z \le x$, define G_z in $\mathscr{F}$ to be a CDF for a degenerate distribution which assigns the value z with probability 1. Then the random variable is $\delta_z = (z, 1)$ and hence $G_0 = F_D = F_{D'}$.

FIG. 6. The Common Ratio Effect and the Distorted Independence Axiom



We now find $\alpha \in [0,1]$ such that $F_{A_3} = g \circ [\alpha F_{A'_1} + (1-\alpha)F_{D'}]$ and $F_{A_4} = g \circ [\alpha F_{A'_2} + (1-\alpha)F_{D'}]$. Define A'_3 and A'_4 as $F_{A'_3} = \alpha F_{A'_1} + (1-\alpha)F_{D'}$ and $F_{A'_4} = \alpha F_{A'_2} + (1-\alpha)F_{D'}$, then $F_{A_3} = g \circ F_{A'_3}$ and $F_{A_4} = g \circ F_{A'_4}$ (See Figure 6). In this case, $\overline{A'_3 A'_4} \parallel \overline{A'_1 A'_2}$ and $A_3 \succ A_4$.

$$\begin{aligned} 1 - \lambda p &= g(\alpha g^{-1}(1-p) + (1-\alpha)) \\ 1 - \lambda q &= g(\alpha g^{-1}(1-q) + (1-\alpha)). \end{aligned}$$

Thus for any $0 < q < p < 1$ and $0 < \lambda < 1$,

$$\alpha = \frac{1 - g^{-1}(1-\lambda p)}{1 - g^{-1}(1-p)} = \frac{1 - g^{-1}(1-\lambda q)}{1 - g^{-1}(1-q)}. \tag{13}$$

Then $\dfrac{1 - g^{-1}(1-\lambda p)}{1 - g^{-1}(1-p)}$ is independent of p. For any $0 < p < 1$ and $0 < \lambda < 1$, $\dfrac{1 - g^{-1}(1-\lambda p)}{1 - g^{-1}(1-p)} = R(\lambda) \in [0,1]$. That is,

$$1 - g^{-1}(1-\lambda p) = R(\lambda)[1 - g^{-1}(1-p)] \tag{14}$$

with $R(1) = 1$. Differentiating (14) with respect to p and λ, we then have

$$\begin{aligned} \lambda[g^{-1}]'(1-\lambda p) &= R(\lambda)[g^{-1}]'(1-p) \\ p[g^{-1}]'(1-\lambda p) &= R'(\lambda)[1 - g^{-1}(1-p)] \end{aligned}$$

Thus $\lambda R'(\lambda) = R'(1)R(\lambda)$ and hence

$$R(\lambda) = \lambda^{R'(1)} \tag{15}$$

When p approaches unity in (14), we have the limit as $1 - g^{-1}(1-\lambda) = R(\lambda)$. It follows, from (15), $g^{-1}(1-\lambda) = 1-\lambda^{R'(1)}$ and $g(1-\lambda^{R'(1)}) = 1-\lambda$. Therefore

$$g(p) = 1 - (1-p)^{\frac{1}{R'(1)}}.$$

Therefore, under DIA, $\overline{A_1'A_2'} \parallel \overline{A_3'A_4'}$ if and only if the distortion function g satisfies $g(p) = 1 - (1-p)^{\gamma}$ where $\gamma > 0$, and in this case both $A_1 \succ A_2$ and $A_3 \succ A_4$ hold together. The value of α in (13) is chosen such that $\overline{A_1'A_2'} \parallel \overline{A_3'A_4'}$ and hence

$$\begin{aligned} \frac{1 - g^{-1}(1-q)}{g^{-1}(1-q) - g^{-1}(1-p)} &= \text{Slope of } \overline{A_1'A_2'} \\ &= \text{Slope of } \overline{A_3'A_4'} = \frac{1 - g^{-1}(1-\lambda q)}{g^{-1}(1-\lambda q) - g^{-1}(1-\lambda p)}. \end{aligned}$$

We can then conclude that the distortion function g is of the form $g(p) = 1 - (1-p)^{\gamma}$. Note that if the form of the distortion function is $g(p) = 1-(1-p)^{\gamma}$ with $\gamma > 0$, then the function $f(p)$ defined in Section 3 becomes p^{γ}, and hence the elasticity of f, which is defined as $p\dfrac{f'(p)}{f(p)}$, equals to γ. Conversely, if $p\dfrac{f'(p)}{f(p)} = \gamma$, then $f(p) = p^{\gamma}$ and $g(p) = 1 - (1-p)^{\gamma}$. Therefore under DIA, both $A_1 \succ A_2$ and $A_3 \succ A_4$ hold if and only if the elasticity of the weights function f is a positive constant.

4.2.2. A Closer Look

The common ratio effect example is also regarded as irrational behavior under IA and EU. However, the inconsistent result holds when the distortion function g satisfies $g(p) = 1 - (1-p)^{\gamma}$ where $\gamma > 0$ under DIA (which is IA when $\gamma = 1$). Under RDEU, the paradox can disappear when the corresponding weights function has an increasing elasticity.

