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Extraneous Risk: Pricing of Non-Systematic Risk*

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We study a simple equilibrium model where aggregate stock market quantity can be affected by non-systematic risk. In the model, investors perceive the growth rate of the aggregate dividend differently. Further, they are confused between the aggregate risk and the extraneous risk so that they form different expectations about the dividend growth rate through extraneous risk. As a result, extraneous risk affects aggregate equilibrium quantities. We derive equilibrium quantities and investigate how an extraneous risk priced via investors' different beliefs can affect the equilibrium at the aggregate level. The model provides a pricing of non-systematic risk in equilibrium without assuming an incomplete financial market or an under diversification.

Key Words: Extraneous risk; Heterogeneous beliefs; General equilibrium.
JEL Classification Numbers: G00, G02, G10, G11, G12.

1. INTRODUCTION

Traditional asset pricing literature has focused on the rational behavior of investors in the economy and said the only relevant (priced) risk is the only the systematic market risk factor. CAPM is the benchmark model on this strand. There have been some other variant models like APT(Arbitrage Pricing Model), Linear Factor Model, and etc. which al-

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lowed some other risk factors other than the market risk. However all of these models explicitly assume the rationality of economic agents. In this tradition, we have two main implications. First, all the economic agents look at the economic phenomena in the same way as they are all rational. Second, only the systematic risk matters in asset pricing.

However there are many empirical evidence indicating these notions do not hold very well. First, we know that difference of opinions among investors can play an important roles in the economy. Traditional literature goes back to Harrison and Kreps (1978), Varian (1985), Varian (1989), Abel (1989), Harris and Raviv (1993) in discrete time analysis. Williams (1977), Wang (1996), Detemple and Murthy (1997) and Basak (2005) provided continuous time analysis with heterogeneous beliefs. Earlier works like Lintner (1969), Miller (1977), Jarrow (1980), show that, in a simple two-period framework, the heterogeneous beliefs among investors make the financial constraints, i.e., short-sale constraint, work more importantly in the model.[1] Also the difference of opinions have been shown to be priced in the cross-section. Diether, Malloy, and Scherbina (2002) find that the analysts' forecasting dispersion is actually being priced. Buraschi and Jiltsov (2006) show that belief heterogeneity can help explain option's trading volume. Xiong and Yan (2010) argue that the belief difference on inflation among investors better explains the volatility amplifications of bond yield and possibly provide an explanation for why hump-shaped forward rate combination can predict the future bond yield. Gallmeyer and Hollifiled (2008) find that short sale constraint can have significant effect on the market equilibrium quantities through difference of opinions.

Second, systematic risk might not be the only priced factor.[2] For instance, non-systematic (extraneous or idiosyncratic) risk can obviously be priced when investors hold under-diversified portfolio. Even in the (seemingly well diversified) portfolio level, there is some empirical evidence indicating non-systematic (or idiosyncratic) risk can be priced. In particular, Ang, Hodrick, Xing, and Zhang (2006) document the evidence that idiosyncratic return volatility is priced in the cross-section. Besides, there is another literature indicating that idiosyncratic risk in consumption (growth) might play an important role in asset pricing. Mankiw (1986) and Constantinides and Duffie (1996) show that counter-cyclical idiosyncratic volatility in consumption growth might be a source of high equity premium. Constantinides and Ghosh (2014) investigate an incomplete market model and argue that counter-cyclical left-skewness in the cross-sectional distribution of household consumption growth has important asset pricing impli-

[1] Recently, Sun and Yang (2003) show that a competitive equilibrium can exist under general conditions such as heterogeneous beliefs and short-sale constraint.

[2] The role of non-systematic risk in economics has long been explored. See Cass and Shell (1983).

cations. Storesletten, Telmer, and Yaron (2004) and Storesletten, Telmer, and Yaron (2007) show that counter-cyclical variation in idiosyncratic labor income variance has important implications in equity premium.

In this study, we explore a simple asset pricing model in a complete financial market in order to price non-systematic risk in the aggregate. In usual, a conventional complete market asset pricing model does not allow the pricing of non-systematic risk in equilibrium due to the diversification. Therefore, pricing of non-systematic risk has been investigated mostly in the setting of incomplete market with idiosyncratic labor income risk or the assumption of under-diversification of portfolios due to some constraints. However we bypass the situation of diversification-away of non-systematic risk in a complete market model by using investors' heterogeneous beliefs. In particular, we introduce two crucial assumptions. First, investors agree to disagree to each other about the mean growth rate of the aggregate dividend process. This is similar to conventional belief difference literature (Varian (1985), Varian (1989), and Basak (2000)). Second, when investors evaluate their expectations about the mean aggregate dividend growth, they rely on non-systematic risk factor as they get confused between aggregate and non-systematic factors. This assumptions is partly an application of rational inattention hypothesis. Rational inattention hypothesis implies that investors show different interpretations when it comes to less important information due to their limited cognitive resources. Thus investors likely to show differences in their interpretations about individual (or non-systematic) information. We further assume that investors can get confused in evaluating mean of aggregate dividend growth. Though we do not have empirical evidence indicating that investors mis-interpolate the mean aggregate dividend growth with non-systematic information, it might be plausible that investors carry out their different interpretations of individual informations to the aggregate level by mistake. Technically speaking, aforementioned assumption is used to let non-systematic risk survive in equilibrium in a complete financial market economy.

