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Bank money, aggregate liquidity, and asset prices

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We build a general equilibrium model to analyze how the ability of banks to create money can affect asset prices and financial stability. In the model, demand for liquidity takes the form of demand for money to make payments. We show that banks can provide elastic aggregate liquidity by creating and lending out deposits, which will reduce the need for people to sell assets and help maintain asset price stability. We also compare two types of liquidity provision mechanisms. The first is liquidity-risk-sharing through a Diamond-and-Dybvig style coalition that pools together people's resources, and the second is liquidity provision by banks though money creation. We show that without elastic aggregate liquidity provided by banks, coalitions can not actually perform their risk-sharing function, their attempt to sell assets to raise liquidity will only make asset prices decrease further, without actually raising more liquidity for shareholders hit by liquidity shocks. However, with banks providing elastic aggregate liquidity, people can indeed achieve better risk-sharing though coalitions. Finally, we show that the central bank can help banks provide liquidity to the market by lending to banks at low interest rates during the inter-bank settlement process, so as to relax the liquidity constraint of banks.

Key Words: Liquidity; Asset prices; Banking; Inside money; Monetary policy; Payment system.
JEL Classification Numbers: E4, E5, G12, G21.

1. INTRODUCTION

This paper shows why the ability of banks to create money is important to financial stability. We show that banks can provide elastic liquidity to the financial system through money creation, which can help stabilize asset

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prices. The ability of banks to expand aggregate liquidity is essential when people try to create mutual-fund-like coalitions to share liquidity risks.

The main motivation of this paper is to extend the recent literature of "liquidity and asset prices" to include money creation by banks. The studies in "liquidity and asset prices" focus on how the limited ability of the market to absorb sales of assets may cause assets to be sold at low prices. For example, in a series of papers[1], Allen and Gale argue that if the amount of cash that buyers can use to buy assets is limited, then when people sell assets, the market price can deviate from the fundamental value. This is the so called "cash-in-the-market-pricing" (Allen and Gale (2005)).

However, most of the current works in liquidity and asset prices are based on non-monetary models (usually the Diamond and Dybvig (1983) framework), and "liquidity" or "cash" is modeled as real consumption goods. For example, in a basic Diamond-and-Dybvig style model, there are three periods, 0,1, and 2. Every agent is endowed with 1 unit of goods in period 0. The goods can be stored as consumption goods with zero return or be invested in long-term projects that will yield a higher return in period 2. Long-term investments can be liquidated into consumption goods in period 1, but with a high liquidation cost. In addition, the timing of consumption of every agent is uncertain. Part of the agents will turn out to be impatient and must consume in period 1, and the remaining agents only consume in period 2. To insure the risks, people can pool their resources into a coalition called as "bank". The bank will store part of the goods as "liquidity", which will be provided to impatient consumers in period 1, and invest the remaining goods in long-term projects. It can be shown that under this arrangement, the expected utility of agents will be higher than under autarky or when agents trade assets on the market.

In the above model, "liquidity" and "cash" are modeled as real short-term consumption goods. However, in reality, the need for liquidity usually means the need to get money to make payments. In addition, banks do not collect and lend out real goods, and what people deposit into and withdraw from banks is money. To be fair, although the Diamond and Dybvig model is a non-monetary model, its purpose is to capture the liquidity problem of an individual bank, and the model serves this purpose quite well. After all, withdrawing money or withdrawing consumption goods create a similar liquidity shortage problem for an individual bank.

However, non-monetary models may be impropriate for modeling the aggregate liquidity problems of the financial system. First, suppose we only include central bank outside money (i.e., currency) in liquidity. If people withdraw currency from banks and use it to buy consumption goods, the currency does not disappear, it is merely transferred to the seller of the

[1] Allen and Gale (1994, 1998, 2000a, 2000b, 2002, 2004a, 2004b, 2005)

goods, which means that consumption does not reduce aggregate liquidity. This is quite different from the case where liquidity is modeled as consumption goods, because in the latter case, liquidity (consumption goods) disappears after people consume it. Second, more generally, if we define money as liquidity, then liquidity is not limited to currency, but should also include inside money (bank deposits) created by banks. As we know, in most modern economy, money created by banks is more than the money issued by the central bank. Since banks can provide liquidity by creating and lending out money, the aggregate liquidity is no longer fixed by the real consumption goods available in the economy. As we will show, this implies that liquidity will be more elastic and asset prices will be more stable. As a result, previous non-monetary models may underestimate the ability of the financial system to accommodate liquidity shocks.[2]

The reason that in this paper we focus on private banks instead of the central bank in providing elastic money is that we want to emphasize the fact that the private market system has a built-in function for providing elastic liquidity, and government agencies such as the central bank is not the only way for providing elastic liquidity. In addition, the relationship between elastic liquidity provided by private banks and asset prices is not well-analyzed.

In the paper, we compare two types of liquidity provision functions of financial intermediaries. In order to isolate the liquidity-risk-sharing function modeled in the Diamond-Dybvig framework and the liquidity provision function by banks through money creation, we include two types of financial intermediaries into the model. We use "investment funds" to capture the first type of function, they are essentially the same as the "banks" in the Diamond-Dybvig style model. But as we will show, the risk-sharing function does not really need to be carried out by a "bank". We also include banks. The difference between banks and non-banks is that debt issued by banks (i.e., bank deposits) are used as means of payment, this makes it possible for banks to lend by creating their own debt: bank deposits. And we model liquidity needs as the need to get money to make payments.

We show that elastic aggregate liquidity provided by banks is important to asset price stability, and is also essential for non-banks to perform their risk-sharing function. In the model, agents can redeem their investment fund shares when they need money to buy goods. Investment funds can raise liquidity by selling assets to people who have idle deposits, or by borrowing from banks. When there is no bank lending, the liquidity that investment funds can raise by selling assets is limited to the aggregate

[2]Gale (2005) extends the framework to include outside money. He shows that if there is cash-in-advance constraint in the financial market, then sales of assets can lead to low asset price. Still, in his model, private institutions cannot supply elastic aggregate liquidity.

money held by people who are not hit by the liquidity shock. We show that higher degree of risk-sharing will cause investment funds to sell more assets, so as to raise more cash for shareholders who are hit by the liquidity shock. However, this action is self-defeating. When aggregate liquidity is limited, selling more assets will only cause the asset price to decrease further, without actually raising more liquidity. As a result, with inelastic aggregate liquidity, the risk-sharing function of coalitions is useless. This function will actually make asset prices more unstable. However, when banks can supply elastic liquidity, people can indeed get better risk-sharing though non-bank coalitions.

Our paper is related to Freeman (1996a,b), who discussed how central banks and private banks can provide liquidity by issuing bank notes. In his model, buyers buy goods from sellers by issuing personal debt, buyers and sellers then go to a centralized location to settle the debt in fiat money. Sellers may have to leave before buyers arrive at the location, so sellers may have to sell their debt at low prices. Freeman showed that the social welfare can be improved if clearing house banks or central banks issue bank notes to buy the debt and then collect the debt when buyers arrive. This will also help stabilize the asset price.

Our paper can be seen as an extension of Freeman's idea. However, there are four differences between our paper and Freeman's papers. First, the function of banks in our model is no longer limited to discounting existing private debt, which is an function that is somewhat unusual to modern banks today. Instead, we model functions of banks that we see everyday: lend money to borrowers, who use the money to make payments, and pay back money later. Second, we compare the liquidity provision functions of non-banks and banks, and examine their relationship. Third, the bank in Freeman's model is more like a central bank and there is no liquidity constraint when it issues bank notes. In our model, banks are subject to liquidity constraints when they make loans. We build a model of inter-bank-settlement to analyze the liquidity constraint of banks and its effect on bank deposit rate, lending rate and asset prices. Fourth, we model how the central bank can help banks to provide liquidity to the financial market by lending to banks during the settlement process, and how the central bank's interest rate policy is transmitted through the banking system to asset prices. [3]

[3]Our paper is also related to the literature of inside money and payment system. Some recent examples are Andolfatto and Nosal (2001), Bullard and Smith (2003), Kiyotaki and Moore (2000), Lester (2005), Williamson (1999). Head and Qiu (2007) analyze the effects of bank money creation on the optimal inflation rate when there is a zero bound for nominal interest rates. However, the topic of our paper, the relationship between inside money and asset prices, is not well-analyzed in the current literature. Some other recent related papers in asset pricing are Gong, Smith and Zou (2007), Miao (2009) and Zhang (2010).

This paper is organized as follows. Section 2 describes the environment. Section 3 characterizes the equilibrium. Section 4 derives numerical results for an example with a log utility function. Section 5 derives results for general utility functions, and also shows how the ability of banks to create money is important for the risk-sharing function of non-bank mutual funds. Section 6 concludes. Proofs and additional results are included in the appendix.

2. THE ENVIRONMENT

2.1. The basic events

Consider an overlapping generations model with random relocation.[4] Time is indexed by $t = 1, 2, ...$ There are two locations in the economy. In each period, a new generation is born at each of the two locations. In each generation, there are three types of agents: "households", "investment fund managers" and "bankers". We normalize the measure of each type of agents to one. Each generation lives for two periods, and there is no population growth. The initial old generation of households in each location at time $t = 1$ is endowed with outside money M.

There is a single good per period. Agents care only about the consumption when they are old. Each household is endowed with e_h units of the good when young, and nothing when old. Young households thus save all their endowment. Households are risk-averse and they have the constant relative risk aversion(CRRA) utility function:

$$U(c) = \frac{c^{1-\sigma}}{1-\sigma} \qquad \sigma \geq 1 \tag{1}$$

Young investment fund managers can costlessly start new investment funds and young bankers can costlessly start new banks. There is free entry for investment funds and banks. Investment funds compete by offering the best contract to shareholders and banks compete by offering the best contract to depositors. In other words, investment funds and banks will select their portfolio and payout policy to maximize the expected utility of agents, subject to the condition that the expected profit is not negative.

Investment fund managers do not have endowments. And we assume away bankruptcy for banks using the following assumption: every banker has endowment e_b when old, which is sufficiently large to absorb the loss. Investment fund managers and bankers are risk neutral.

Consumption goods are non-storable but can be invested to produce new goods in the next period. Real risky investments can only be made by

[4]The basic random relocation setup follows Champ, Smith and Williamson (1996).

investment funds. The gross return rate for the risky project is

$$R_k = A \tag{2}$$

that is, one unit of consumption goods invested will turn into A units of goods in the next period. A is aggregate productivity. There are two values of A: A_H(high), A_L(low) with equal probability. A is i.i.d. in each period.

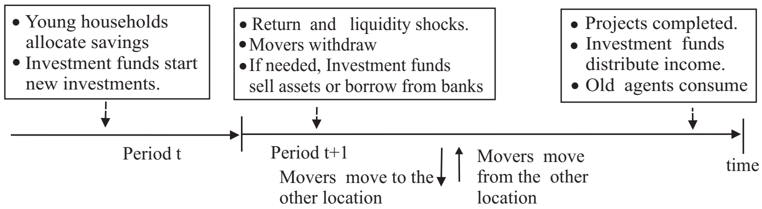


FIG. 1. Events

The main events are shown in Figure 1. The initial portfolio allocation is as follows. We assume that each young household can invest in at most one local investment fund and one local bank. Young households sell part of their endowment to the old generation to earn money balance $\frac{M}{P}$. They deposit money balance $a\frac{M}{P}(0 \leq a \leq 1)$ into banks, where a is chosen optimally. They then invest the remaining wealth, money balance $(1-a)\frac{M}{P}$ and real goods $e_h - \frac{M}{P}$, into investment funds. Investment funds optimally allocate their wealth over money balance and real investments. In the symmetric equilibrium, investment funds use the real goods to make investments and deposit their money into banks. If $1-a=0$, investment funds do not hold money, and if $1-a>0$, investment funds hold both real assets and bank deposits.

The actual number of households is higher than the number of investment funds and banks, and the actual number of investments is higher than the number of banks. This means each investment fund has many shareholders and each bank has many accounts of households and investment funds.

After real investments are made, we enter period $t+1$. At the beginning of period $t+1$, the productivity shock A is publicly observed. The liquidity shock is also realized. A random fraction π of the old households (denoted as "movers") must move to the other location and consume there. π is distributed over $[0, \overline{\pi}]$, where $\overline{\pi} < 1$ is the upper bound of the distribution. The distribution function is $F(\pi)$. π is symmetric in the two locations. π is independently and identically distributed, so each old household has the same *ex ante* probability to be a mover.

Movers cannot carry goods across locations. The value of bank deposits, however, can be verified across locations, so movers can use bank deposits to make payments. More specifically, when movers move from location i

to location j, they still keep their deposits in the banks in location i. And when they need to buy consumption goods in location j, they can pay using their deposits.

The value of other assets cannot be verified across locations. In particular, investment fund shares are not accepted as means of payment across locations. We assume that movers must have their deposits ready in their banking account when they move to the other location. As a result, movers must redeem their investment fund shares into bank deposits before move.

Investment funds optimally choose the payment to movers given each level of liquidity shock, so as to maximize the expected utility of the shareholders. The redemption process is as follows. After the shocks are realized, movers must send a withdrawal notice to their investment fund; Then the financial market opens. If the investment fund cannot meet the withdrawal needs with its own holdings of bank deposits, it can raise cash by selling assets to non-movers who have idle deposits. The transaction cost on the financial market is assumed to be zero. When assets are sold, only the ownership is transferred to the buyers, the production process is not stopped. The investment fund will collect the return and pay it to the owners of the assets at the end of the period. The fund can also choose to borrow from banks. The fund then pays movers by transferring bank deposits to them. Movers receive the payment only after the transactions on the financial market are completed, so movers can not use the cash they have just received to buy the risky assets that investment funds sell on the financial market.

After the redemption, movers move to the other location. At the end of $t+1$, risky projects are completed. Movers in each location use their bank deposits to buy consumption goods. Investment funds allocate returns to their shareholders and also repay the bank loan. Banks pay interest rates. Bankers consume the net income and old non-movers consume all their wealth.

There is no advantage in using currency in transactions, so we simply assume that agents and investment funds always use bank deposits to make payments. Although people are allowed to withdraw their deposits, in the equilibrium, no one will actually choose to withdraw central bank currency from banks.

Banks must settle inter-bank balances with central bank money. Banks keep their reserves in the central bank deposit account, and the deposit rate paid by the central bank is normalized to zero. There is no official reserve requirement and banks can freely choose the reserve level. We assume only banks can borrow from the central bank, and the central bank is not allowed to use money to buy private risky assets.

2.2. The environment for bank lending and settlement

2.2.1. The basic steps for bank lending

In our model, the borrowers are the investment funds. When they borrow loan L from banks at the beginning of $t+1$, banks will credit their deposit account by L; this will increase the outstanding bank deposit by L. Investment funds then use their money to meet the withdrawal of movers. At the end of period $t+1$, investment funds sell their goods and then use bank deposit $L(1+r^l)$ to repay the bank loan, where r^l is the lending rate. This will reduce the outstanding bank deposit by $L(1+r^l)$. Figure 2 illustrates the money flows. The details of money flows are shown in the appendix using the balance sheet of banks.

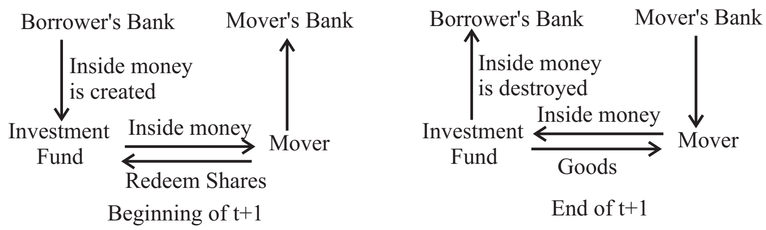


FIG. 2. The flow of inside money.

2.2.2. Bank lending and settlement

We focus on the symmetric case, and we assume that at the end of period t depositors and investment funds are equally distributed among the banks. In addition, the shareholders of each investment fund are equally distributed among all banks. There is no interest payment to depositors for holding deposits between the end of t and the beginning of $t+1$.