As for the explanation for the Allais paradox, we turn to the unit triangle to explain the common ratio effect by directly using DIA, which is illustrated in Figure 6. As we know,

$$\text{Slope of } \overline{A_1'A_2'} = \frac{1-g^{-1}(1-q)}{g^{-1}(1-q)-g^{-1}(1-p)};$$
$$\text{Slope of } \overline{A_3'A_4'} = \frac{1-g^{-1}(1-\lambda q)}{g^{-1}(1-\lambda q)-g^{-1}(1-\lambda p)}.$$

Define a function $h:[0,1]\to[0,1]$ by $h(p)=1-g^{-1}(1-p)$, then

$$\text{Slope of } \overline{A_1'A_2'} = \frac{h(q)}{h(p)-h(q)} = \frac{1}{\frac{h(p)}{h(q)}-1};$$
$$\text{Slope of } \overline{A_3'A_4'} = \frac{h(\lambda q)}{h(\lambda p)-h(\lambda q)} = \frac{1}{\frac{h(\lambda p)}{h(\lambda q)}-1}.$$

Now we can find the relation between the function $f(p)=1-g(1-p)$ and $h(p)=1-g^{-1}(1-p)$ as follows: $h(p)=1-g^{-1}(1-p)$ if and only if $g^{-1}(p)=1-h(1-p)$ if and only if $p=g^{-1}(g(p))=1-h(1-g(p))$ if and only if $1-p=h(1-g(p))$ if and only if $h^{-1}(1-p)=1-g(p)$ if and only if $h^{-1}(p)=1-g(1-p)=f(p)$ if and only if $h(f(p))=p$.

Since $h(f(p))=p$ implies $h'(f(p))f'(p)=1$ and $f'(p)=\dfrac{1}{h'(h^{-1}(p))}$, the elasticity of f at p is

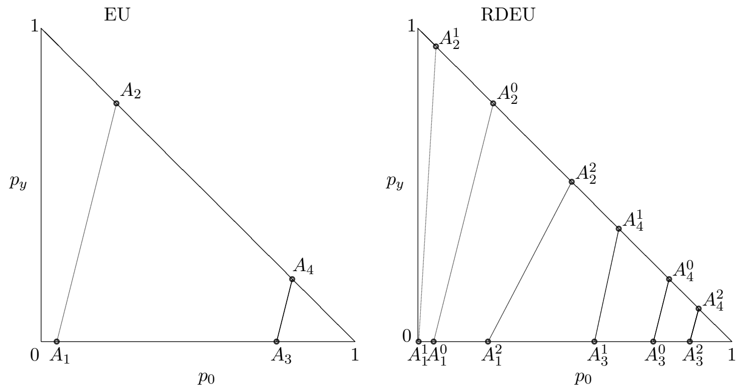
$$p\frac{f'(p)}{f(p)} = p\frac{1}{h^{-1}(p)h'(h^{-1}(p))} = \frac{h(h^{-1}(p))}{h^{-1}(p)h'(h^{-1}(p))} = \frac{1}{h^{-1}(p)\frac{h'(h^{-1}(p))}{h(h^{-1}(p))}}$$

which is the inverse of the elasticity of h at $h^{-1}(p)$.

As we know, the elasticity of the weights function f is a positive constant, $p\dfrac{f'(p)}{f(p)}=\gamma$, if and only if $f(p)=p^{\gamma}$ if and only if $h(p)=p^{\frac{1}{\gamma}}$ if and only if Slope of $\overline{A_1'A_2'}=\frac{1}{\left[\frac{p}{q}\right]^{\frac{1}{\gamma}}-1}=$ Slope of $\overline{A_3'A_4'}$ if and only if $\overline{A_1'A_2'}\parallel\overline{A_3'A_4'}$ if and only if $A_1\succ A_2$ and $A_3\succ A_4$. In the right-hand-side of Figure 7 we use superscript ^{0} to replace $'$ in A' to denote for this case. As we know, under RDEU, $A_1\succ A_2$ and $A_4\succ A_3$ are compatible if and only if the elasticity of the weights function f is a positive constant. Does this result hold under DIA?

[1] The elasticity of the weights function f is increasing if and only if the elasticity of the weights function h is decreasing if and only if $\dfrac{h(\lambda p)}{h(p)}>\dfrac{h(\lambda q)}{h(q)}$ if and only if $\dfrac{h(\lambda p)}{h(\lambda q)}>\dfrac{h(p)}{h(q)}$ if and only if Slope of $\overline{A_1'A_2'}>$ Slope of $\overline{A_3'A_4'}$.

FIG. 7. Rationalization of the Common Ratio Effect by the distorted independence axiom. A^0, A^1, and A^2 correspond to the cases that the elasticity of the weights function f is constant, increasing, and decreasing, respectively.



In the right-hand-side of Figure 7 we use superscript ^{1} to replace $'$ in A'. Thus, the elasticity of the weights function f is increasing if and only if Slope of $\overline{A_1^1 A_2^1}$ > Slope of $\overline{A_3^1 A_4^1}$. In this case, it is possible that $A_1 \succ A_2$ and $A_4 \succ A_3$ are compatible.

[2] The elasticity of the weights function f is decreasing if and only if the elasticity of the weights function h is increasing if and only if $\dfrac{h(\lambda p)}{h(p)} < \dfrac{h(\lambda q)}{h(q)}$ if and only if $\dfrac{h(\lambda p)}{h(\lambda q)} < \dfrac{h(p)}{h(q)}$ if and only if Slope of $\overline{A_1' A_2'}$ < Slope of $\overline{A_3' A_4'}$. In the right-hand-side of Figure 7 we use superscript ^{2} to replace $'$ in A'. Thus, the elasticity of the weights function f is decreasing if and only if Slope of $\overline{A_1^2 A_2^2}$ < Slope of $\overline{A_3^2 A_4^2}$. In this case, $A_1 \succ A_2$ and $A_3 \succ A_4$ are compatible.

We summarize the above results from the view of DIA as follows.

1. The weights function f satisfies $f(p) = p^\gamma$ where $\gamma > 0$ if and only if $\overline{A_1' A_2'} \parallel \overline{A_3' A_4'}$. In this case both $A_1 \succ A_2$ and $A_3 \succ A_4$ hold together.

2. The weights function f is of increasing elasticity if and only if Slope of $\overline{A_1' A_2'}$ > Slope of $\overline{A_3' A_4'}$, and it is possible that $A_1 \succ A_2$ and $A_4 \succ A_3$ are compatible.

3. The weights function f is of decreasing elasticity if and only if Slope of $\overline{A_1' A_2'}$ < Slope of $\overline{A_3' A_4'}$, then $A_1 \succ A_2$ and $A_3 \succ A_4$ are compatible.

Machina (1987) uses the unit triangle to explain the Allais paradox and the common ratio effect. In his diagrams, prospects are fixed and form parallelograms, but indifference curves fan out. In this paper, we examine the examples from a different angle. We transform the unit triangle under the framework of DIA, as in Figures 4 and 7. Here, indifference curves keep parallel but lines of the compared prospect pairs are fanning in in the following two cases — [1] the distortion function g is concave for the Allais paradox and [2] the weights function f is of increasing elasticity for the common ratio effect. Then the behavioral patterns in these paradoxes may be rational.

5. CONCLUSIONS AND REMARKS

In this paper, we build the distorted theory under risk, which is also called the RDEU theory (Quiggin 1982, Segal 1989, and Quiggin and Wakker 1994). We first provide a brief outline of Quiggin's (1982) RDEU formula. Following Quiggin's (1982) assumption that the transformation of CDFs is a continuous function of the whole probability distribution, we show, by induction to reduce the dimensions of lotteries, that the RDEU formula holds over lotteries. Besides, this formula can also be proved for general random variables. The rank-dependent expected utility on risks is represented by mathematical expectations of a utility function with respect to a transformation of probabilities on the set of outcomes.

Then we axiomatize the distorted theory of rank-dependent expected utility. We notice that the function which distorts decision-makers' beliefs is also a CDF. It follows that the distorted CDF is the compound of the distortion function and the CDF of a risky prospect. From this approach we construct our DIA, and hence provide the representation theorem of RDEU by modifying EU (Fishburn 1982 and Yaari 1987).

The RDEU formula can be used to explain the Allais paradox and the common ratio effect (Segal 1987a, 1989), and even the Ellsberg paradox (Segal 1987b). While the Allais paradox and the common ratio effect are inconsistent with EU, they may accord with RDEU. As we know, IA is a special case of DIA. Consequently, the paradoxes under IA may constitute rational behaviors under our DIA. In this paper, we transform the unit triangle in Machina (1987) to the framework of DIA, in which the indifference curves remain parallel but the positions of prospects change. We show that when the distortion function take specific forms, lines of compared prospect pairs fan in and thus the behavioral pattern in these examples may be rational.

APPENDIX A

[Proof of Theorem 1]: We prove the expressions (2)-(3) by induction. It is very easy to check the cases of $N = 1, 2, 3$. Supposing the result holds for N, we want to prove that it also holds for $N + 1$.

For the case of $N+1$, $X^{N+1} = (x_1^{N+1}, p_1^{N+1}; \cdots ; x_{N+1}^{N+1}, p_{N+1}^{N+1})$ with $\sum_{n=1}^{N+1} p_n^{N+1} = 1$. Then

$$RDEU(X^{N+1}) = \sum_{n=1}^{N+1} H_n^{N+1}(p_1^{N+1}, \cdots, p_{N+1}^{N+1}) U(x_n^{N+1})$$

and $\sum_{n=1}^{N+1} H_n^{N+1}(p_1^{N+1}, \cdots, p_{N+1}^{N+1}) = 1$.