At the beginning of period $t+1$, the productivity shock and the liquidity shock are realized, and then banks announce their deposit rate and lending rate. Banks are competitive and they take the market lending rate as given, and offer the highest deposit rate r^d subject to the zero expected profit condition.

Each unit of bank loan incurs a management cost δ to the bank. Denote the net real lending rate as r^l, where $r^l \geq \delta$, and denote the gross lending rate as $R = 1 + r^l$.

An investment fund will borrow from the bank only when the borrowing cost is less than or equal to the cost for selling assets on the financial market. If the fund sells the asset, for each unit of asset with value R_k, the fund can get Q_k. If the fund borrows from the bank, for each unit of loan with future payment R_k, the fund can borrow $\frac{R_k}{1+r^l}$. Thus, when bank loans are needed, we must have

$$Q_k = \frac{R_k}{1 + r^l} \tag{3}$$

that is, investment funds will borrow from banks only when the market price decreases to $\frac{R_k}{1+r^l}$. Since $r^l \geq \delta$, so $R \geq 1 + \delta$ and we must have $Q_k \leq \frac{R_k}{1+\delta}$ when bank loans are needed.

There is no transaction cost for non-movers to purchase assets. When investment funds need additional money, they will sell their assets to non-movers first. As long as non-movers' cash is enough to absorb all the sales of assets, assets will be sold at their fundamental value($Q_k = R_k$). When non-movers' cash is not enough to absorb all the sales, then Q_k will be lower than R_k. Once Q_k decreases to $\frac{R_k}{1+\delta}$, investment funds will start to borrow from banks. We use π_1 to denote the liquidity shock π at which the cash of non-movers is binding, and we use π_2 to denote the π at which investment funds start to borrow from banks.

When $\pi \leq \pi_2$, there is no need for bank loans. The deposit rate will be zero because there is no income for banks. (We assume there is no cost for managing the deposits and allowing the depositors to use the payment facility). Since there is no bank loan, the level of deposits is the same as the level of reserves. This means all deposits are backed by reserves, so banks can never run out of reserves during the settlement process because the maximum outflow of payment is equal to the level of deposits.

When $\pi > \pi_2$, investment funds need to borrow loans in addition to selling assets on the financial market. The main steps for lending and settlement are as follows (see Figure 3).

Let D_0 denote the deposit and reserve balance for each bank at the beginning of $t + 1$. After the shocks, the financial market opens, and the investment funds sell assets to non-movers. Non-movers will use all their own money holdings to buy assets because when bank loans are needed, the return for using money to buy assets, $\frac{R_k}{Q_k}$, is equal to the lending rate $1 + r^l$, which is higher than the deposit rate. So after the transactions, non-movers transfer all their deposits to the investment funds. (This is "settlement 1" in Figure 3). Because all bank deposits are still backed by reserves, banks will not face any liquidity constraint in the settlement process. In the symmetric case, each fund sells the same amount of assets, and after "settlement 1", the deposit balance in each bank is still the same as the initial deposit D_0.

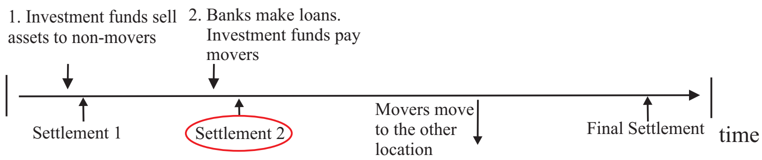


FIG. 3. The lending and settlement process

After all the payments by non-movers are completed, the financial market closes. We then enter step 2 in which banks make loans to the investment funds. Investment funds will borrow from the banks they have the account with as long as the lending rate is not higher than the lending rate by other banks. If an investment fund borrows from bank i, it keeps the newly borrowed money with bank i before making payments to movers.

When investment funds make payments to movers, we have another settlement, which is "settlement 2" in Figure 3. Since banks have created new deposits during lending, the payment may not be fully covered by initial reserves and banks may need to borrow from the central bank (the details will be explained later).

Movers can switch banks after receiving the payments; this will force banks to be competitive when offering deposit rates. In the symmetric case, all banks offer the same deposit rate and no mover will switch banks. As a result, in "settlement 2", we need only to consider the inter-bank payments caused by investment funds paying their movers. After the redemption process is completed, movers move to the other location.

We ignore the liquidity constraint for banks during the transactions at the end of $t+1$. We assume there is only one settlement based on net balance. In can be shown that after all transactions are completed, the deposits in each bank will be fully backed by reserves. This means the liquidity constraint will not be binding because the maximum payment by depositors is equal to the deposit level. As a result, we need only to consider the liquidity constraint in "settlement 2".

Inter-bank payments in "settlement 2" are settled according to the "Real-Time Gross Settlement" method. More specifically, we assume that there are N banks in the economy, with N being a very large number. The redemption process will be separated into N subperiods. We normalize the total time length of the redemption process to 1. The time length of each subperiod is $\frac{1}{N}$. In each subperiod, one of the banks that haven't make payments is randomly chosen to make payments to other banks, and the inter-bank balance is settled right away (i.e, the transfer of reserve happens right away). During this process, any negative balance of reserve must be met by borrowing from the central bank.

There is no inter-bank loan market, and banks can borrow from the central bank when they need extra reserves. We also assume that the central bank consumes the interest of its loan by purchasing consumption goods, so the outside money is always restored to M at the end of each period. The details of monetary payments and settlements are explained in the appendix.

3. THE EQUILIBRIUM

3.1. The portfolio choice

Young households optimally choose the share of wealth invested in bank deposits, which we denote as ω. The remaining wealth (share $1-\omega$) is invested in investment funds. Investment funds optimally choose the share of portfolio invested in bank deposits, which we denote as α. And the remaining $1-\alpha$ of the portfolio is invested in risky assets. After shocks are realized, investment funds optimally choose r_m, the return rate paid to movers, and r_n, the return rate paid to non-movers. α, r_m and r_n should maximize the expected utility of shareholders.

Let v_m and v_n denote the value of the portfolio of movers and non-movers

$$v_m = s\left[\omega + (1-\omega)r_m\right](1+r^d) \tag{4}$$

$$v_n = s\left[\omega\frac{R_k}{Q_k} + (1-\omega)r_n\right] \tag{5}$$

$s = e_h$ is the saving by each young household. For movers, $s\omega$ is the initial bank deposit and $s(1-\omega)r_m$ is the deposit withdrawn from the investment fund. The deposit interest, if turns out to be positive, is paid at the end of the period. For non-movers, the return for the initial deposit can be written as $\frac{R_k}{Q_k}$. The reason is as follows. Non-movers can use part or all of their deposits to buy assets when investment funds sell assets. The market asset price is Q_k and the future payment is R_k. When the deposit of non-movers is not binding, the market price will be equal to the fundamental price and we have $\frac{R_k}{Q_k} = 1$, and we can write the return of deposit of non-movers as $\frac{R_k}{Q_k}$. If $Q_k < R_k$, the return from buying assets is higher than holding the deposit, so non-movers will use all their deposits to buy assets, and the return can still be written as $\frac{R_k}{Q_k}$. Non-movers are not affected by any positive deposit rate. Because, as we will show later, whenever the deposit rate is positive, the return rate $\frac{R_k}{Q_k}$ is equal to the gross lending rate, which is higher than the deposit rate. In this case, non-movers will use all their deposits to buy assets.

Investment funds can choose the best payout policy contingent on the realized shocks. For each value of the realized shocks, an investment fund will maximize the expected utility of its shareholders

$$\pi\frac{(v_m)^{1-\sigma}}{1-\sigma}+(1-\pi)\frac{(v_n)^{1-\sigma}}{1-\sigma} \tag{6}$$

subject to its budget constraints and the constraint $r_m \leq r_n$(payment to movers cannot be higher than non-movers, otherwise non-movers will pretend to be movers and withdraw.) We have the following result

PROPOSITION 1. *If $Q_k = R_k$, then it is optimal to set $v_m = v_n$ and $r_m = r_n = \alpha + (1-\alpha)R_k$. When $Q_k < R_k$, if the constraint $r_m \leq r_n$ is not binding, then the optimal policy is to set*

$$\frac{v_m}{v_n} = \frac{(\omega + (1-\omega)r_m)(1+r^d)}{\omega\frac{R_k}{Q_k} + (1-\omega)r_n} = \left(\frac{Q_k(1+r^d)}{R_k}\right)^{\frac{1}{\sigma}} \tag{7}$$

Given ω, Q_k and r^d, $\frac{r_m}{r_n}$ is increasing in σ. If the constraint $r_m \leq r_n$ is binding, then the optimal policy is $r_m = r_n$. For the log utility function($\sigma = 1$), the optimal policy is $r_m = \alpha + (1-\alpha)Q_k$ and $r_n = \alpha\frac{R_k}{Q_k} + (1-\alpha)R_k$.

Proof. See the appendix. ▌

The intuition is as follows. When $Q_k = R_k$, the investment fund can give movers and non-movers equal payment, and all shareholders will have the same consumption. When $Q_k < R_k$, it is costly to raise cash, and the investment fund may not want to fully smooth shareholders' consumption. When people are more risk averse (σ is higher), the fund will provide more consumption smoothing, and the value of $\frac{r_m}{r_n}$ will tend to be higher. With the log utility function, the optimal r_m is equal to the the market value of the fund's asset.

In the remaining part of this section and section 4, we consider the example of the log utility function. This utility function gives simpler results of portfolio choices and payout policy. The result for more general utility functions will be discussed in section 5, where we will discuss how different consumption smoothing due to different σ will affect the volatility of asset prices.

With the log utility function, we find that it is optimal to let the households hold all the riskless assets ($\alpha = 0$). The basic reason is that when households hold more deposits by themselves, investment funds can pay less money to movers during redemption, as a result, banks are less likely to borrow costly loans from the central bank during the settlement process. The detailed explanation is provided in the appendix. Since $r_m = \alpha + (1-\alpha)Q_k$ and $\alpha = 0$, we have $r_m = Q_k$.

3.2. Bank's problem

3.2.1. The expected borrowing from the central bank

Let L_i denote the total loan made by bank i and $L_j = L$ denote the loan made by each of the other banks $j \neq i$. When loans are made, the deposit balance for bank i becomes $D_0 + L_i$ and the deposit balance for all other banks $j \neq i$ becomes $D_0 + L_j$, where L_i and L_j are new deposits created during lending.

Then investment funds use all their deposits to pay the movers. We use X to denote the payment made by each bank, then

$$X_i = (1-\pi)D_0 + L_i \tag{8}$$

$$X_j = (1-\pi)D_0 + L_j \tag{9}$$

where $(1-\pi)D_0$ is the deposits raised by investment funds by selling assets to non-movers. (The remaining deposits πD_0 belong to movers.)

Recall that the settlement process is divided into N subperiods and in each subperiod a bank is chosen to make the payment to other banks. Since we assume there are N banks and the shareholders of each fund are evenly distributed among all banks, when bank i is chosen to make the payment, the payment made to the shareholders in the same bank is $\frac{1}{N}X_i$, the payment made to each of the other $N-1$ banks is also $\frac{1}{N}X_i$, and the total payment outflow is $\frac{N-1}{N}X_i$. The pattern is symmetric for all other banks $j \neq i$.

Let $FL(k,n)$ denote the accumulated outflow of payments in subperiod k if a bank is chosen to make the payment to other banks in subperiod n. Banks are required to borrow from the central bank as long as $FL(k,n) > D_0$. Let $b(k,n)$ denote the central bank loan in period k.

$$b(k,n) = \max(0, FL(k,n) - D_0) \tag{10}$$

Higher L_i means higher payment X_i, which will make $b(k,n)$ more likely to be positive.

Recall that each of the N subperiods has a time length of $\frac{1}{N}$. For a bank that makes its payment in period n, the accumulated borrowing over the settlement process (normalized by time length) is defined as

$$\bar{b}_n = \frac{1}{N}\Sigma_{k=1}^{N} b(k,n) \tag{11}$$

The total interest cost for the bank is $r^c\bar{b}_n$, where r^c is the central bank lending rate. Since n is uniformly distributed over $[1, N]$, before the settlement process starts, the expected future borrowing is defined as

$$Eb = \frac{1}{N}\Sigma_{n=1}^{N}\bar{b}_n \tag{12}$$

When N is large, we can get a closed-form solution for Eb; the result is as follows:

Proposition 2. *Suppose N is very large. Given L_j, when $Eb > 0$, it can be written as*

$$\begin{aligned} Eb(L_i) \;=\; & \frac{1}{6}\frac{(X_i - X_j - D_0)^3}{X_j^2} + \frac{1}{2}\frac{(X_i - X_j - D_0)^2}{X_j} \\ & + \frac{1}{2}(X_i - X_j - D_0) + \frac{1}{6}X_j \end{aligned} \tag{13}$$

In the symmetric case($L_i = L_j = L$), $Eb(L) > 0$ when $L > \pi D_0$(when $X > D_0$).

Proof. See the appendix ∎

In the symmetric case, the payment X is $(1-\pi)D_0 + L$. If $L > \pi D_0$, then $X > D_0$ and the payment will not be fully covered by the initial reserve, and $Eb > 0$.

3.2.2. Loan supply

After inter-bank payments are completed, bank i's deposit balance is

$$\begin{aligned} & \underbrace{(D_0 + L_i)}_{\text{deposit after loan making}} - \underbrace{\frac{N-1}{N}X_i}_{\text{payment outflow}} + \underbrace{\frac{N-1}{N}X_j}_{\text{payment inflow}} \\ \approx\; & (D_0 + L_i) - X_i + X_j = D_0 + L_j \end{aligned} \tag{14}$$

And the reserve balance is

$$\begin{aligned} &\underbrace{D_0}_{\text{initial reserve}} - \underbrace{\frac{N-1}{N}X_i}_{\text{payment outflow}} + \underbrace{\frac{N-1}{N}X_j}_{\text{payment inflow}} \\ \approx\ & D_0 - (X_i - X_j) = D_0 - (L_i - L_j) \end{aligned} \tag{15}$$

We assume that banks pay the interest of the central bank loan $r^c\overline{b}_n$ at the end of the settlement process. For simplicity, we focus on the case in which the reserve balance after paying the interest, $D_0 - (L_i - L_j) - r^c\overline{b}_n$, is positive. That is, D_0 is high enough to cover marginal increases in L_i and the interest cost $r^c\overline{b}_n$. (Since we focus on the symmetric case, $L_i - L_j$ means small marginal deviations of L_i from L_j.)[5] So banks do not need to borrow any central bank loan after the settlement process is completed, and we only need to care about the loan borrowed during the settlement process.

Bank profit is given by

$$\Pi = [D_0 - (L_i - L_j) - r^c\overline{b}_n] + L_i(R - \delta) - (1 + r^d)(D_0 + L_j) \tag{16}$$

The first term is the remaining reserves, the second term is the value of bank loan, and the third term is the gross payment to deposits, where the deposit level is given by (14). The expected profit is

$$E\Pi = [D_0 - (L_i - L_j)] - r^c Eb + L_i(R - \delta) - (1 + r^d)(D_0 + L_j) \tag{17}$$

If the liquidity shock is low and no central bank loan is needed($Eb = 0$), then (17) becomes

$$E\Pi = [D_0 - (L_i - L_j)] + L_i(R - \delta) - (1 + r^d)(D_0 + L_j) \tag{18}$$

In the equilibrium, bank i should not be able to increase the profit by changing L_i. The first order condition is

$$\frac{\partial E\Pi}{\partial L_i} = -1 + R - \delta = 0 \tag{19}$$

[5] In order for this assumption to hold, we only need D_0 to be higher than the maximum $r^c\overline{b}_n$. In the appendix, we show that the bank which makes the payment in subperiod 1 will have the highest borrowing. In the symmetric case, the accumulated borrowing for the first bank is $(1 - \frac{D_0}{X})(-D_0) + \frac{1-(\frac{D_0}{X})^2}{2}X < \frac{1}{2}X$. So we only need $D_0 > \frac{r^c X}{2}$. This condition can be easily met as long as D_0 is not extremely low and the lending rate r^c is very high.

which gives the loan supply curve when $Eb = 0$

$$R = 1 + \delta \tag{20}$$

In the symmetric case we have $L_i = L_j = L$, and the deposit rate can be computed by applying the zero expected profit condition ($E\Pi = 0$) to equation (18), the result is $r^d = 0$.