As x_1^{N+1} approaches x_2^{N+1}, $U(x_1^{N+1})$ approaches $U(x_2^{N+1})$. Then $RDEU(X^{N+1})$ approaches $RDEU(X^N)$ when $x_2^{N+1} = x_1^N$, $x_n^{N+1} = x_{n-1}^N$ for $n = 3, \cdots, N+1$, and $p_n^{N+1} = p_{n-1}^N$ for $n = 3, \cdots, N+1$ (in this case, $p_1^{N+1} + p_2^{N+1} = p_1^N$). In the limit, $x_1^{N+1} = x_2^{N+1}$, $X^{N+1} = X^N$ and $RDEU(X^{N+1}) = RDEU(X^N)$. Thus we have

$$\begin{aligned} & \sum_{n=1}^{N} H_n^N(p_1^N, \cdots, p_N^N) U(x_n^N) \\ = \quad & [H_1^{N+1}(p_1^{N+1}, p_2^{N+1}, p_2^N, \cdots, p_N^N) + H_2^{N+1}(p_1^{N+1}, p_2^{N+1}, p_2^N, \cdots, p_N^N)] U(x_1^N) \\ & + \sum_{n=3}^{N+1} H_n^{N+1}(p_1^{N+1}, p_2^{N+1}, p_2^N, \cdots, p_N^N) U(x_{n-1}^N). \end{aligned}$$

Since $x_1^N, \cdots, x_N^N$ are chosen arbitrarily and $(H_1^{N+1}, \cdots, H_{N+1}^{N+1})$ is independent of $(x_1^N, \cdots, x_N^N)$, the coefficients on $U(x_1^N), \cdots, U(x_3^N)$ must be equal. Hence,

$$\begin{aligned} H_1^N(p_1^N, \cdots, p_N^N) &= H_1^{N+1}(p_1^{N+1}, p_2^{N+1}, p_2^N, \cdots, p_N^N) + H_2^{N+1}(p_1^{N+1}, p_2^{N+1}, p_2^N, \cdots, p_N^N) \\ H_n^N(p_1^N, \cdots, p_N^N) &= H_{n+1}^{N+1}(p_1^{N+1}, p_2^{N+1}, p_2^N, \cdots, p_N^N) \quad for \quad n = 2, \cdots, N. \end{aligned} \tag{A.1}$$

From (1) and (A.1) we have, for $n = 2, \cdots, N$,

$$\begin{aligned} & H_{n+1}^{N+1}(p_1^{N+1}, \cdots, p_{N+1}^{N+1}) = H_n^N(p_1^N, \cdots, p_N^N) \\ = \ & g\left(\sum_{n'=1}^{n} p_{n'}^N\right) - g\left(\sum_{n'=1}^{n-1} p_{n'}^N\right) = g\left(\sum_{n'=1}^{n+1} p_{n'}^{N+1}\right) - g\left(\sum_{n'=1}^{n} p_{n'}^{N+1}\right). \end{aligned} \tag{A.2}$$

We next consider the cases of $k = 2, \cdots, N-1$. As x_k^{N+1} approaches x_{k+1}^{N+1}, $U(x_k^{N+1})$ approaches $U(x_{k+1}^{N+1})$. Then $RDEU(X^{N+1})$ approaches $RDEU(X^N)$ when $x_n^{N+1} = x_n^N$ for $n = 1, \cdots, k-1$, $x_{k+1}^{N+1} = x_k^N$, $x_n^{N+1} = x_{n-1}^N$ for $n = k+2, \cdots, N+1$, $p_n^{N+1} = p_n^N$ for $n = 1, \cdots, k-1$, and

$p_n^{N+1} = p_{n-1}^N$ for $n = k+2, \cdots, N+1$ (in this case, $p_k^{N+1} + p_{k+1}^{N+1} = p_k^N$). In the limit, $x_k^{N+1} = x_{k+1}^{N+1}$, $X^{N+1} = X^N$ and $RDEU(X^{N+1}) = RDEU(X^N)$, so that

$$\begin{aligned}
&\sum_{n=1}^{N} H_n^N(p_1^N, \cdots, p_N^N) U(x_n^N) \\
= &\sum_{n=1}^{k-1} H_n^{N+1}(p_1^N, \cdots, p_{k-1}^N, p_k^{N+1}, p_{k+1}^{N+1}, p_{k+1}^N, \cdots, p_N^N) U(x_n^N) \\
&+[H_k^{N+1}(p_1^N, \cdots, p_{k-1}^N, p_k^{N+1}, p_{k+1}^{N+1}, p_{k+1}^N, \cdots, p_N^N) \\
&\qquad +H_{k+1}^{N+1}(p_1^N, \cdots, p_{k-1}^N, p_k^{N+1}, p_{k+1}^{N+1}, p_{k+1}^N, \cdots, p_N^N)] U(x_k^N) \\
&+\sum_{n=k+2}^{N+1} H_n^{N+1}(p_1^N, \cdots, p_{k-1}^N, p_k^{N+1}, p_{k+1}^{N+1}, p_{k+1}^N, \cdots, p_N^N) U(x_{n-1}^N).
\end{aligned}$$