If $Eb > 0$, then the first order condition is

$$\frac{\partial E\Pi}{\partial L_i} = -1 - r^c \frac{\partial Eb(L_i)}{\partial L_i} + (R - \delta) = 0 \tag{21}$$

Using (8) and (13), we have

$$\frac{\partial Eb(L_i)}{\partial L_i} = \frac{1}{2}\left(1 + \frac{X_i - X_j - D_0}{X_j}\right)^2 \tag{22}$$

and (21) becomes

$$R = 1 + \delta + r^c \frac{1}{2}\left(1 + \frac{X_i - X_j - D_0}{X_j}\right)^2 \tag{23}$$

This is the loan supply curve of bank i given L_j. In the symmetric case ($L_i = L_j = L$), the supply curve when $Eb > 0$ becomes:

$$\begin{aligned} R &= 1 + \delta + r^c \frac{1}{2}\left(1 - \frac{D_0}{X}\right)^2 & (24)\\ &= 1 + \delta + r^c \frac{1}{2}\left(1 - \frac{D_0}{(1-\pi)D_0 + L}\right)^2 & (25) \end{aligned}$$

R is increasing in L. From (25), we can see that when $L > \pi D_0$, $\frac{D_0}{(1-\pi)D_0+L} < 1$ and so $R > 1 + \delta$.

3.3. The equilibrium lending rate and loan level

The demand curve for bank loan can be derived from the payout policy of the investment fund. Since with the log utility function $\alpha = 0$, all assets of the investment fund are in the form of risky assets. Denote the level of risky assets as Z_k. Since investment funds pay movers the market value of the fund's assets, the total payout is $\pi Z_k Q_k$. This should equal the cash collected from non-movers, $(1-\pi)D_0$, plus the loan borrowed from the bank. Thus, we have

$$\pi Z_k Q_k(\pi) = (1-\pi)D_0 + L(\pi) \tag{26}$$

The market price for asset is

$$Q_k(\pi) = \frac{R_k}{R(\pi)} \tag{27}$$

where $R(\pi)$ is the bank lending rate when the liquidity shock is π. Substitute $Q_k(\pi)$ into equation (26) and we have the demand curve for bank loan

$$R(\pi) = \frac{\pi Z_k R_k}{(1-\pi)D_0 + L(\pi)} = \frac{\pi Z_k R_k}{X} \tag{28}$$

When $Eb = 0$, the loan supply curve is $R = 1 + \delta$, and $Q_k = \frac{R_k}{1+\delta}$. Using (26), the equilibrium loan level is

$$\begin{aligned} L &= \underbrace{\pi Z_k Q_k}_{\text{Total redemption}} - \underbrace{(1-\pi)D_0}_{\text{cash raised from nonmovers}} && (29) \\ &= \pi Z_k \frac{R_k}{1+\delta} - (1-\pi)D_0 = S\left[\pi(1-\omega)\frac{R_k}{1+\delta} - (1-\pi)\omega\right] && (30) \end{aligned}$$

When $Eb > 0$, (25) and (28) give the following result:

PROPOSITION 3. *The equilibrium $L(\pi)$ and $R(\pi)$ are*

$$L^*(\pi) = S\left(\frac{r^c\omega + \pi(1-\omega)R_k + \sqrt{(r^c\omega + \pi(1-\omega)R_k)^2 - 2(1+\delta+\frac{r^c}{2})r^c\omega^2}}{2(1+\delta+\frac{r^c}{2})} - (1-\pi)\omega\right) \tag{31}$$

$$R^*(\pi) = \frac{2(1+\delta+\frac{r^c}{2})\pi(1-\omega)R_k}{r^c\omega + \pi(1-\omega)R_k + \sqrt{(r^c\omega + \pi(1-\omega)R_k)^2 - 2(1+\delta+\frac{r^c}{2})r^c\omega^2}} \tag{32}$$

Proof. See the appendix. ▮

Given the equilibrium $L(\pi)$ and $R(\pi)$, we can solve for $Eb(\pi)$ from equation (13) by setting $L_i = L_j = L^*(\pi)$, and then solve for the equilibrium deposit rate $r^d(\pi)$ from equation (17) by setting $E\Pi = 0$. The result is as follows:

PROPOSITION 4. *Suppose N is large, when $Eb \geq 0$, the expected central bank loan is*

$$Eb = \frac{-D_0^3}{6X^2} + \frac{D_0^2}{2X} - \frac{D_0}{2} + \frac{X}{6} \tag{33}$$

and the equilibrium deposit rate is[6]

$$
\begin{aligned}
r^d(\pi) &= \frac{\frac{Lr^c}{2}(1-\frac{D_0}{X})^2 - r^c Eb}{D_0+L} \\
&= \frac{\frac{Lr^c}{2}(1-\frac{D_0}{X})^2 - r^c(\frac{-D_0^3}{6X^2}+\frac{D_0^2}{2X}-\frac{D_0}{2}+\frac{X}{6})}{D_0+L}
\end{aligned}
\tag{34}
$$

where $X = (1-\pi)D_0 + L$, and $L = L^*(\pi)$ is the equilibrium loan level.

Once the equilibrium loan level is decided, the aggregate deposit level in the economy is also decided. The aggregate deposit (i.e, money supply) is $D_0+L(\pi)$, where D_0 is the initial deposit balance, and $L(\pi)$ is the deposits that are created by banks during lending. Note that the aggregate bank loan level is elastic and is not limited by the monetary funds saved by depositors. Instead, it is the lending activities of banks that decide the aggregate deposits in the economy.

The distribution of asset prices are as follows.

PROPOSITION 5. *The distribution of Q_k is*

$$
Q_k(\pi) = \begin{cases}
R_k & : \ \pi \le \pi_1 = \frac{\omega}{\omega+(1-\omega)R_k} \qquad \text{(Nonmovers' cash is not binding)} \\
\frac{\omega(1-\pi)}{\pi(1-\omega)} & : \ \pi_1 < \pi < \pi_2 = \frac{\omega}{\omega+(1-\omega)\frac{R_k}{1+\delta}} \qquad \text{(Nonmovers' cash is binding)} \\
\frac{R_k}{1+\delta} & : \ \pi_2 \le \pi \le \pi_3 = \frac{\omega}{(1-\omega)\frac{R_k}{1+\delta}} \qquad \text{(Bank loan } L>0, Eb=0) \\
\frac{R_k}{R(\pi)} & : \ \pi > \pi_3 \qquad \text{(Bank loan } L>0, Eb>0)
\end{cases}
\tag{35}
$$

Proof. See the appendix. ▌

Below π_1, non-movers' cash is not binding, and Q_k is equal to the fundamental value R_k. Between π_1 and π_2, non-movers' cash is binding but it is not worthwhile to borrow from banks. Above π_2, the price on the market is low enough and investment funds will choose to borrow from banks. Between π_2 and π_3, the bank loan level is still low and there is no need to borrow from the central bank, and the lending rate is $1+\delta$. When $\pi > \pi_3$, the loan level is high and the expected central bank loan is positive, this will lead to a higher lending rate $R(\pi)$, and a lower equilibrium asset price.

3.4. The first order condition for ω

[6]We replace R in (17) using the loan supply curve (25).

Using v_m and v_n from (4) and (5), the expected utility for a representative household i is

$$\begin{aligned} EU_i \;=\; & \frac{1}{2}\int_0^1 \Big\{\pi \ln\left[(1+r^d)s\left(\omega_i + (1-\omega_i)Q_{k,H}\right)\right] \\ & +(1-\pi)\ln\left[s\frac{R_{k,H}}{Q_{k,H}}(\omega_i + (1-\omega_i)Q_{k,H})\right]\Big\} dF(\pi) \\ & +\frac{1}{2}\int_0^1 \Big\{\pi \ln\left[(1+r^d)s\left(\omega_i + (1-\omega_i)Q_{k,L}\right)\right] \\ & +(1-\pi)\ln\left[s\frac{R_{k,L}}{Q_{k,L}}(\omega_i + (1-\omega_i)Q_{k,L})\right]\Big\} dF(\pi) \end{aligned} \tag{36}$$

The household will choose ω_i to maximize his expected utility. The first order condition is

$$\frac{\partial EU_i}{\partial \omega_i} \;=\; \frac{1}{2}\int_0^1 \frac{1-Q_{k,H}}{\omega_i + (1-\omega_i)Q_{k,H}} + \frac{1-Q_{k,L}}{\omega_i + (1-\omega_i)Q_{k,L}} dF(\pi) = 0 \tag{37}$$

3.5. The equilibrium

The equilibrium of the model can be defined as follows. Given the initial portfolio choice in t, after the shocks are realized in $t+1$, investment funds optimally choose the payout policy. Banks optimally choose the lending rate to maximize the profit; they also choose the deposit rate subject to the zero expected-profit condition. The asset market clears, the assets sold by investment funds are equal to the assets purchased by non-movers. The bank loan market clears. The loan borrowed by investment funds is equal to the loan lent by banks.

At the end of period $t+1$, the goods market clears. The consumption of movers is v_m, and the consumption of non-movers is v_n. The aggregate consumption is equal to the total goods produced by investment funds $e_h(1-\omega)R_k$ plus the goods $\frac{M}{P}$ purchased from the next young generation.

Given the expected outcome in period $t+1$, at the end of period t, households and investment funds optimally choose their portfolios. With the log utility function, all riskless assets are held by households, and the value of riskless assets is equal to the real balance of outside money:

$$\frac{M}{P} = \omega e_h \tag{38}$$

This equation determines the equilibrium price level. In the stationary equilibrium, P is the same in each period.

3.6. When bank lending is exogenously shut-down

If there is no bank lending, then the cash in the economy will be limited to the amount of outside money held by households. The market asset price will be lower than the fundamental price when non-movers' cash is binding. In the appendix, we show that with the log utility function, when there is no bank lending, we can still assume that all initial cash is held by households. The distribution of Q_k is

$$Q_k(\pi) = \begin{cases} R_k & : \quad \pi \leq \pi_1 \quad (\text{Non} - \text{movers}' \text{ cash is not binding}) \\ \frac{\omega(1-\pi)}{\pi(1-\omega)} & : \quad \pi > \pi_1 \quad (\text{Non} - \text{movers}' \text{ cash is binding}) \end{cases} \tag{39}$$

which is simply (35) without bank lending.

4. NUMERICAL RESULTS: $U = \ln C$

This section considers an example with $U = \ln c$. The purpose is to restrict σ at 1 and then focus on liquidity provision by banks through money creation. (The effects of different σ on consumption smoothing will be discussed in section 5.)

4.1. Parameter values

Table 1 shows the parameter values. The return of the risky assets are $R_{k,H} = A_H$ and $R_{k,L} = A_L$. We set the household endowment e_h at 1, the loan management cost of banks δ at 3% and the central bank lending rate r^c at 3%.

TABLE 1.

Values of parameters

A_H	A_L	e_h	δ	r^c
1.21	0.85	1	0.03	0.03

We assume that the liquidity shock is distributed according to

$$\pi = 0.9\theta^{\varphi} \tag{40}$$

where θ is uniform over $[0, 1]$. φ is used to adjust the density of π. With higher φ, the density of π will be more concentrated on low values and households will hold lower monetary balance. We use $\varphi = 6$. At this level, D_0 is low enough and we can see clearly the effects when banks borrow from the central bank.

4.2. Results

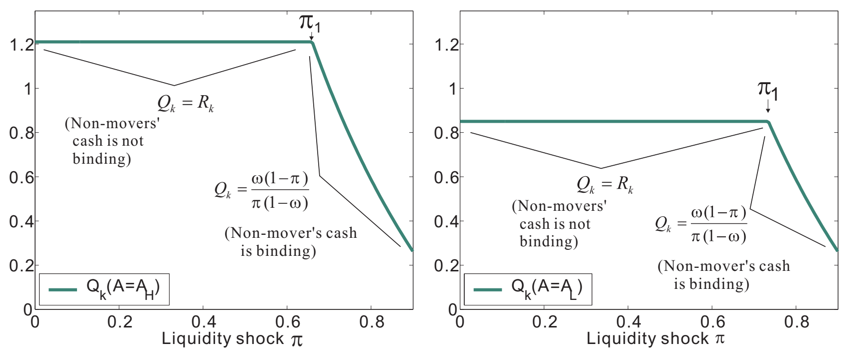
Table 2 shows ω^* and the expected utility of households. We show four cases. Case 1 is when all assets (including mutual fund shares) can be used to buy goods across locations. In this case, movers do not need to withdraw from investment funds. It can be shown that we will have a standard portfolio choice problem with one riskless and one risky asset. The riskless asset is money, its return is always 1, and the risky asset is simply the risky investment; its return is R_k. In case 2, we assume there is no bank lending. In case 3, bank lending is allowed. We show two cases:(3a) and (3b). The difference is that in (3a), the interest rate is constant, while in (3b), the interest rate is lower for $\pi > 0.5$.

Since the aggregate household wealth is set at 1, ω^* is also equal to the aggregate real money balance $\frac{M}{P} = D_0$. Table 2 shows that when money is more elastic (when bank lending is allowed, and when the lending rate of central bank loan is lower), households will hold less money balance and their expected utility will also be higher.

TABLE 2.

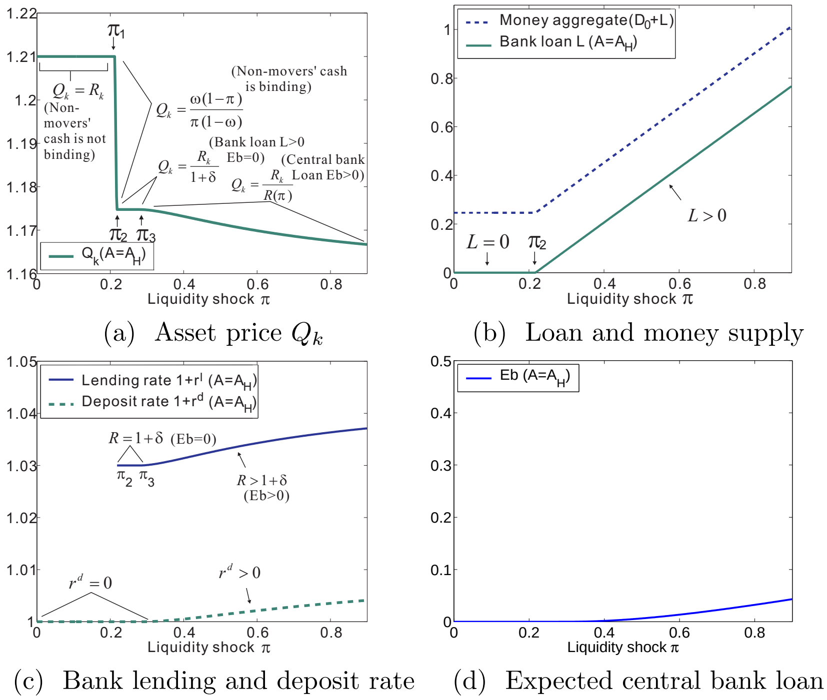
Numerical example

	ω^*	$E \ln c$(households)
(1) All assets can be used to make payments	0.0476	0.0141
(2) No bank lending (inelastic money)	0.7013	0.0059
(3) With bank lending (elastic inside money)		
(3a) Central bank sets $r^c = 0.03$ for all levels of π	0.2460	0.0118989
(3b) CB sets $r^c = 0.02$ for $\pi \geq 0.5$. Policy pre-announced	0.2438	0.0119093



(a) Asset price, $A = A_H$ (b) Asset price, $A = A_L$

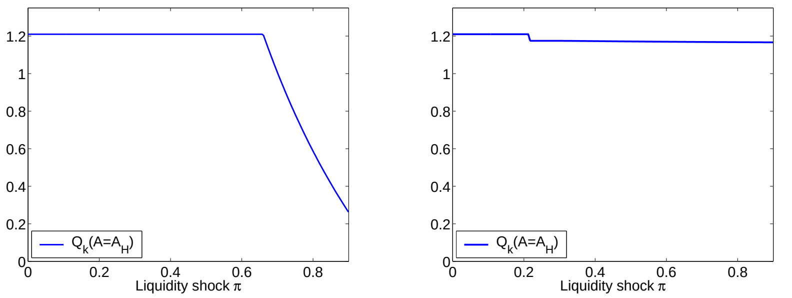
FIG. 4. Q_k in case 2: no bank loan.



(a) Asset price Q_k (b) Loan and money supply

(c) Bank lending and deposit rate (d) Expected central bank loan

FIG. 5. Q_k, L, R, r^d and Eb in case 3a (elastic money). $A = A_H$ is shown.



(a) cases 1(no inside money) (b) 3a(with inside money)

FIG. 6. Compare Q_k in cases 1(no inside money) and 3a(with inside money). $A = A_H$

Figure 4 shows the market price of risky assets in case 2 (equation 39) where bank lending is shut down. An important result is that there is a big decrease in the asset price when the liquidity shock is high. The reason is that once non-movers have used all their cash to buy assets, investment funds can not raise any additional cash by selling assets, and it is the asset

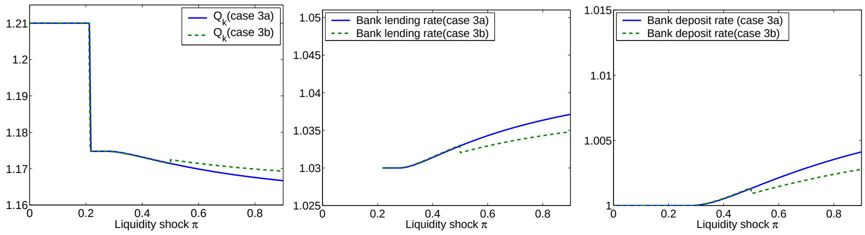


FIG. 7. Compare cases 3a (elastic money with constant r^c) and 3b (elastic money with lower r^c for $\pi > 0.5$). $A = A_H$.

price that must decrease, until investment funds find that the asset price is so low that it is not optimal to sell more assets.

Figure 5 shows the results for case 3a when bank lending is allowed. The results for $A = A_H$ are shown (The results for $A = A_L$ are similar). Q_k is decided according to equation (35). Note that the aggregate money supply $D_0 + L(\pi)$ is stochastic. Higher liquidity shock will cause investment funds to borrow more, which will in turn lead to higher money supply.

Figure 6 compares asset prices in case 2 (bank lending is shut down) and case 3a (bank lending is allowed). The asset price is more stable with elastic inside money.

Figure 7 compares case 3a, where the central bank lending rate r^c is fixed, and case 3b, where r^c is lower for $\pi > 0.5$. In case 3b, the lower central bank lending rate will lead to lower bank lending rates, lower bank deposit rates, and higher asset prices.

4.3. The transmission of central bank interest rate policy

The transmission of the interest rate policy is summarized in Figure 8. When the central bank adjusts its lending rate, it will affect the expected cost of banks to borrow central bank loans during the settlement process, which will in turn affect the bank lending rate and the equilibrium loan level. When the central bank sets a lower lending rate, the equilibrium lending rate will be lower and the loan level will be higher. The lower lending rate helps to support higher asset prices, because investment funds can choose to borrow from banks instead of selling assets on the market. The higher bank loan level will create a higher demand for settlement balances. Since the central bank meets the demand for settlement balances by providing loans to banks, the supply of reserves is thus endogenously decided by the equilibrium demand for reserves.

The aggregate money supply is not directly controlled by the central bank. Since the changes in money supply are simply the changes in bank

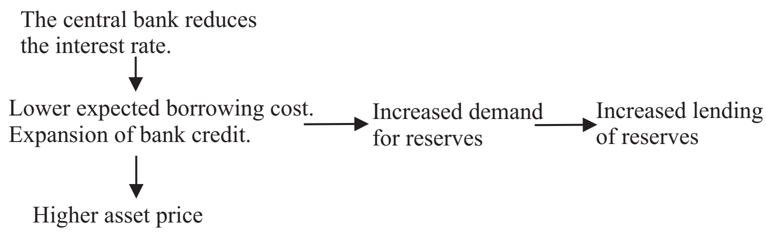


FIG. 8. The transmission of central bank interest rate policy

deposits, the money supply is decided by the equilibrium bank loan level. Higher liquidity shock will cause higher demand for bank loans, and higher equilibrium bank loans means more deposits are created, thus a higher money supply.

4.4. Implication

Our results have an important implication for the elasticity of liquidity. When demand for liquidity takes the form of demand for means of payment, it can be met with the creation of additional inside money (i.e., bank deposit). The availability of liquidity can no longer be measured by the existing outside money (i.e., central bank money), and certainly not by short-term real consumption goods. The supply of liquidity here is more elastic compared to models where liquidity takes the form of consumption goods. In those models, aggregate liquidity is fixed by the initially stored short-term consumption goods. Thus, non-monetary models may underestimate the ability of the financial system to meet the increased demand for liquidity.

5. THE RELATIONSHIP BETWEEN LIQUIDITY-RISK SHARING THROUGH COALITIONS AND BANK MONEY CREATION

Our model includes two types of liquidity provision mechanisms: liquidity-risk-sharing through mutual funds and liquidity provision through money creation. Liquidity risk-sharing means early withdrawers of liquidity will get a higher payment than if there were no mutual funds. In the previous analysis, we use the log utility case to illustrate the second mechanism (liquidity provision through money creation). But with log utility, the first mechanism (liquidity-risk-sharing) is not actually used by households. This is because when $U = \ln c$, the payment r_m is equal to the market price of the fund's asset. If investment funds allocate all assets equally to shareholders and then movers sell assets directly to non-movers, then movers will get the same payment.

If $\sigma > 1$, r_m will be set above the market price of the fund's asset. That is, given the market price Q_k, movers will get more than what they would get if they sold the assets by themselves. In this case, investment funds perform risk-sharing (i.e., consumption smoothing) for shareholders.

We ask the following questions:

1. Can risk-sharing investment funds help movers to achieve higher consumption? How does the liquidity provision function of banks affect the liquidity-risk-sharing function of investment funds?

2. How will the risk-sharing function of coalitions affect the volatility of asset prices on the market?

We show that if bank lending is exogenously shut down, since the aggregate money is limited to the money held by people in their initial portfolio, investment funds will not actually help movers to achieve higher consumption; the liquidity-risk-sharing function is useless. The attempt of investment funds to provide higher payment than the market value of the fund's asset will only cause lower asset prices and so add more volatility into the market. But with banks supplying elastic aggregate money, investment funds can indeed help movers to achieve higher consumption.

We will first illustrate the effects of σ on asset prices by assuming that the initial portfolios for different σ are the same. This allows us to isolate and compare the effects of different σ on risk-sharing and asset prices. The reason that we fix the initial portfolio is that different σ can lead to different initial portfolio choices, which makes the effects of σ on asset prices not directly comparable. We will later discuss the outcome where people choose different optimal portfolios for different σ.

5.1. When ω and α are exogenously fixed

We first consider the no-bank-lending case. We define κ as the total share of bank deposit in a household's portfolio (i.e., both the deposit held directly by the household and the deposit held indirectly through the investment fund).

$$\kappa = \omega + (1-\omega)\alpha \tag{41}$$

We first show two basic results.

1. If there is no bank lending, then once total investment in the riskless asset, κ, is decided, the consumption level of movers and non-movers are

determined. The consumption level is not affected by risk-sharing provided by investment funds.

2. Once the initial distribution of money among investment funds and households is given, the distribution of r_m is decided. r_m does not depend on the risk-sharing policy, but risk-sharing will cause higher volatility in asset prices.

These two results are proved in the following two propositions. The results are then illustrated using a numerical example.

PROPOSITION 6. *In the symmetric equilibrium, the distribution of v_m and v_n only depends on κ. Set $e_h = 1$, when $\pi \leq \pi_1$, $v_m = v_n = \kappa + (1-\kappa)R_k$. When $\pi > \pi_1$, $v_m = \frac{\kappa}{\pi}$ and $v_n = \frac{(1-\kappa)R_k}{1-\pi}$. $\pi_1 = \frac{\kappa}{\kappa+(1-\kappa)R_k}$.*

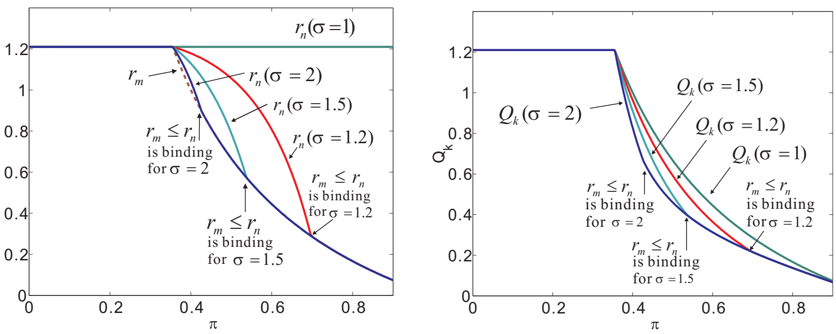
Proof. See the appendix. ▌

The intuition is as follows. When we set $e_h = 1$, the total wealth of the economy can be written as $\kappa + (1-\kappa)R_k$. When the aggregate cash constraint is not binding, everyone gets $\kappa + (1-\kappa)R_k$. Once the cash constraint is binding, movers can only carry all the cash in the economy κ with them to the other location, since the size of movers is π, so the consumption of every mover is $\frac{\kappa}{\pi}$. The aggregate consumption of non-movers will be the remaining risky assets $(1-\kappa)R_k$, and every non-mover consumes $\frac{(1-\kappa)R_k}{1-\pi}$. To determine π_1, note that at π_1, each mover still receives the average wealth $\kappa + (1-\kappa)R_k$, so π_1 is determined by the ratio between the aggregate cash κ and the aggregate wealth $\kappa + (1-\kappa)R_k$. We can see that given κ, the consumption level of movers is independent of the payout policy of the investment fund.

PROPOSITION 7. *In the symmetric case, if we fix the initial portfolio choice ω and α, then the distribution of r_m will be the same for different σ. But the distribution of Q_k will be different. Between π_1 and π_{bind}(the π above which the constraint $r_m \leq r_n$ is binding), Q_k is lower for higher σ.*

Proof. See the appendix. ▌

The intuition is as follows. Once the aggregate liquidity is decided by ω and α, then the distribution of r_m will be uniquely decided. For $\pi > \pi_1$, when σ is higher, people are more risk averse, and the investment fund would like to set a higher r_m given the market value of the fund's asset, so as to smooth the consumption of movers and non-movers. But since the payment to movers is limited by the liquidity in the economy, it is the price Q_k(and the market value of the fund's asset) that must adjust in order for the optimal payout policy to be satisfied. In other words, if σ is higher, then Q_k must decrease more in order to satisfy the optimal payout policy (i.e., the fund finds that the price is so low that it is not optimal to sell more assets). On the microeconomic level, each investment fund takes the price Q_k as given and tries to liquidate assets to raise cash in order to provide liquidity insurance to movers. But if the aggregate liquidity is limited, then the effort of investment funds is self-defeating, it will only cause the asset prices to be lower, without actually providing more liquidity to movers. The fund will sell more assets to non-movers on the market, only to raise the same amount of liquidity.



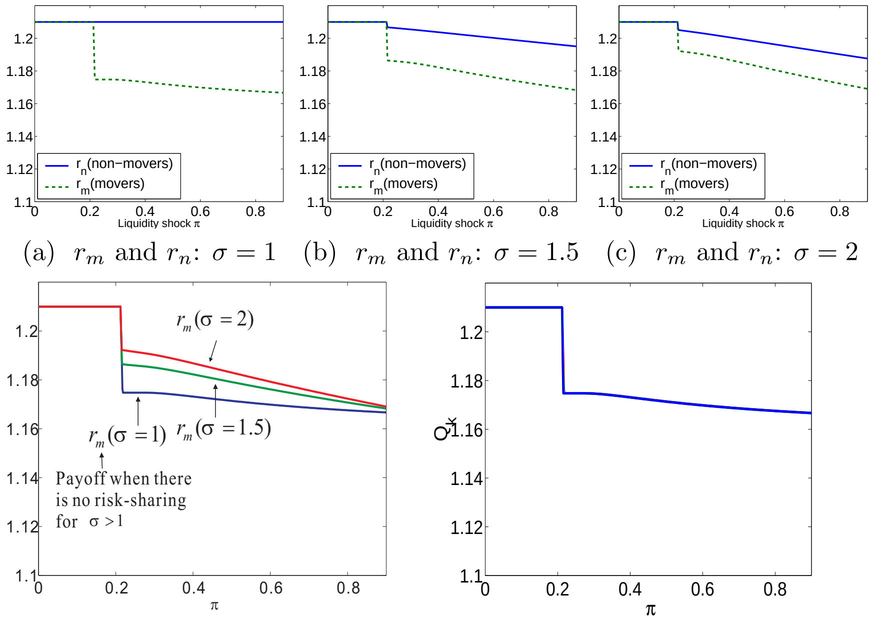
(a) r_m and r_n for different σ (b) Asset price Q_k

FIG. 9. Payout policy and Q_k when the initial portfolio is given by $[\omega = 0.4, \alpha = 0]$. No bank lending. The results for $A = A_H$ are shown.

The results of proposition 7 are illustrated in Figure 9, where we use the example $[\omega = 0.4, \alpha = 0]$(we set a low ω so that we can see clearly what will happen when Q_k goes to low levels.) Figure 9 shows that given the initial portfolio, the actual payment to movers r_m will not be affected by σ. In addition, when σ is higher, since investment funds try to liquidate more assets, this will cause the asset price Q_k to decrease more quickly(figure (b)). Since liquidations of assets are costly, this will also cause r_n, the payment to the remaining shareholders, to decrease more quickly(figure (a)). Higher σ (higher consumption smoothing) implies that given the wealth of non-movers, the fund will try to make r_m and r_n closer to each other. In this case, when the aggregate liquidity is binding, r_m and r_n are made closer by reducing r_n without actually raising r_m.

Note that once the constraint $r_m \leq r_n$ is binding, r_m is set equal to r_n, and the liquidation will no longer depend on σ, but is determined by the constraint $r_m = r_n$. As a result, Q_k will be the same for different σ. The payment to movers will be lower than the optimal payment that when r_m were allowed to be higher than r_n. This lower payment to movers will reduce the need to liquidate assets. In this case, limiting the payment to movers does not really reduce the actual payment r_m received by movers, but it helps stabilize the asset price.

With bank lending With banks supplying elastic aggregate money, the aggregate liquidity is no longer limited to the existing cash in the economy. In this case, investment funds can help people to smooth their consumption.



(a) r_m and r_n: $\sigma = 1$ (b) r_m and r_n: $\sigma = 1.5$ (c) r_m and r_n: $\sigma = 2$

(d) r_m (e) Q_k for $\sigma = 1$, 1.5 and 2.0

FIG. 10. Payout policy and Q_k when the initial portfolio is fixed at $[\omega = 0.2460, \alpha = 0]$. Bank lending is allowed. $A = A_H$. (a),(b) and (c) show the optimal r_m and r_n for different σ. Figure (d) compares r_m for different σ. Figure (e) shows the asset price. The differences in Q_k for different σ are very small so that we can see only one curve for Q_k in the Figure.

The results are shown in Figure 10, where the initial portfolio is fixed at $[\omega = 0.2460, \alpha = 0]$.[7] The findings can be summarized as follows:

[7]It can be shown that this is the optimal equilibrium for $\sigma = 1$ when people can freely choose the portfolio.

1. r_m, r_n and the asset price Q_k are more stable compared to the no-bank-lending case. (Note that the scale for Q_k is smaller than that in Figure 9).

2. The levels of r_m are no longer the same for different σ. When σ is higher, investment funds can provide higher r_m to movers.