Then

$$\begin{aligned}
H_n^N(p_1^N, \cdots, p_N^N) &= H_n^{N+1}(p_1^N, \cdots, p_{k-1}^N, p_k^{N+1}, p_{k+1}^{N+1}, p_{k+1}^N, \cdots, p_N^N), \quad \text{(A.3)} \\
&\quad n = 1, \cdots, k-1 \\
H_k^N(p_1^N, \cdots, p_N^N) &= H_k^{N+1}(p_1^N, \cdots, p_{k-1}^N, p_k^{N+1}, p_{k+1}^{N+1}, p_{k+1}^N, \cdots, p_N^N) \\
&\quad +H_{k+1}^{N+1}(p_1^N, \cdots, p_{k-1}^N, p_k^{N+1}, p_{k+1}^{N+1}, p_{k+1}^N, \cdots, p_N^N) \\
H_n^N(p_1^N, \cdots, p_N^N) &= H_{n+1}^{N+1}(p_1^N, \cdots, p_{k-1}^N, p_k^{N+1}, p_{k+1}^{N+1}, p_{k+1}^N, \cdots, p_N^N), \quad \text{(A.4)} \\
&\quad n = k+1, \cdots, N
\end{aligned}$$

From (2) and (A.3) we have

$$H_1^{N+1}(p_1^{N+1}, \cdots, p_{N+1}^{N+1}) = H_1^N(p_1^N, \cdots, p_N^N) = g(p_1^N) = g(p_1^{N+1}). \quad \text{(A.5)}$$

From (3) and (A.3) we have, for $n = 2, \cdots, k-1$,

$$\begin{aligned}
&H_n^{N+1}(p_1^{N+1}, \cdots, p_{N+1}^{N+1}) = H_n^N(p_1^N, \cdots, p_N^N) \\
= &\; g\left(\sum_{n'=1}^{n} p_{n'}^N\right) - g\left(\sum_{n'=1}^{n-1} p_{n'}^N\right) = g\left(\sum_{n'=1}^{n} p_{n'}^{N+1}\right) - g\left(\sum_{n'=1}^{n-1} p_{n'}^{N+1}\right). \quad \text{(A.6)}
\end{aligned}$$

From (3) and (A.4) we have, for $n = k+1, \cdots, N$,

$$\begin{aligned}
&H_{n+1}^{N+1}(p_1^{N+1}, \cdots, p_{N+1}^{N+1}) = H_n^N(p_1^N, \cdots, p_N^N) \\
= &\; g\left(\sum_{n'=1}^{n} p_{n'}^N\right) - g\left(\sum_{n'=1}^{n-1} p_{n'}^N\right) = g\left(\sum_{n'=1}^{n+1} p_{n'}^{N+1}\right) - g\left(\sum_{n'=1}^{n} p_{n'}^{N+1}\right). \quad \text{(A.7)}
\end{aligned}$$

As x_N^{N+1} approaches x_{N+1}^{N+1}, $U(x_N^{N+1})$ approaches $U(x_{N+1}^{N+1})$. Then $RDEU(X^{N+1})$ approaches $RDEU(X^N)$ when $x_n^{N+1} = x_n^N$ for $n = 1, \cdots, N-1$, $x_{N+1}^{N+1} =$

x_N^N, and $p_n^{N+1} = p_n^N$ for $n = 1, \cdots, N-1$ (in this case, $p_N^{N+1} + p_{N+1}^{N+1} = p_N^N$). In the limit, $x_N^{N+1} = x_{N+1}^{N+1}$, $X^{N+1} = X^N$ and $RDEU(X^{N+1}) = RDEU(X^N)$, so that

$$\sum_{n=1}^{N} H_n^N(p_1^N, \cdots, p_N^N)U(x_n^N) = \sum_{n=1}^{N-1} H_n^{N+1}(p_1^N, \cdots, p_{N-1}^N, p_N^{N+1}, p_{N+1}^{N+1})U(x_n^N)$$
$$+[H_N^{N+1}(p_1^N, \cdots, p_{N-1}^N, p_N^{N+1}, p_{N+1}^{N+1}) + H_{N+1}^{N+1}(p_1^N, \cdots, p_{N-1}^N, p_N^{N+1}, p_{N+1}^{N+1})]U(x_N^N).$$

Then

$$\begin{aligned} H_n^N(p_1^N, \cdots, p_N^N) &= H_n^{N+1}(p_1^N, \cdots, p_{N-1}^N, p_N^{N+1}, p_{N+1}^{N+1}) \qquad \text{(A.8)} \\ &\quad \text{for} \quad 1 = 2, \cdots, N-1 \\ H_N^N(p_1^N, \cdots, p_N^N) &= H_N^{N+1}(p_1^N, \cdots, p_{N-1}^N, p_N^{N+1}, p_{N+1}^{N+1}) \\ &+ H_{N+1}^{N+1}(p_1^N, \cdots, p_{N-1}^N, p_N^{N+1}, p_{N+1}^{N+1}) \end{aligned}$$