The key figure is 10(d). Recall that r_m under the log utility function is the market price of the fund's asset. This is also the payment that movers would get if all investment funds allocated their assets to shareholders and shareholders traded assets by themselves. That is, if $\sigma > 1$ but there is no risk-sharing, movers would get $r_m(\sigma = 1)$ for the investments allocated into investment funds. But here, we can see that when $\sigma > 1$, the payment to movers is higher than $r_m(\sigma = 1)$, which means movers achieve higher consumption than if there were no investment funds to provide risk-sharing.

5.2. When ω and α are chosen freely

We have the following results for the equilibrium portfolio choice. Our numerical results for non-log-utility function ($\sigma > 1$) show that $r_m \leq r_n$ is more likely to be binding when α is low, and in the equilibrium, people will choose α high enough such that $r_m \leq r_n$ is not binding, which means investment funds can freely choose its optimal payout policy and are not constrained by the condition $r_m \leq r_n$. Once $r_m \leq r_n$ is not binding, the optimal portfolio can be defined by the optimal κ, the total money balance held by households. When there is no bank lending, there are many combinations of α and ω that will give the same optimal equilibrium κ, all of them are equally optimal. When there is bank lending, it is optimal to choose the highest ω(lowest α) that are consistent with the optimal κ. This means it is optimal to let the households to hold more deposits by themselves. This can reduce the payments made by investment funds to movers, thus reducing the possibility of banks to borrow costly central bank loans. For the log utility, $U = \ln c$, the constraint $r_m \leq r_n$ is never binding and it is optimal for households to hold all the deposits.

Here are some explanations for the above results. Recall that both households and investment funds try to maximize the expected utility of households. Suppose $r_m \leq r_n$ is not binding, which means the payout policy of investment funds can be chosen freely. We show in the appendix that, when there is no bank lending, the expected utility of households depends on the total deposits κ, it does not matter whether these deposits are held by households or by investment funds. The reason is as follows. The distribution of asset prices are uniquely decided by the aggregate liquidity κ. When the aggregate liquidity is not binding, the consumption level of every household is the same, which is equal to the value of aggregate deposits plus the fundamental value of risky investments. When the aggregate liquidity is binding, investment funds will give all deposits back to movers, plus the deposits raised from non-movers, the outcome will be the same if households hold all initial deposits by themselves. As a result, once the optimal κ is decided, it does not matter whether the cash is held by households or by investment funds. The result when there is bank lending is similar. However, with bank lending, interbank payment can be higher than the bank's reserve level, and it would be better for households to hold more deposits, so as to reduce the payment flows and the need for banks to borrow from the central bank during the settlement process.

Even when we allow the initial portfolio to be chosen freely, the basic results are still similar to those in section 5.1 where the initial portfolio allocation are given exogenously. For example, when there is no bank lending, the distribution of consumption is determined by κ, and investment funds will not actually help people to smooth consumption. Once the cash constraint is binding, liquidations will only cause the asset price to decrease further. When there is bank lending, asset price will be more stable and investment funds can help people better smooth consumption.

There is one difference: When portfolios are chosen freely, the equilibrium κ tends to higher for higher σ. This is because when σ is higher, once the cash is binding, investment funds will try to liquidate more assets (or to take more loans), which will lead to a lower asset price. This implies a higher return for using deposits to buy assets, so people will hold fewer risky assets and more riskless deposits in their initial portfolio. The detailed numerical results are omitted.

5.3. Summary and comparison with partial equilibrium models

Our results show that the ability of banks to relax the aggregate liquidity constraint is important for non-bank investment funds to provide insurance for liquidity risks. Without banks, investment funds cannot actually provide more liquidity to people who need liquidity early, and risk-sharing will only make asset prices more volatile. But with banks supplying elastic aggregate money, people who need liquidity early can indeed achieve higher consumption through the risk-sharing function provided by investment funds.

Note that we get the above results because we use a general equilibrium model to endogenously decide the asset price. Many partial equilibrium models use the assumption that the liquidation price is exogenously given as a constant proportion of the fundamental value; an example is $Q_k = \lambda_q R_k$, where $\lambda_q < 1$ is a constant. These models essentially assume that risk-sharing coalitions can sell unlimited amounts of assets at the same price $\lambda_q R_k$. While in our model, if there is no bank lending, then once the aggregate liquidity is binding, sales of assets will not raise additional liquidity for investment funds, they will cause only lower asset prices. Our model shows more clearly why the function of banks to expand aggregate liquidity is important to financial stability.

6. CONCLUSION

This paper studied the role of banks in providing liquidity through inside money creation. Using a general equilibrium model, we showed that the ability of banks to expand aggregate liquidity is important to financial stability.

First, we showed that elastic inside money can help maintain stable asset prices. Second, we compared the liquidity provision function of Diamond-Dibvig style coalitions that people formed to share liquidity risks, and the liquidity provision function of banks. We show that the function of banks in providing elastic aggregate liquidity is important for the function of non-bank coalitions in providing risk sharing. If there is no elastic aggregate liquidity, then non-bank coalitions cannot actually provide higher liquidity to shareholders who need liquidity early. The risk-sharing function will be useless and will actually cause bigger drops in asset prices during liquidity shocks. However, with banks supplying elastic aggregate liquidity, risk-sharing coalitions can indeed help people better insure their liquidity risks. Finally, we showed that the central bank can help banks to provide liquidity to the financial market by lending to banks at low costs during the inter-bank settlement process, so as to relax the liquidity constraint of banks.

APPENDIX A

A.1. THE DETAILED TRANSACTION STEPS

In this part, we use bank balance sheet to show the transactions and monetary flows in the model. Recall that the inter-bank settlement at the end of period $t+1$ is carried out based on net balance. But for illustrative purposes, we show the gross payment flows. We normalize the initial bank equity to zero. We put the interest costs and interest income of the bank under the entry “Bank Equity”.

TABLE 1.

Monetary flows

Balance sheet of a representative commercial bank in location i (when $L > 0$)						
	Asset Side		Liability side			
	Reserve	Loan (Fund)	Deposit (movers)	Deposit (non-movers)	Deposit (Fund)	Bank Equity
Balance before the liquidity shock						
Balance	D_0	0	πD_0	$(1-\pi)D_0$	0	0
Changes in each account after the liquidity shock						
Investment fund sells assets to non-movers				$-(1-\pi)D_0$	$+(1-\pi)D_0$	
Bank makes loans to the investment fund		$+L$			$+L$	
Movers redeem fund shares			$+L+$ $(1-\pi)D_0$		$-L-$ $(1-\pi)D_0$	
Bank pays the borrowing cost	$-r^c\bar{b}_n$					$-r^c\bar{b}_n$
Changes in each account at the end of the Period						
The central bank consumes the interest	$+r^c\bar{b}_n$				$+r^c\bar{b}_n$	
Bank pays the deposit interest			$+r^d(D_0+L)$			$-r^d(D_0+L)$
Movers(i) buy goods in location j			$-(L+D_0)\times$ $(1+r^d)$			
Movers(j) buy goods in location i					$+(L+D_0)\times$ $(1+r^d)$	
Investment fund repays the bank loan		$-L$			$-L(1+r^l)$	$+Lr^l$
Bank spends the interest income					$+[Lr^l - r^c\bar{b}_n$ $-r^d(D_0+L)]$	$-[Lr^l - r^c\bar{b}_n$ $-r^d(D_0+L)]$
Final Balance						
Balance	D_0	0	0	0	D_0	0

The initial deposit and reserve balance is D_0, the total deposit held by movers is πD_0, and the total deposit held by non-movers is $(1-\pi)D_0$. After the shocks, investment funds sell assets to non-movers and raise cash $(1-\pi)D_0$. Investment funds also borrow L from the bank. After the loan is made, ‘loan” and “deposit” on the balance sheet increase by the same amount L, and the new deposit L is created. Investment funds then use all their deposits $L + (1 - \pi)D_0$ to meet the redemption needs of movers, and movers end up holding all the deposits $D_0 + L$. After the redemption process is completed, the bank pays the interest cost $r^c \bar{b}_n$ of the central bank loan.

At the end of the period, the central bank consumes the interest income by using central bank money to buy goods from investment funds. The reserve balance of the bank and the deposit of investment funds will increase by the same amount. Banks then pay deposit interest to movers. Movers then use their deposits to buy goods. When movers from location i buy goods in location j, their deposit balance decreases by $(D_0 + L)(1 + r^d)$. And when movers from location j buy goods from the investment funds in location i, the deposit balance of those investment funds will increase by $(D_0 + L)(1 + r^d)$.

Investment funds then repay the bank loan. The outstanding loan is reduced by L, and the outstanding deposit is reduced by $L(1 + r^l)$, and the interest income of the bank is Lr^l. The bank then spends the income $[Lr^l - r^c \bar{b}_n - r^d(D_0 + L)]$ to buy goods from investment funds.[1] When the bank spends the income, the bank pays the sellers(investment funds) by crediting their deposit accounts.

After all the above steps are completed, investment funds will then transfer the deposit balance D_0 to non-movers, who will then use the deposit to purchase goods from the young generation.

A.2. ADDITIONAL RESULTS: THE GENERAL CASE $\sigma \geq 1$, NO BANK LENDING

A.2.1. The optimal payout policy of the investment fund

LEMMA 1. *If $Q_k = R_k$, then the optimal policy is to set $v_m = v_n$, and $r_m = r_n = \alpha + (1 - \alpha)R_k$. When $Q_k < R_k$, if the constraint $r_m \leq r_n$ is not binding, then the optimal policy is to set*

$$\frac{v_m}{v_n} = \frac{\omega + (1 - \omega)r_m}{\omega \frac{R_k}{Q_k} + (1 - \omega)r_n} = \left(\frac{Q_k}{R_k}\right)^{\frac{1}{\sigma}} \tag{A.1}$$

Given ω and Q_k, $\frac{r_m}{r_n}$ is increasing in σ. If the constraint $r_m \leq r_n$ is binding, then the optimal policy is $r_m = r_n$. For the log utility function($\sigma =$

[1]Note that the income is usually positive because the lending rate r^l includes the management cost for bank loans. If δ is not too small, then Lr^l should be higher than the interest costs of the bank. In case where the income is negative, the bank can sell its endowment to absorb the loss.

1*), we have*

$$r_m = \alpha + (1-\alpha)Q_k \tag{A.2}$$

$$r_n = \alpha\frac{R_k}{Q_k} + (1-\alpha)R_k = r_m\frac{R_k}{Q_k} \tag{A.3}$$

The meaning of the above result is as follows. First, as long as $Q_k = R_k$, there is no cost to raise cash by selling assets, and it is optimal to fully insure the liquidity risk and give movers and non-movers the same return. Second, if $Q_k < R_k$, since it is costly to raise cash, the investment fund may not provide full insurance. When people are more risk averse(higher σ), it is optimal to set a higher $\frac{r_m}{r_n}$, which means to give a higher payment to movers. When the constraint $r_m \leq r_n$ is binding, it is optimal to set $r_m = r_n$. With the log utility function, the optimal r_m is simply to pay the market value of the fund's asset.

Proof: Let v_m denote the value of movers' portfolio and v_n the value of non-movers' portfolio. We have

$$v_m = s\left[\omega + (1-\omega)r_m\right] \tag{A.4}$$

$$v_n = s\left[\omega\frac{R_k}{Q_k} + (1-\omega)r_n\right] \tag{A.5}$$

The expected utility of the household is

$$\frac{1}{2}\int_0^1 \left[\pi\frac{(v_{m,H})^{1-\sigma}}{1-\sigma} + (1-\pi)\frac{(v_{n,H})^{1-\sigma}}{1-\sigma}\right]dF(\pi) + \frac{1}{2}\int_0^1 \left[\pi\frac{(v_{m,L})^{1-\sigma}}{1-\sigma} + (1-\pi)\frac{(v_{n,L})^{1-\sigma}}{1-\sigma}\right]dF(\pi) \tag{A.6}$$

where H and L are productivity shocks. Since r_m is chosen after the shocks are realized, the fund can choose the best r_m for each level of the shock. So the fund maximizes

$$\pi\frac{(v_m)^{1-\sigma}}{1-\sigma} + (1-\pi)\frac{(v_n)^{1-\sigma}}{1-\sigma} \tag{A.7}$$

subject to its budget constraints and the constraint $r_m \leq r_n$(payment to movers can not be higher than non-movers, otherwise non-movers will pretend to be movers and withdraw.)

We first analyze the case when the fund does not need to sell assets. The budget constraints are

$$\pi r_m = \phi\alpha \tag{A.8}$$

$$(1-\pi)r_n = (1-\phi)\alpha + (1-\alpha)R_k \tag{A.9}$$

where $\phi\alpha$ is the riskless asset used to pay movers, $(1-\phi)\alpha$ is the unused riskless asset and $(1-\alpha)R_k$ is the value of the risky assets. Using the budget constraints to replace r_m and r_n in v_m and v_n (equation (A.4) and (A.5)), the fund's problem (A.7) can be written as (we eliminate the common "s" from v_m and v_n)

$$\pi\frac{(\omega+(1-\omega)\frac{\phi\alpha}{\pi})^{1-\sigma}}{1-\sigma}+(1-\pi)\frac{(\omega+(1-\omega)\frac{(1-\phi)\alpha+(1-\alpha)R_k}{1-\pi})^{1-\sigma}}{1-\sigma} \tag{A.10}$$

Here, we use $\frac{R_k}{Q_k}=1$ since there is no liquidation of assets. Taking the derivative with respect to ϕ and simplifying the terms, we get

$$\frac{1}{(\omega+(1-\omega)\frac{\phi\alpha}{\pi})^{\sigma}}-\frac{1}{(\omega+(1-\omega)\frac{(1-\phi)\alpha+(1-\alpha)R_k}{1-\pi})^{\sigma}}=0 \tag{A.11}$$

that is, $\frac{1}{v_m^\sigma}-\frac{1}{v_n^\sigma}=0$, which means $v_m=v_n$ and $r_m=r_n=\alpha+(1-\alpha)R_k$.

Next, suppose the fund needs to sell assets. Let η denote the share of risky assets that is liquidated. The budget constraints are

$$\pi r_m = \alpha+(1-\alpha)\eta Q_k \tag{A.12}$$
$$(1-\pi)r_n = (1-\alpha)(1-\eta)R_k \tag{A.13}$$

The fund maximizes

$$\begin{aligned}&\pi\frac{(\omega+(1-\omega)\frac{\alpha+(1-\alpha)\eta Q_k}{\pi})^{1-\sigma}}{1-\sigma}\\ +\ &(1-\pi)\frac{(\omega\frac{R_k}{Q_k}+(1-\omega)\frac{(1-\alpha)(1-\eta)R_k}{1-\pi})^{1-\sigma}}{1-\sigma}\end{aligned} \tag{A.14}$$

Taking the derivative with respect to η and simplifying the terms, we get

$$\frac{Q_k}{(\omega+(1-\omega)\frac{\alpha+(1-\alpha)\eta Q_k}{\pi})^{\sigma}}-\frac{R_k}{(\omega\frac{R_k}{Q_k}+(1-\omega)\frac{(1-\alpha)(1-\eta)R_k}{1-\pi})^{\sigma}}=0 \tag{A.15}$$

which can be written as

$$\frac{\omega+(1-\omega)r_m}{\omega\frac{R_k}{Q_k}+(1-\omega)r_n}=\left(\frac{Q_k}{R_k}\right)^{\frac{1}{\sigma}} \tag{A.16}$$

When $R_k=Q_k$, we have $r_m=r_n$.

Now consider $\frac{Q_k}{R_k}<1$. When $\sigma>1$, given $\frac{Q_k}{R_k}$, $\left(\frac{Q_k}{R_k}\right)^{\frac{1}{\sigma}}$ is increasing in σ. So higher σ will increase the level of r_m relative to r_n. That is, people share more liquidity risks when they are more risk averse. When $\sigma \to \infty$, $\left(\frac{Q_k}{R_k}\right)^{\frac{1}{\sigma}} \to 1$, and equation (A.16) would imply that $r_m > r_n$. Once the constraint $r_m \leq r_n$ is binding, the fund sets $r_m = r_n$.

When $\sigma = 1$, (A.16) becomes

$$\frac{\omega + (1-\omega)r_m}{\omega \frac{R_k}{Q_k} + (1-\omega)r_n} = \frac{Q_k}{R_k} \tag{A.17}$$

which gives $r_n = r_m \frac{R_k}{Q_k}$. Substitute this into (A.12) and (A.13) and we get $\eta = \pi - \frac{\alpha(1-\pi)}{(1-\alpha)Q_k}$, which gives $r_m = \alpha + (1-\alpha)Q_k$ and $r_n = \alpha \frac{R_k}{Q_k} + (1-\alpha)R_k$.

A.2.2. The values of r_m, r_n and Q_k in the symmetric equilibrium

In this part, we take the initial portfolio choice α and ω as given and solve for r_m, r_n and Q_k in the symmetric equilibrium.

We can separate π into three ranges: $[0, \pi_1]$, $[\pi_1, \pi_{bind}]$ and $[\pi_{bind}, \overline{\pi}]$. Non-movers' cash is binding for $\pi \geq \pi_1$. And for $\pi > \pi_{bind}$, the constraint $r_m \leq r_n$ is binding.

Below π_1, we have $Q_k = R_k$ and $r_m = r_n = \alpha + (1-\alpha)R_k$. At π_1, the payment to movers is equal to the cash collected from non-movers plus the cash held by the fund, and we have

$$\pi_1 Z_p r_m = Z_f + (1-\pi_1)D^h \tag{A.18}$$

$$\Rightarrow \quad \pi_1 = \frac{D^h + Z_f}{D^h + Z_p r_m} = \frac{\omega + (1-\omega)\alpha}{\omega + (1-\omega)(\alpha + (1-\alpha)R_k)}$$
$$= \frac{\kappa}{\kappa + (1-\kappa)R_k} \tag{A.19}$$

where Z_p is the wealth allocated to the investment fund. Z_f is the deposits held by the investment fund. D^h is the deposit held by every household, and $(1-\pi_1)D^h$ is the total deposits collected from non-movers by selling assets. So π_1 only depends on κ.

For $\pi > \pi_1$, in the symmetric equilibrium, we have

$$\pi Z_p r_m = Z_f + (1-\pi)D^h$$
$$\Rightarrow \quad r_m = \frac{Z_f + (1-\pi)D^h}{\pi Z_p} = \frac{(1-\omega)\alpha + (1-\pi)\omega}{\pi(1-\omega)} \tag{A.20}$$

Having solved r_m, we can use (A.12), (A.13) and (A.16) to solve for equilibrium Q_k and r_n.

Between $[\pi_1, \pi_{bind}]$, the constraint $r_m \leq r_n$ is not binding. Using (A.12) and (A.13), we can write r_n as a function of r_m and Q_k

$$r_n = \frac{(1-\alpha)R_k}{1-\pi} - \frac{R_k(\pi r_m - \alpha)}{Q_k(1-\pi)} = \frac{(1-\alpha)R_k}{1-\pi} - \frac{R_k}{Q_k}\frac{\omega}{1-\omega} \tag{A.21}$$

Then substitute r_m(A.20) and r_n(A.21) into (A.16), we have

$$\frac{\omega + (1-\omega)\frac{(1-\omega)\alpha+(1-\pi)\omega}{\pi(1-\omega)}}{\omega\frac{R_k}{Q_k} + (1-\omega)\left(\frac{(1-\alpha)R_k}{1-\pi} - \frac{R_k}{Q_k}\frac{\omega}{1-\omega}\right)} = \left(\frac{Q_k}{R_k}\right)^{\frac{1}{\sigma}} \tag{A.22}$$

Arranging terms, we get

$$\frac{\omega + (1-\omega)\alpha}{\pi} = \left(\frac{Q_k}{R_k}\right)^{\frac{1}{\sigma}} \frac{(1-\omega)(1-\alpha)R_k}{1-\pi}$$

$$\implies \quad Q_k = R_k^{1-\sigma}\left(\frac{\kappa(1-\pi)}{\pi(1-\kappa)}\right)^{\sigma} \tag{A.23}$$

Substitute Q_k back to (A.21) and we get the solution for r_n

$$r_n = \frac{(1-\alpha)R_k}{1-\pi} - \left(\frac{R_k\pi(1-\kappa)}{\kappa(1-\pi)}\right)^{\sigma}\frac{\omega}{1-\omega} \tag{A.24}$$

Also, from (A.23), we have

$$\frac{Q_k}{R_k} = \left(\frac{\kappa(1-\pi)}{R_k\pi(1-\kappa)}\right)^{\sigma} \tag{A.25}$$

This ratio is equal to 1 at π_1. For $\pi > \pi_1$, the RHS is lower than 1. So given κ, R_k and π, Q_k will be lower for higher σ.

At π_{bind}, we have $r_m = r_n$. Using (A.20) and (A.24), we have

$$\begin{aligned} & r_m = r_n \\ \implies \quad & \frac{(1-\omega)\alpha+(1-\pi)\omega}{\pi(1-\omega)} = \frac{(1-\alpha)R_k}{1-\pi} - \left(\frac{R_k\pi(1-\kappa)}{\kappa(1-\pi)}\right)^{\sigma}\frac{\omega}{1-\omega} \end{aligned} \tag{A.26}$$

This equation implicitly defines π_{bind}.

Above π_{bind}, r_m is still (A.20), and we have

$$r_m = r_n = \frac{(1-\omega)\alpha+(1-\pi)\omega}{\pi(1-\omega)} \tag{A.27}$$

Using the budget constraints (A.12) and (A.13), we get

$$Q_k = \frac{(1-\pi)\omega}{(1-\omega)(1-\alpha) - \frac{(1-\pi)((1-\omega)\alpha+(1-\pi)\omega)}{\pi R_k}} \tag{A.28}$$

So the distribution for Q_k is

$$Q_k(\pi) = \begin{cases} R_k & : \quad \pi \leq \pi_1 = \frac{\kappa}{\kappa+(1-\kappa)R_k} \\ R_k^{1-\sigma}\left(\frac{\kappa(1-\pi)}{\pi(1-\kappa)}\right)^{\sigma} & : \quad \pi_1 < \pi \leq \pi_{bind} \\ \text{equation A.28} & : \quad \pi_1 > \pi_{bind} \end{cases} \tag{A.29}$$

We can see that if the constraint $r_m \leq r_n$ is not binding, then the distribution of Q_k only depends on κ. This can be seen from (A.28) where Q_k for $\pi < \pi_{bind}$ only depends on κ(remember that π_1(A.19) only depends on κ).

With the log utility function, $r_n = \frac{R_k}{Q_k} r_m$, so the constraint $r_m \leq r_n$ is never binding. As a result, under the log utility function, the distribution of Q_k only depends on κ.

A.2.3. The first order conditions for ω and α

The above analysis takes the initial portfolio choice as given. This part derives the optimal conditions for the portfolio choice of representative household and investment fund. The representative household and the investment fund will take the choices of other agents and the distribution of Q_k as given.

For notational convenience, we set $s = 1$, so $v_m = \omega + (1-\omega)r_m$ and $v_n = \omega\frac{R_k}{Q_k} + (1-\omega)r_n$. The expected utility is

$$\begin{aligned} EU &= \frac{1}{2}\int_0^1 \left[\pi\frac{(v_{m,H})^{1-\sigma}}{1-\sigma} + (1-\pi)\frac{(v_{n,H})^{1-\sigma}}{1-\sigma}\right. \\ &\quad \left. +\pi\frac{(v_{m,L})^{1-\sigma}}{1-\sigma} + (1-\pi)\frac{(v_{n,L})^{1-\sigma}}{1-\sigma}\right] dF(\pi) \end{aligned} \tag{A.30}$$

The first order condition for ω is

$$\begin{aligned} \frac{\partial EU}{\partial \omega} &= \frac{1}{2}\int_0^1 \frac{\pi(1-r_{m,H})}{(\omega+(1-\omega)r_{m,H})^{\sigma}} + \frac{(1-\pi)(\frac{R_{k,H}}{Q_{k,H}} - r_{n,H})}{(\omega\frac{R_{k,H}}{Q_{k,H}} + (1-\omega)r_{n,H})^{\sigma}} dF(\pi) \\ &\quad + \frac{1}{2}\int_0^1 \frac{\pi(1-r_{m,L})}{(\omega+(1-\omega)r_{m,L})^{\sigma}} \\ &\quad + \frac{(1-\pi)(\frac{R_{k,L}}{Q_{k,L}} - r_{n,L})}{(\omega\frac{R_{k,L}}{Q_{k,L}} + (1-\omega)r_{n,L})^{\sigma}} dF(\pi) \end{aligned} \tag{A.31}$$

The first order condition for α is

$$\begin{aligned}\frac{\partial EU}{\partial \alpha} &= \frac{(1-\omega)}{2}\int_0^1 \pi\frac{\frac{\partial r_{m,H}}{\partial \alpha}}{v^{\sigma}_{m,H}} + (1-\pi)\frac{\frac{\partial r_{n,H}}{\partial \alpha}}{v^{\sigma}_{n,H}} \\ &\quad +\pi\frac{\frac{\partial r_{m,L}}{\partial \alpha}}{v^{\sigma}_{m,L}} + (1-\pi)\frac{\frac{\partial r_{n,L}}{\partial \alpha}}{v^{\sigma}_{n,L}} dF(\pi) \end{aligned} \tag{A.32}$$

We still need to decide $\frac{\partial r_m}{\partial \alpha}$ and $\frac{\partial r_n}{\partial \alpha}$. When $\pi \leq \pi_1$, since $r_m = r_n = \alpha + (1-\alpha)R_k$, we have

$$\frac{\partial r_m}{\partial \alpha} = \frac{\partial r_n}{\partial \alpha} = 1 - R_k \tag{A.33}$$

For $\pi > \pi_1$, we first need to solve for r_m and r_n by taken Q_k as given.

First, for $\pi \in [\pi_1, \pi_{bind}]$, using (A.12) and (A.13), we can write r_m as $\frac{\alpha+(1-\alpha)\eta Q_k}{\pi}$ and r_n as $\frac{(1-\alpha)(1-\eta)R_k}{1-\pi}$. Substituting them into (A.16) and arranging terms, we get

$$\eta = \frac{-\omega-(1-\omega)\frac{\alpha}{\pi} + (\frac{Q_k}{R_k})^{\frac{1}{\sigma}}\left[\omega\frac{R_k}{Q_k} + \frac{(1-\omega)(1-\alpha)R_k}{1-\pi}\right]}{\frac{(1-\omega)(1-\alpha)Q_k}{\pi} + (\frac{Q_k}{R_k})^{\frac{1}{\sigma}}\frac{(1-\omega)(1-\alpha)R_k}{1-\pi}} \tag{A.34}$$

Substitute η into (A.12) and (A.13) and we get

$$\begin{aligned} r_m &= \frac{1}{\pi}(\alpha + Q_k(1-\alpha)\eta) \qquad\qquad (A.35)\\ &= \frac{1}{\pi}\left(\alpha + Q_k\frac{-\omega-(1-\omega)\frac{\alpha}{\pi} + (\frac{Q_k}{R_k})^{\frac{1}{\sigma}}\left[\omega\frac{R_k}{Q_k} + \frac{(1-\omega)(1-\alpha)R_k}{1-\pi}\right]}{\frac{(1-\omega)Q_k}{\pi} + (\frac{Q_k}{R_k})^{\frac{1}{\sigma}}\frac{(1-\omega)R_k}{1-\pi}}\right) \end{aligned}$$

$$\begin{aligned} r_n &= \frac{R_k(1-\alpha)(1-\eta)}{1-\pi} \qquad\qquad (A.36)\\ &= \frac{R_k}{1-\pi}\left((1-\alpha) - \frac{-\omega-(1-\omega)\frac{\alpha}{\pi} + (\frac{Q_k}{R_k})^{\frac{1}{\sigma}}\left[\omega\frac{R_k}{Q_k} + \frac{(1-\omega)(1-\alpha)R_k}{1-\pi}\right]}{\frac{(1-\omega)Q_k}{\pi} + (\frac{Q_k}{R_k})^{\frac{1}{\sigma}}\frac{(1-\omega)R_k}{1-\pi}}\right) \end{aligned}$$

So

$$\frac{\partial r_m}{\partial \alpha} = \frac{1}{\pi}\left(1 + Q_k\left(\frac{-(1-\pi)-\pi(\frac{Q_k}{R_k})^{\frac{1}{\sigma}}R_k}{(1-\pi)Q_k + \pi(\frac{Q_k}{R_k})^{\frac{1}{\sigma}}R_k}\right)\right) \tag{A.37}$$

$$\frac{\partial r_n}{\partial \alpha} = \frac{R_k}{1-\pi}\left(-1 - \left(\frac{-(1-\pi)-\pi(\frac{Q_k}{R_k})^{\frac{1}{\sigma}}R_k}{(1-\pi)Q_k + \pi(\frac{Q_k}{R_k})^{\frac{1}{\sigma}}R_k}\right)\right) \tag{A.38}$$

In the symmetric equilibrium, $(\frac{Q_k}{R_k})^{\frac{1}{\sigma}} = \frac{\kappa(1-\pi)}{R_k\pi(1-\kappa)}$(equation A.25). Rearranging terms, we get

$$\frac{\partial r_m}{\partial \alpha} = \frac{1}{\pi}\frac{\kappa(1-Q_k)}{\kappa+(1-\kappa)Q_k} \tag{A.39}$$

$$\frac{\partial r_n}{\partial \alpha} = \frac{R_k}{1-\pi}\frac{(1-\kappa)(1-Q_k)}{\kappa+(1-\kappa)Q_k} \tag{A.40}$$

For $\pi > \pi_{bind}$, since $r_m = r_n$, using (A.12) and (A.13) and taking Q_k as given, we get

$$r_m = r_n = \frac{R_k(\alpha+(1-\alpha)Q_k)}{(1-\pi)Q_k+\pi R_k} \tag{A.41}$$

and we have

$$\frac{\partial r_m}{\partial \alpha} = \frac{\partial r_n}{\partial \alpha} = \frac{R_k(1-Q_k)}{(1-\pi)Q_k+\pi R_k} \tag{A.42}$$

A.2.4. The equilibrium when $r_m \leq r_n$ is not binding

This part considers the features of the equilibrium portfolio choice when the constraint $r_m \leq r_n$ is not binding. We have the following result: First, the response curves of the household and the investment fund overlap with each other. Second, the equilibrium is defined by κ. As long as κ is equal to the equilibrium κ, then people can choose different combinations of $[\omega, \alpha]$.

First, we explain why the response curves overlap with each other when the constraint $r_m \leq r_n$ is not binding. The response curves are simply the first order conditions of ω and α(equation A.31 and A.32). Denote the response curve of the household and the investment fund as $R_{household}(\alpha)$ and $R_{fund}(\omega)$. Let $\omega(\alpha)$ denote the optimal choice of the household by taken α as given and $\alpha(\omega)$ the optimal choice of the investment fund by taking ω as given. Then on the response curves, we have $\alpha(\omega(\alpha_0)) = \alpha_0$ and $\omega(\alpha(\omega_0)) = \omega_0$. The reason is that given the distribution of Q_k, the portfolio of movers and non-movers can be written as functions of κ. So when the investment fund chooses the best α given ω, or when the household chooses ω given α, they essentially choose the best κ.