From (2) and (A.8) we have

$$H_1^{N+1}(p_1^{N+1}, \cdots, p_{N+1}^{N+1}) = H_1^N(p_1^N, \cdots, p_N^N) = g(p_1^N) = g(p_1^{N+1}). \qquad \text{(A.9)}$$

From (3) and (A.8) we have, for $n = 2, \cdots, N-1$,

$$\begin{aligned} &H_n^{N+1}(p_1^{N+1}, \cdots, p_{N+1}^{N+1}) = H_n^N(p_1^N, \cdots, p_N^N) \qquad \text{(A.10)} \\ &= g\left(\sum_{n'=1}^{n} p_{n'}^N\right) - g\left(\sum_{n'=1}^{n-1} p_{n'}^N\right) = g\left(\sum_{n'=1}^{n} p_{n'}^{N+1}\right) - g\left(\sum_{n'=1}^{n-1} p_{n'}^{N+1}\right). \end{aligned}$$

Summarizing (A.2), (A.5), (A.6), (A.7), (A.9) and (A.10), we have

$$\begin{aligned} H_1^{N+1}(p_1^{N+1}, \cdots, p_{N+1}^{N+1}) &= g(p_1^{N+1}) \\ H_n^{N+1}(p_1^{N+1}, \cdots, p_{N+1}^{N+1}) &= g\left(\sum_{n'=1}^{n} p_{n'}^{N+1}\right) - g\left(\sum_{n'=1}^{n-1} p_{n'}^{N+1}\right), \text{ for } n = 2, \cdots, N+1. \end{aligned}$$

[**Proof of Theorem 2**]: For $X \in \mathscr{L}$, its CDF is F_X on $[m, M]$. Then F_X can be produced from the limit of a non-decreasing sequence of CDFs of discrete random variables. For any natural number $N = 1, 2, \cdots$ we define a function $F_N : [m, M] \to [0, 1]$ as

$$F_N(x) = \begin{cases} F_X(x_k), & if \;\; x \in [x_k, x_{k+1}) \qquad \text{for} \qquad k = 0, 1, \cdots, 2^N - 1 \\ 1, & if \;\; x = M \end{cases}$$

where $x_k = m + \dfrac{k}{2^N}(M-m)$ for $k = 0, 1, \cdots, 2^N$ (hence $x_0 = m$ and $x_{2^N} = M$). That is to say, the sequence $\{x_k : k = 1, \cdots, 2^N - 1\}$ divides the interval $[m, M)$ into 2^N equal-length small intervals $[x_k, x_{k+1})$ for $k =$

$0, 1, \cdots, 2^N-2$ and $[x_{2^N-1}, x_{2^N}]$. Then $m = x_0 < x_1 < \cdots < x_{2^N-1} < x_{2^N} = M$ and $[m, M) = \sum_{k=0}^{2^N-2}[x_k, x_{k+1}) + [x_{2^N-1}, x_{2^N}]$ where the sum of sets means union of the disjoint sets. Thus F_N is the CDF of the discrete random variable $X^{2^N+1} = (x_0, F_X(x_0); x_1, F_X(x_1)-F_X(x_0); \cdots ; x_{2^N-1}, F_X(x_{2^N-1})-F_X(x_{2^N-2}); x_{2^N}, F_X(x_{2^N})-F_X(x_{2^N-1}))$. By construction,

$$\lim_{N\to\infty} F_N(x) = F_X(x) \qquad \text{for} \qquad x \in \{x_k : k = 0, 1, \cdots, 2^N \ \textit{and}\ N = 1, 2, \cdots\}.$$

The set $\{x_k : k = 0, 1, \cdots, 2^N$ and $N = 1, 2, \cdots\}$ is dense in $[m, M]$, then

$$\lim_{N\to\infty} F_N(x) = F_X(x) \qquad \text{for} \qquad x \in [m, M].$$

That is, the non-decreasing sequence of CDFs $\{F_N : N = 1, 2, \cdots\}$ converges to F_X. It follows that, from the continuity of function g,

$$\lim_{N\to\infty} g(F_N(x)) = g(F_X(x)) \qquad \text{for} \qquad x \in [m, M].$$

That is, the non-decreasing sequence of CDFs $\{g\circ F_N : N = 1, 2, \cdots\}$ converges to $g\circ F_X$.

Since U is a continuous and increasing von Neumann - Morgenstern utility function on $[m, M]$, then we have, by Helly Theorem in Chow and Ticher (1988),

$$\lim_{N\to\infty} \int_{[m,M]} U(x)dg(F_N(x)) = \int_{[m,M]} U(x)dg(F_X(x)).$$

[Proof of Proposition 2]: The CDF of the random variable $F_X(X)$ is, for $p \in [0, 1]$,

$$P\{F_X(X) \le p\} = P\{X \le F_X^{-1}(p)\} = F_X(F_X^{-1}(p)) = p.$$

Therefore $F_X(X)$ follows the uniform distribution on $[0, 1]$.