The portfolio of movers is $v_m = \omega + (1-\omega)r_m$ and the portfolio of non-movers is $v_n = \omega\frac{R_k}{Q_k} + (1-\omega)r_n$. For $\pi \leq \pi_1$, $r_m = r_n = \alpha + (1-\alpha)R_k$, and so

$$v_m = v_n = \omega + (1-\omega)(\alpha+(1-\alpha)R_k) = \kappa + (1-\kappa)R_k \tag{A.43}$$

For $\pi > \pi_1$, when Q_k is given, the solutions for r_m and r_n are (A.35) and (A.36). After some arrangement of equations, we get

$$v_m = \omega + (1-\omega)r_m = \frac{(\frac{Q_k}{R_k})^{\frac{1}{\sigma}}\frac{R_k}{Q_k}(\kappa + (1-\kappa)Q_k)}{(1-\pi) + \pi(\frac{Q_k}{R_k})^{\frac{1}{\sigma}}\frac{R_k}{Q_k}} \tag{A.44}$$

$$v_n = \omega\frac{R_k}{Q_k} + (1-\omega)r_n = \frac{\frac{R_k}{Q_k}(\kappa + (1-\kappa)Q_k)}{(1-\pi) + \pi(\frac{Q_k}{R_k})^{\frac{1}{\sigma}}\frac{R_k}{Q_k}} \tag{A.45}$$

So given Q_k, v_m and v_n can be written as functions of κ.

This implies that given the equilibrium level of κ, every combination of $[\omega, \alpha]$ that gives the same κ is an equilibrium. To see this, note that we've shown that in the symmetric equilibrium, when the constraint $r_m \leq r_n$ is not binding, π_1 and Q_k only depend on κ. From the previous analysis, we know that households and investment funds try to maximize EU given v_m and v_n specified in (A.44) and (A.45). v_m and v_n only depend on κ and the distribution of Q_k. In the equilibrium, once κ is given, the distribution of Q_k is decided, so people will be indifferent between the different combinations of $[\omega, \alpha]$ that are consistent with the optimal κ.

A.3. ADDITIONAL RESULTS: THE GENERAL CASE $\sigma \geq 1$, WITH BANK LENDING

A.3.1. The optimal payout policy when bank loan is allowed

This part proves proposition 1, the optimal payout policy given the initial portfolio choice.

Proof of proposition 1: When $Q_k = R_k$, there is no bank borrowing and $r^d = 0$, and the problem is the same as in the no lending case.

Let η_1 denote the share of assets sold on the financial market and let η_2 denote the share of assets used as collateral to borrow from banks. Define $\eta = \eta_1 + \eta_2$. When $Q_k < R_k$, the budget constraints are

$$\begin{aligned} \pi r_m &= \alpha + (1-\alpha)\eta_1 Q_k + (1-\alpha)\eta_2 Q_k \\ &= \alpha + (1-\alpha)\eta Q_k && \text{(A.46)} \\ (1-\pi)r_n &= (1-\alpha)(1-\eta)R_k && \text{(A.47)} \end{aligned}$$

Set $s = 1$. We have

$$v_m = [\omega + (1-\omega)r_m](1 + r^d) \tag{A.48}$$

$$v_n = \omega\frac{R_k}{Q_k} + (1-\omega)r_n \tag{A.49}$$

and the fund's problem (A.7) becomes

$$\pi \frac{\left[(\omega + (1-\omega)\frac{\alpha+(1-\alpha)\eta Q_k}{\pi})(1+r^d)\right]^{1-\sigma}}{1-\sigma} + (1-\pi)\frac{(\omega\frac{R_k}{Q_k} + (1-\omega)\frac{(1-\alpha)(1-\eta)R_k}{1-\pi})^{1-\sigma}}{1-\sigma} \tag{A.50}$$

Taking the derivative with respect to η and simplifying the terms, we get

$$\frac{Q_k(1+r^d)}{\left[(\omega + (1-\omega)\frac{\alpha+(1-\alpha)\eta Q_k}{\pi})(1+r^d)\right]^{\sigma}} - \frac{R_k}{(\omega\frac{R_k}{Q_k} + (1-\omega)\frac{(1-\alpha)(1-\eta)R_k}{1-\pi})^{\sigma}} = 0 \tag{A.51}$$

which can be written as (7). We can also write it as

$$\frac{\omega + (1-\omega)r_m}{\omega\frac{R_k}{Q_k} + (1-\omega)r_n} = \left(\frac{Q_k}{R_k}\right)^{\frac{1}{\sigma}} (1+r^d)^{\frac{1}{\sigma}-1} \tag{A.52}$$

When $\frac{Q_k}{R_k} < 1$, if $r^d = 0$, it is clear that the RHS of (A.52) is increasing in σ. r^d is positive only when investment funds borrow positive loans from banks. In this case, $\frac{Q_k}{R_k} = \frac{1}{1+r^l}$, and the RHS of (A.52) can be written as $\frac{Q_k}{R_k}\left(\frac{1+r^l}{1+r^d}\right)^{1-\frac{1}{\sigma}}$, which is increasing in σ since $r^l > r^d$.

When $\sigma = 1$, it is easy to see that the result for r_m and r_n is the same as in the no-bank-lending case.

A.3.2. The bank loan supply

We first prove proposition 2.

Proof: Table **??** shows the accumulated flow of payments. For example, column 1 shows the result if bank i is chosen to make the payment in subperiod 1. In subperiod 1(row 1), the outflow of payment is $\frac{(N-1)X_i}{N}$. In each subperiod $k > 1$, bank i receives $\frac{X_j}{N}$. Similarly, column 2 shows the result if the bank makes the payment in period 2. The bank receives $\frac{X_j}{N}$ in $k = 1$ (row 1), makes the payment $\frac{(N-1)X_i}{N}$ in $k = 2$(row 2), and receives $\frac{X_j}{N}$ in each of the subperiods $k > 2$.

In Table **??**, in each column n, the maximum accumulated payment is $\frac{(N-1)X_i}{N} - \frac{(n-1)X_j}{N}$, which happens in period $k = n$(the diagonal of the matrix) when a bank is chosen to make the payment. And for $k > n$, the accumulated payment is $\frac{(N-1)X_i}{N} - \frac{(k-1)X_j}{N}$. If N is very large, then

$$\begin{pmatrix}
\frac{(N-1)X_i}{N} & -\frac{X_j}{N} & -\frac{X_j}{N} & \cdots & -\frac{X_j}{N} \\
\frac{(N-1)X_i}{N} - \frac{X_j}{N} & \frac{(N-1)X_i}{N} - \frac{X_j}{N} & -\frac{2X_j}{N} & \cdots & -\frac{2X_j}{N} \\
\frac{(N-1)X_i}{N} - \frac{2X_j}{N} & \frac{(N-1)X_i}{N} - \frac{2X_j}{N} & \frac{(N-1)X_i}{N} - \frac{2X_j}{N} & \cdots & -\frac{3X_j}{N} \\
\frac{(N-1)X_i}{N} - \frac{3X_j}{N} & \frac{(N-1)X_i}{N} - \frac{3X_j}{N} & \frac{(N-1)X_i}{N} - \frac{3X_j}{N} & \cdots & -\frac{4X_j}{N} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\frac{(N-1)X_i}{N} - \frac{(N-2)X_j}{N} & \frac{(N-1)X_i}{N} - \frac{(N-2)X_j}{N} & \frac{(N-1)X_i}{N} - \frac{(N-2)X_j}{N} & \cdots & -\frac{(N-1)X_j}{N} \\
\frac{(N-1)X_i}{N} - \frac{(N-1)X_j}{N} & \frac{(N-1)X_i}{N} - \frac{(N-1)X_j}{N} & \frac{(N-1)X_i}{N} - \frac{(N-1)X_j}{N} & \cdots & \frac{(N-1)X_i}{N} - \frac{(N-1)X_j}{N}
\end{pmatrix}$$

TABLE 2.

Accumulated flow of payments. The accumulated flow in subperiod k(row k) is the total outflow minus the total inflow up to that subperiod. Column n shows the accumulated flow if bank i makes the payment in subperiod n.

$\frac{(N-1)X_i}{N} \approx X_i$. We set $1 - \frac{n-1}{N}$ as λ_{max} and $1 - \frac{k-1}{N}$ as λ, then for $k \geq n$, we can write

$$FL_{\max} = X_i + (\lambda_{\max} - 1)X_j \tag{A.53}$$

$$FL(k) = X_i + (\lambda - 1)X_j \tag{A.54}$$

The central bank loan is

$$b(k) = \max(FL(k) - D_0, 0) \tag{A.55}$$

Note that in Table ??, in each column, the accumulated flow $FL(k)$ for $k \geq n$ is the same as the $FL(k)$ in the previous column. Let $\underline{\lambda}$ denote the level of λ at which $b(k) = 0$. Using (A.54) and (A.55), we get

$$\underline{\lambda} = \frac{D_0 + X_j - X_i}{X_j} \qquad \underline{\lambda} \in [0, 1] \tag{A.56}$$

$b(k) > 0$ if $\lambda > \underline{\lambda}$.

When N is large, we can take λ as continuous, and the expected loan can be written as

$$\begin{aligned} Eb(L_i) &= \int_{\underline{\lambda}}^{1} \int_{\underline{\lambda}}^{\lambda_{\max}} b(k) d\lambda d\lambda_{\max} \\ &= \int_{\underline{\lambda}}^{1} \int_{\underline{\lambda}}^{\lambda_{\max}} \left([X_i + (\lambda - 1)X_j] - D_0 \right) d\lambda d\lambda_{\max} \end{aligned} \tag{A.57}$$

The integral of $b(k)$ over $[\underline{\lambda}, \lambda_{\max}]$ is the borrowing for each realized n(i.e., each column of the matrix). The integral over $[\underline{\lambda}, 1]$ denotes the changes in $\lambda_{\max}$ caused by the changes in n(i.e., different columns of the matrix). $b(k)$ is positive only when λ and $\lambda_{\max}$ are $> \underline{\lambda}$.

$$
\begin{aligned}
Eb(L_i) &= \int_{\underline{\lambda}}^{1} \int_{\underline{\lambda}}^{\lambda_{\max}} [X_i - X_j - D_0 + \lambda X_j]\, d\lambda d\lambda_{\max} \\
&= \int_{\underline{\lambda}}^{1} \left[(\lambda_{\max} - \underline{\lambda})(X_i - X_j - D_0) + \frac{\lambda_{\max}^2 - \underline{\lambda}^2}{2} X_j \right] d\lambda_{\max} \\
&= \left(\frac{\lambda_{\max}^2}{2}\Big|_{\underline{\lambda}}^{1} - \lambda_{\max}\Big|_{\underline{\lambda}}^{1}\underline{\lambda} \right)(X_i - X_j - D_0) + \frac{1}{2}\left(\frac{\lambda_{\max}^3}{3}\Big|_{\underline{\lambda}}^{1} - \lambda_{\max}\Big|_{\underline{\lambda}}^{1}\underline{\lambda}^2 \right) X_j \\
&= \left(\frac{1-\underline{\lambda}^2}{2} - (1-\underline{\lambda})\underline{\lambda} \right)(X_i - X_j - D_0) + \frac{1}{2}\left(\frac{1-\underline{\lambda}^3}{3} - (1-\underline{\lambda})\underline{\lambda}^2 \right) X_j
\end{aligned} \tag{A.58}
$$

Replacing $\underline{\lambda}$ with (A.56) and arranging terms, we get

$$
\begin{aligned}
Eb(L_i) &= \frac{1}{6}\frac{(X_i - X_j - D_0)^3}{X_j^2} + \frac{1}{2}\frac{(X_i - X_j - D_0)^2}{X_j} \\
&\quad + \frac{1}{2}(X_i - X_j - D_0) + \frac{1}{6}X_j
\end{aligned} \tag{A.59}
$$

■

In the general case, we have

$$X_i = Z_f + (1-\pi)D^h + L_i \tag{A.60}$$

$$X_j = Z_f + (1-\pi)D^h + L_j \tag{A.61}$$

where Z_f is the riskless asset of the investment fund, $(1-\pi)D^h$ is the money collected from non-movers. The method is the same and it can be shown that in the symmetric case we have

$$R = 1 + \delta + r^c \frac{1}{2}\left(1 - \frac{D_0}{X}\right)^2 \tag{A.62}$$

$$
\begin{aligned}
r^d &= \frac{\frac{Lr^c}{2}(1 - \frac{D_0}{X})^2 - r^c Eb}{D_0 + L} \\
&= \frac{\frac{Lr^c}{2}(1 - \frac{D_0}{X})^2 - r^c(\frac{-D_0^3}{6X^2} + \frac{D_0^2}{2X} - \frac{D_0}{2} + \frac{X}{6})}{D_0 + L}
\end{aligned} \tag{A.63}
$$

where D_0 is $D^h + Z_f$ and $X = Z_f + (1-\pi)D^h + L$.

A.3.3. The equilibrium solutions for r_m, r_n, Q_k, L, R and r^d

This part derives the equilibrium solutions in period $t+1$ by taken ω and α as given. We first consider the case in which the constraint $r_m \leq r_n$ is not binding.

Recall that at π_2, investment funds start to borrow from banks. And at π_3, banks start to borrow from the central bank. Everything for $\pi < \pi_2$ is the same as in the no-bank-lending case. In order to get the solution for $\pi \geq \pi_2$, we first decide π_2 and π_3.

Derive π_2 and π_3

π_2 can be decided as follows. At π_2, $R = 1+\delta$, $Q_k = \frac{R}{1+\delta}$ and $L = 0$. At the same time, $\frac{Q_k}{R_k}$ should satisfy (A.25), and so we have

$$\frac{Q_k}{R_k} = \frac{1}{1+\delta} = \left(\frac{\kappa(1-\pi)}{R_k \pi_2 (1-\kappa)}\right)^{\sigma} \tag{A.64}$$

$$\Rightarrow \quad \pi_2 = \frac{\kappa}{R_k(1-\kappa)\left(\frac{1}{1+\delta}\right)^{\frac{1}{\sigma}} + \kappa} \tag{A.65}$$

π_3 can be decided as follows. At π_3, $R = 1+\delta$, $r^d = 0$ and $Q_k = \frac{R_k}{1+\delta}$. At π_3, $X = D_0$. Since $X = Z_f + (1-\pi)D^h + L$ and $D_0 = Z_f + D^h$, so $L = \pi D^h$. Thus, we have

$$\pi Z_p r_m = Z_f + (1-\pi)D^h + \pi D^h \tag{A.66}$$

$$\Rightarrow \pi r_m = \alpha + (1-\pi)\frac{\omega}{1-\omega} + \pi\frac{\omega}{1-\omega} = \alpha + \frac{\omega}{1-\omega} \tag{A.67}$$

$$\Rightarrow r_m = \frac{1}{\pi}\left(\alpha + \frac{\omega}{1-\omega}\right) \tag{A.68}$$

Comparing (A.46) and (A.67), we have

$$(1-\alpha)\eta Q_k = \frac{\omega}{1-\omega} \Rightarrow \eta = \frac{\omega}{(1-\omega)(1-\alpha)Q_k} \tag{A.69}$$

Substituting η into (A.47) and we have

$$r_n = \frac{1}{1-\pi}\left((1-\alpha) - \frac{\omega}{(1-\omega)Q_k}\right) R_k \tag{A.70}$$

Then substitute (A.68) and (A.70) into the optimal payout policy (A.52) and we have

$$\frac{\omega + (1-\omega)\frac{1}{\pi}\left(\alpha + \frac{\omega}{1-\omega}\right)}{\omega(1+\delta) + (1-\omega)\frac{1}{1-\pi}\left((1-\alpha) - \frac{\omega}{(1-\omega)Q_k}\right) R_k} = \left(\frac{1}{1+\delta}\right)^{\frac{1}{\sigma}} \tag{A.71}$$

Arranging terms, we get

$$\omega\left[(1+\delta)^{1-\frac{1}{\sigma}}-1\right]\pi^2-\pi\left[\kappa-\omega+(1+\delta)^{-\frac{1}{\sigma}}(1-\kappa)R_k\right]+\kappa=0 \quad \text{(A.72)}$$

When $\sigma=1$, the solution is $\frac{\kappa}{\kappa-\omega+(1-\kappa)\frac{R_k}{1+\delta}}$. When $\sigma>1$, the smaller one of the two solutions is π_3. Note that given κ, π_3 is affected by ω. For example, if $\omega=0$ (all riskless assets are held by the investment fund), then $\pi_3=\pi_2$.

The distribution for Q_k takes the following form:

$$Q_k(\pi)=\begin{cases} R_k & : \ \pi\le\pi_1=\pi_1=\frac{\kappa}{\kappa+(1-\kappa)R_k} \\ R_k^{1-\sigma}\left(\frac{\kappa(1-\pi)}{\pi(1-\kappa)}\right)^{\sigma} & : \ \pi_1<\pi<\pi_2=\frac{\kappa}{R_k(1-\kappa)\left(\frac{1}{1+\delta}\right)^{\frac{1}{\sigma}}+\kappa} \\ \frac{R_k}{1+\delta} & : \ \pi_2\le\pi\le\pi_3 \\ \frac{R_k}{R(\pi)} & : \ \pi>\pi_3 \end{cases} \quad \text{(A.73)}$$

Equilibrium solutions over π_2 and π_3

Over $[\pi_2,\pi_3]$, $R=1+\delta$, $r^d=0$ and $Q_k=\frac{R_k}{1+\delta}$. We still need to decide r_m, r_n and L. Since $r^d=0$, (A.52) is the same as (A.16) , and the solution for η, r_m and r_n are simply (A.34), (A.35) and (A.36) with $Q_k=\frac{R_k}{1+\delta}$. Knowing η, we can decide L from the budget constraint (A.46). Since L is equal to the total external cash minus the cash from non-movers, so

$$L=Z_p(1-\alpha)\eta Q_k-(1-\pi)D^h=S[(1-\omega)(1-\alpha)\eta Q_k-(1-\pi)\omega] \quad \text{(A.74)}$$

Equilibrium solutions for $\pi>\pi_3$

We will set L as the variable that we try to solve, and we express all other variables as a function of L. The equilibrium is defined by the following conditions. 1.The budget constraints (A.46) and (A.47); 2.The optimal payout policy (A.52); 3.Asset Price on the financial market: $Q_k=\frac{R_k}{R}$; 4.The cash paid to movers is equal to the fund's own money plus the money raised from the financial market and the bank.

$$\pi Z_p r_m = Z_f+(1-\pi)D^h+L \quad \text{(A.75)}$$

5.The loan supply curve (A.62) which defines the relationship between L and R; 6.r^d(equation A.63) derived from the zero expected profit condition. We can write (A.62) as $R(L)$ and (A.63) as $r^d(L)$. Then $Q_k(L)=\frac{R_k}{R(L)}$. We can also write (A.75) as

$$\begin{aligned} r_m &= \frac{1}{\pi Z_p}(Z_f+(1-\pi)D^h+L) \\ &= \frac{1}{\pi}\left(\alpha+(1-\pi)\frac{\omega}{1-\omega}+\frac{L}{S(1-\omega)}\right) \end{aligned} \quad \text{(A.76)}$$

which we define as $r_m(L)$. Then using the two budget constraints (A.46) and (A.47), we have

$$r_n = \frac{R_k(1-\alpha)}{1-\pi} - \frac{R(L)(\pi r_m(L) - \alpha)}{1-\pi} \tag{A.77}$$

which we define as $r_n(L)$. Substitute $r_m(L)$, $r_n(L)$, $Q_k(L)$, $R(L)$, and $r^d(L)$ into the optimal payout policy (A.52), and we can get an equation in which the *only* unknown is L:

$$\frac{\omega + (1-\omega)r_m(L)}{\omega R(L) + (1-\omega)r_n(L)} = \left(\frac{1}{R(L)}\right)^{\frac{1}{\sigma}} (1 + r^d(L))^{\frac{1}{\sigma}-1} \tag{A.78}$$

where $R(L)$, $r_d(L)$, $r_m(L)$ and $r_n(L)$ are (A.62), (A.63), (A.76), and (A.77). This equation implicitly defines the equilibrium L. After deciding L, all other variables can then be decided.

When $r_m \leq r_n$ is binding

Let π_{bind} denote the π above which the constraint $r_m \leq r_n$ is binding. We first consider the case when $\pi_1 < \pi_{bind} < \pi_2$. At π_2, r_m is still (A.20), and r_n is (A.41) with $Q_k = \frac{R_k}{1+\delta}$, equating r_m and r_n gives the value of π_2. At π_3, $Q_k = \frac{R_k}{1+\delta}$. r_m is (A.68) and r_n is (A.70). Equating r_m and r_n gives π_3.

For equilibrium values of variables. For $\pi \leq \pi_{bind}$, everything is the same as in the non-binding case. For $[\pi_{bind}, \pi_2]$, we use (A.27) and (A.28). Over $[\pi_2, \pi_3]$, r_m and r_n are (A.41) with $Q = \frac{R_k}{1+\delta}$. For $\pi > \pi_3$, we can solve the equilibrium using the same method as in the non-binding case, the only difference is that instead of using condition (A.78), we use the condition $r_m(L) = r_n(L)$.

If $\pi_{bind} \in [\pi_2, \pi_3]$ or $\pi_{bind} > \pi_3$, then we can decide π_{bind} using simulation methods. The method for deciding the equilibrium values is the same as explained above.

A.3.4. The first order conditions for ω and α

This part derives the optimal conditions for portfolio choices of households and investment funds. Using EU(A.30), v_m(A.48) and v_n(A.49), we get

$$\begin{aligned}\frac{\partial EU}{\partial \omega} &= \frac{1}{2}\int_0^1 \frac{\pi(1-r_{m,H})(1+r_H^d)}{[(\omega+(1-\omega)r_{m,H})(1+r_H^d)]^\sigma} + \frac{(1-\pi)(\frac{R_{k,H}}{Q_{k,H}} - r_{n,H})}{(\omega\frac{R_{k,H}}{Q_{k,H}} + (1-\omega)r_{n,H})^\sigma} dF(\pi) \\ &+ \frac{1}{2}\int_0^1 \frac{\pi(1-r_{m,L})(1+r_L^d)}{[(\omega+(1-\omega)r_{m,L})(1+r_L^d)]^\sigma} + \frac{(1-\pi)(\frac{R_{k,L}}{Q_{k,L}} - r_{n,L})}{(\omega\frac{R_{k,L}}{Q_{k,L}} + (1-\omega)r_{n,L})^\sigma} dF(\pi)\end{aligned} \tag{A.79}$$

The first order condition for α is

$$\begin{aligned}\frac{\partial EU}{\partial \alpha} &= \frac{(1-\omega)}{2}\int_0^1 \left(\pi \frac{\frac{\partial r_{m,H}}{\partial \alpha}(1+r_H^d)}{v_{m,H}^{\sigma}} + (1-\pi)\frac{\frac{\partial r_{n,H}}{\partial \alpha}}{v_{n,H}^{\sigma}}\right. \\ &\quad \left. +\pi \frac{\frac{\partial r_{m,L}}{\partial \alpha}(1+r_L^d)}{v_{m,L}^{\sigma}} + (1-\pi)\frac{\frac{\partial r_{n,L}}{\partial \alpha}}{v_{n,L}^{\sigma}}\right) dF(\pi) \qquad \text{(A.80)}\end{aligned}$$

We need to decide $\frac{\partial r_m}{\partial \alpha}$ and $\frac{\partial r_n}{\partial \alpha}$. When $\pi \leq \pi_1$, since $r_m = r_n = \alpha + (1-\alpha)R_k$, we get $\frac{\partial r_m}{\partial \alpha} = \frac{\partial r_n}{\partial \alpha} = 1 - R_k$. For $\pi > \pi_1$, we first need to solve for r_m and r_n by taken Q_k and r^d as given. Note that equations (A.46) and (A.47) are the same as (A.12) and (A.13), and the only difference between (A.16) and (A.52) is that the RHS is changed from $\left(\frac{Q_k}{R_k}\right)^{\frac{1}{\sigma}}$ into $\left(\frac{Q_k}{R_k}\right)^{\frac{1}{\sigma}} (1+r^d)^{\frac{1}{\sigma}-1}$. It turns out that we only need to modify the solutions of η, r_m, r_n in the no-lending case(A.34, A.35 and A.36) by changing $\left(\frac{Q_k}{R_k}\right)^{\frac{1}{\sigma}}$ into $\left(\frac{Q_k}{R_k}\right)^{\frac{1}{\sigma}} (1+r^d)^{\frac{1}{\sigma}-1}$. And so $\frac{\partial r_m}{\partial \alpha}$ and $\frac{\partial r_n}{\partial \alpha}$ are equations (A.37) and (A.38) with the term $\left(\frac{Q_k}{R_k}\right)^{\frac{1}{\sigma}}$ replaced by $\left(\frac{Q_k}{R_k}\right)^{\frac{1}{\sigma}} (1+r^d)^{\frac{1}{\sigma}-1}$.

If the constraint $r_m \leq r_n$ is binding, then for $\pi > \pi_{bind}$, $\frac{\partial r_m}{\partial \alpha}$ and $\frac{\partial r_n}{\partial \alpha}$ are the same as (A.42).

A.4. THE PROOF OF THE REMAINING PROPOSITIONS

Proof of proposition 3: Eliminating R from (25) and (28), we get a quadratic equation for X.

$$X^2\left(1+\delta+\frac{r^c}{2}\right) - X\left(\frac{r^c}{2}D_0 + \pi Z_k R_k\right) + \frac{r^c}{2}D_0^2 = 0 \qquad \text{(A.81)}$$

Z_k is risky investments of the investment fund. After we solve for X, we have $L^*(\pi) = X-(1-\pi)D_0$. $R^*(\pi)$ is decided according to (28). Finally, we can simplify the results using the relationship $D_0 = \omega S$ and $Z_k = (1-\omega)S$.
Proof of proposition 5: Below π_1, the cash constraint of non-movers is not binding and the cash of non-movers is more than enough to absorb the sale of assets, thus, $Q_k = R_k$. At π_1, non-movers use all their cash to buy assets, and Q_k is still equal to R_k, we get

$$\pi_1 Z_k R_k = (1-\pi_1)D_0 \qquad \text{(A.82)}$$

Using $D_0 = \omega S$ and $Z_k = (1-\omega)S$, we get the solution for π_1 in (35). Non-movers use all their deposits to buy assets if $\pi \geq \pi_1$.

Between π_1 and π_2, Q_k is lower than R_k, but since $Q_k > \frac{R_k}{1+\delta}$, it is still not worthwhile for the investment funds to borrow from banks, and Q_k is decided according to

$$\text{optimal payout} = \text{redemption} \;\Rightarrow\; \pi Z_k Q_k = (1-\pi) D_0 \tag{A.83}$$

Q_k decreases to $\frac{R_k}{1+\delta}$ at π_2, and investment funds start to borrow from banks when $\pi > \pi_2$. π_2 can be decided by replacing Q_k in (A.83) with $Q_k = \frac{R_k}{1+\delta}$. Between π_2 and π_3, the lending rate is $R = 1+\delta$(equation 20), and $Q_k = \frac{R_k}{1+\delta}$.

π_3 is the level of π above which the expected central bank loan is positive. At π_3, Eb is exactly zero and $L(\pi_3) = \pi_3 D_0$. Since $Q_k(\pi_3)$ is still $\frac{R_k}{1+\delta}$, we have

$$\begin{aligned} \pi_3 &= \frac{\text{Redemption}}{Z_k Q_k} = \frac{(1-\pi_3)D_0 + L(\pi_3)}{Z_k \frac{R_k}{1+\delta}} \\ &= \frac{D_0}{Z_k \frac{R_k}{1+\delta}} = \frac{\omega}{(1-\omega)\frac{R_k}{1+\delta}} \end{aligned} \tag{A.84}$$

Above π_3, $Eb > 0$, and $Q_k = \frac{R_k}{R(\pi)}$, where $R(\pi)$ is defined in (32).

Proof of proposition 6: For notational convenience, we set household endowment at 1. First, given κ, the total wealth of the economy $\kappa + (1-\kappa)R_k$ is decided. In addition, π_1 only depends on κ. At π_1, both r_m and r_n are still equal to the fundamental value of the fund: $r_m = r_n = \alpha + (1-\alpha)R_k$, and we also know that the payment to movers is equal to the cash of the investment fund Z_f plus the cash collected from non-movers $(1-\pi)D^h$, so we have

$$\begin{aligned} & \pi_1 Z_p r_m = Z_f + (1-\pi_1)D^h \\ \Rightarrow \quad & \pi_1 = \frac{D^h + Z_f}{D^h + Z_p r_m} = \frac{\omega + (1-\omega)\alpha}{\omega + (1-\omega)(\alpha + (1-\alpha)R_k)} \\ & = \frac{\kappa}{\kappa + (1-\kappa)R_k} \end{aligned} \tag{A.85}$$

For $\pi \leq \pi_1$,

$$v_m = v_n = \omega + (1-\omega)(\alpha + (1-\alpha)R_k) = \kappa + (1-\kappa)R_k. \tag{A.86}$$

For $\pi > \pi_1$, movers carry all the cash κ with them, which means the risky assets will become the wealth of non-movers.

$$\pi v_m = \kappa \;\Rightarrow\; v_m = \frac{\kappa}{\pi} \tag{A.87}$$

$$(1-\pi)v_n = (1-\kappa)R_k \;\Rightarrow\; v_n = \frac{(1-\kappa)R_k}{1-\pi} \tag{A.88}$$

Thus, the distribution of v_m and v_n only depends on κ.
Proof of proposition 7: Once ω and α are given, κ is given, and π_1 is uniquely decided. For $\pi \leq \pi_1$, we have $r_m = r_n = \alpha + (1-\alpha)R_k$. For $\pi > \pi_1$, all the cash owned by investment funds and non-movers are used to pay movers, and we have

$$\begin{aligned} & \pi Z_p r_m = Z_f + (1-\pi)D^h \\ \Rightarrow \quad & r_m = \frac{Z_f + (1-\pi)D^h}{\pi Z_p} = \frac{(1-\omega)\alpha + (1-\pi)\omega}{\pi(1-\omega)} \end{aligned} \tag{A.89}$$

which is the same for different σ. We've shown that between π_1 and π_{bind}, we have

$$\frac{Q_k}{R_k} = \left(\frac{\kappa(1-\pi)}{R_k \pi(1-\kappa)}\right)^{\sigma} \tag{A.90}$$

This ratio is equal to 1 at π_1. For $\pi > \pi_1$, since $Q_k < R_k$, the right-hand-side should be lower than 1, which also implies that given κ, R_k and π, Q_k will be lower for higher σ.

Further explanations for the results in section 5.2
Using the same method as in the no-bank-lending case, we can prove the result that when $r_m \leq r_n$ is not binding, the optimal choice is defined by the optimal κ. The details are omitted. We have proved in proposition 1 that when $U = \ln c$, we have $\frac{r_n}{r_m} = \frac{R_k}{Q_k} \geq 1$, so $r_m \leq r_n$ can never be binding.

Our numerical results show that people will choose the initial portfolio such that $r_m \leq r_n$ is not binding. The example for $\sigma = 2$ (without bank lending) is shown in Figure 11. There exists a level of $\alpha = \alpha_{bind}$. When $\alpha < \alpha_{bind}$, the constraint $r_m \leq r_n$ is binding for positive probability. We find that it will cause $R_{fund}(\omega)$(the optimal choice of the investment fund given ω of households) to be slightly higher than $R_{household}(\alpha)$(the optimal choice of the household given α of investments). This means equilibrium will not be reached for $\alpha < \alpha_{bind}$, because in this range, people will reduce ω and increase α. For $\alpha \geq \alpha_{bind}$, the constraint $r_m \leq r_n$ is not binding. The two response curves overlap with each other, and the equilibrium is defined by the equilibrium κ. The results when there is bank lending can be analyzed using similar methods, and the details are omitted.

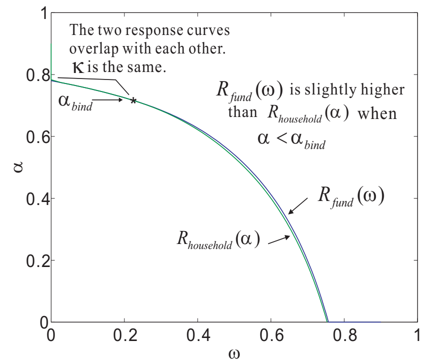


FIG. 11. Response curves when $\sigma = 2$

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