If the random variable θ follows the uniform distribution on $[0, 1]$, for any CDF F,

$$P\{F^{-1}(\theta) \le x\} = P\{\theta \le F(x)\} = F(x).$$

Therefore $F^{-1}(\theta)$ follows the CDF F.

[Proof of Theorem 4]: Define a binary relation $\succ^*$ on the family $\mathscr{F}$ of CDFs as follows:

$$F_1 \succ^* F_2 \qquad \text{if and only if} \qquad g \circ F_1 \succ g \circ F_2$$

for all F_1 and F_2 in $\mathscr{F}$. Clearly, if X_1 and X_2 are random variables in $\mathscr{L}$, then

$$X_1 \succ^* X_2 \qquad \text{if and only if} \qquad g \circ F_{X_1} \succ g \circ F_{X_2}.$$

Checking Axioms A1 - A4, we find that they hold for $\succsim^*$ on $\mathscr{F}$ if and only if they hold for $\succsim$ on $\mathscr{F}^\circ$. Axioms A1, A2, and A4 are straightforward. Now we check Axiom A3 as follows.

Let $g\circ F$ and $g\circ F'$ belong to $\mathscr{F}^\circ$. Since g satisfies Lipschitz condition, then there exists a positive number $K_1 > 0$ such that, for p_1 and p_2 in $[0,1]$, $|g(p_1) - g(p_2)| \leq K_1|p_1 - p_2|$.

$$\|g\circ F - g\circ F'\| = \int_{[m,M]} |[g\circ F](x) - [g\circ F'](x)|dx = \int_{[m,M]} |g(F(x)) - g(F'(x))|dx$$
$$\leq \int_{[m,M]} K_1|F(x) - F'(x)|dx = K_1 \int_{[m,M]} |F(x) - F'(x)|dx = K_1\|F - F'\|.$$

Let F and F' belong to $\mathscr{F}$. Since g^{-1} satisfies Lipschitz condition, then there exists a positive number $K_2 > 0$ such that, for p_1 and p_2 in $[0,1]$, $|g^{-1}(p_1) - g^{-1}(p_2)| \leq K_2|p_1 - p_2|$.

$$\|F - F'\| = \int_{[m,M]} |F(x) - F'(x)|dx = \int_{[m,M]} |g^{-1}(g(F(x))) - g^{-1}(g(F'(x)))|dx$$
$$\leq \int_{[m,M]} K_2|g(F(x)) - g(F'(x))|dx = K_2 \int_{[m,M]} |[g\circ F](x) - [g\circ F'](x)|dx$$
$$= K_2\|g\circ F - g\circ F'\|.$$

Then the L_1-norms are equivalent on $\mathscr{F}$ and $\mathscr{F}^\circ$.

Furthermore, $\succ^*$ satisfies Axiom A5EU if and only if $\succ$ satisfies Axiom A5.

If F_1, F_2 and F belong to $\mathscr{F}$ and α is a real number satisfying $0 < \alpha < 1$, and $F_1 \succ^* F_2$, then $g\circ F_1 \succ g\circ F_2$. Since $g\circ F_1$, $g\circ F_2$ and $g\circ F$ belong to $\mathscr{F}^\circ$, then, by DIA A5, $\alpha[g\circ F_1]\oplus(1-\alpha)[g\circ F] \succ \alpha[g\circ F_2]\oplus(1-\alpha)[g\circ F]$. That is, $g\circ[\alpha F_1 + (1-\alpha)F] \succ g\circ[\alpha F_2 + (1-\alpha)F]$ from the definition of mixture operation. Hence $\alpha F_1 + (1-\alpha)F \succ^* \alpha F_2 + (1-\alpha)F$.

Conversely, if $g\circ F_1$, $g\circ F_2$ and $g\circ F$ belong to $\mathscr{F}^\circ$ and α is a real number satisfying $0 < \alpha < 1$, and $g\circ F_1 \succ g\circ F_2$, then $F_1 \succ^* F_2$. It follows that, by IA A5EU, $\alpha F_1 + (1-\alpha)F \succ^* \alpha F_2 + (1-\alpha)F$. That is, $g\circ[\alpha F_1 + (1-\alpha)F] \succ g\circ[\alpha F_2 + (1-\alpha)F]$. Thus $\alpha[g\circ F_1] \oplus (1-\alpha)[g\circ F] \succ \alpha[g\circ F_2] \oplus (1-\alpha)[g\circ F]$ from the definition of mixture operation.

Hence, from Theorem 3, it follows that $\succ$ satisfies Axioms A1 - A5 if and only if $\succ^*$ has the appropriate expected utility representation. In other words, $\succ$ satisfies Axioms A1 - A5 if and only if there exists a continuous and non-decreasing real function u, defined on $[m, M]$, such that, for all X_1 and X_2 belonging to $\mathscr{L}$,

$$X_1 \succ X_2 \qquad \Longleftrightarrow \qquad E[u(F_{X_1}^{-1}(\xi))] > E[u(F_{X_2}^{-1}(\xi))].$$

Let H be any member of $\mathscr{F}$. Then, the equation

$$\begin{aligned} E[u(F^{-1}(\xi))] &= \int_{[m,M]} u(x)dF_{F^{-1}(\xi)}(x) \\ &= \int_{[m,M]} u(x)d[g\circ F](x) = \int_{[m,M]} u(x)dg(F(x)) \end{aligned}$$

holds, and this proves the first part of the theorem.

Now, applying the second part of Theorem 3 to $\succ^*$, we find that u can be selected so as to satisfy the preference equation

$$(m, 1-u(x); M, u(x)) \sim^* (x, 1)$$

for $m \leq x \leq M$, which produces (7). This completes the proof of the theorem.

REFERENCES

Allias, M., 1953. 'Le Comportement de l'Homme Rationnel Devant le Risque: Critique des Postulats et Axiomes de l'Ecole Americaine. *Econometrica* **21**, 503-546.

Chew, S. H., 1985. An Axiomatization of Rank Dependent Quasi-Linear Mean Generalizing the Gini Mean and the Quasi-Linear Mean. Economics Working paper No. 156, Johns Hopkins University.

Chew, S. H., E. Karni, and Z. Safra, 1987. Risk Aversion in the Theory of Expected Utility with Rank Dependent Probabilities. *Journal of Economic Theory* **42**, 370-381.

Chow, Y. S. and H. Ticher, 1988. *Probability Theory: Independence, Interchangeability, Martingales.* Springer-Verlag.

Fishburn, P. C., 1982. *The Foundations of Expected Utility.* Dordrecht: Reidel.

Handa, J., 1977. Risk, probabilities and A New Theory of Cardinal Utility. *Journal of Political Economy* **85**, 97-122.

Kahneman, D. and A. Tversky, 1979. Prospect Theory: An Analysis of decision under Risk. *Econometrica* **47**, 263-291.

Karmarkar, U., 1979. Subjectively Weighted Utility and the Allais paradox. *Organisational Behavior and Human Performance* **24**, 67-72.

Karni, E., 1987. Generalized Expected Utility Analysis of Risk Aversion with State-dependent Preferences. *International Economic Review* **28**, 229-240.

Karni, E. and D. Schmeidler, 1991. Utility Theory with Uncertainty. In *Handbook of Mathematical Economics.* Volume 4, Chapter 33, Page 1763-1831 (Edited by W. Hildenbrand and H. Sonnenschein). Elsevier Science Publishers B.V., 1991.

MacCrimmon, K. R. and S. Larsson, 1979. Utility Theory: Axioms versus "Paradoxes". in: M. Allais and O. Hagen, eds., *Expected Utility Hypothesis and the Allais Paradox.* Dordrecht: Reidel.

Machina, M. J., 1982 "Expected Utility" Analysis Without the Independence Axiom. *Econometrica* **50**, 277-323.

Machina, M. J., 1987. Choice Under Uncertainty: Problems Solved and unsolved. *Journal of Economic Perspectives* **1**, 121-154.

Miao, J., 2004. A Note on Consumption and Savings under Knightian Uncertainty. *Annals of Economics and Finance* **5**, 299-311.

Miao, J., 2009. Ambiguity, Risk and Portfolio Choice under Incomplete Information. *Annals of Economics and Finance* **10**, 257-279.

Quiggin, J., 1982. A Theory of Anticipated Utility. *Journal of Economic Behavior and Organization* **3**, 323-343.

Quiggin, J., 1991. Comparative Statics for Rank-Dependent Expected Utility Theory. *Journal of Risk and Uncertainty* **4**, 339-350.

Quiggin, J. and P. Wakker, 1994. The Axiomatic Basis of Anticipated Utility: A Clarification. *Journal of Economic Theory* **64**, 486-499.

Röell, A., 1987. Risk Aversion in Quiggin and Yaari's Rank-Order Model of Choice under Uncertainty. *Economic Journal* **97** (Supplement), 143-159.

Segal, U., 1987. Some remarks on Quiggin's Anticipated utility. *Journal of Economic Behavior and Organization* **8**, 145-54.

Segal, U., 1987. The Ellsberg Paradox and Risk Aversion: An Anticipated Utility Approach. *International Economic Review* **28**, 175-202.

Segal, U., 1989. Anticipated Utility: A Measure Representation Approach. *Annals of Operations Research* **19**, 359-373.

Wakker, P., 1993. Counterexamples to Segal's Measure Representation Theorem. *Journal of Risk and Uncertainty* **6**, 91-98.

Yaari, M. E., 1987. The Dual Theory of Choice under Risk. *Econometrica* **55**, 95-116.

Zou, L., 2006. An Alternative to Prospect Theory. *Annals of Economics and Finance* **7**, 1-27.