Extraneous risk is priced in equilibrium via investors' different beliefs. Both aggregate and individual risky assets are exposed to extraneous risk. Investors' equilibrium consumptions are explicitly affected by extraneous risk via belief difference Riskless interest rate becomes lower than the conventional counterpart due to precautionary savings motif from extra consumption volatility caused by extraneous risk factor. Though aggregate expected excess return is explicitly impacted by extraneous risk via belief difference, there are several different possibilities on how aggregate expected return can be high compared to the conventional counterpart. The key that the pricing of extraneous risk depends on the magnitude of investors' belief difference.

This paper is organized as follows: section 2 introduces our model. In section 3, we derive equilibrium quantities and discuss equilibrium impacts of idiosyncratic cash ow risk qualitatively. Details of proofs and additional equilibrium results are in the appendix.

2. THE ECONOMY

In order to study the market-wide influence of an extraneous risk, we consider a continuous-time, pure-exchange security market economy with heterogeneous investors. Our model is similar to Basak (2005) and Gallmeyer and Hollifiled (2008).[3] There is a single perishable consumption good that acts as a numeraire. For modeling simplicity in investor heterogeneity, we assume that there are just two types of investors in the economy: one is optimistic, and the other is pessimistic in their beliefs. This simple model is set to highlight how an extraneous risk can enter the equilibrium via investors' different beliefs.[4]

2.1. Information structure and investors' perceptions

Two investors ($i = 1, 2$) commonly observe the aggregate cash flow or the dividend process. We assume that the exact aggregate cash flow process is written as follows.

$$d\delta_t = \mu_{\delta,t}dt + \sigma_{\delta,A}dB_A, \tag{1}$$

where $\delta_t \equiv \ln D_t$ and D_t is the aggregate cash flow at time t. Also there is a mean-reverting process describing the movement of the drift as follows.

$$d\mu_{\delta,t} = (\bar{\mu}_\delta - \mu_{\delta,t})dt + \sigma_{\mu_\delta,A}\mu_{\delta,t}dB_A. \tag{2}$$

Now we introduce main assumptions. Investors disagree to each other about the mean of the drift, $\bar{\mu}_\delta$.[5] Thus they have different evaluation or perception of this value so that it can be represented as $\bar{\mu}_\delta^{(i)}$, $i = 1, 2$. Furthermore, investors are subject to confusion when they form their own value, $\bar{\mu}_\delta^{(i)}$. In principle, investors are supposed to form their expectation, $\bar{\mu}_\delta^{(i)}$, based on aggregate or economy-wide systematic risk, dB_A. However,

[3]Early works on investors' heterogeneous beliefs are Lintner (1965), Miller (1977), and Harrison and Kreps (1978). Detemple and Murthy (1997) study the effect of belief dierences in a production economy. For exchange economies, see Zapatero (1998), Basak (2000), Basak (2005), Buraschi and Jiltsov (2006), Jouini and Napp (2007), Gallmeyer and Hollifiled (2008), David (2008), Dumas, Kurshev, and Uppal (2009), Weinbaum (2009) and others.

[4]As will be clear later, the absolute value of their beliefs are not important. Only the magnitude of their belief differences matter.

[5]The theoretical foundation for this "agree-to-disagree" assumption can be found in Varian (1985), Harris and Raviv (1993), and Morris (1994).

we assume that investors are confused between aggregate risk and extraneous risk at the formation of their expectations.[6] An extraneous risk can be thought of as a risk source that is (statistically) orthogonal to aggregate risk. Also it stems from pure individual firm level information that is not relevant to aggregate information so that they can be technically diversified away. In this study, we assume that an extraneous risk represents individual or idiosyncratic risk that is independent of aggregate risk so that we can keep the model as simple as possible. This will be clear later when we introduce an individual asset's price process. Thus investor i's perceived drift process changes as follows.

$$d\mu_{\delta,t} = \left(\bar{\mu_\delta}^{(i)} - \mu_{\delta,t}\right) dt + \sigma_I dB_I^{(i)}, \quad i = 1, 2, \tag{3}$$

where dB_I represents an extraneous risk that is independent of the aggregate risk dB_A.

Our assumption that investors show difference in their opinions on the mean of the drift can partially be justified by a rational inattention hypothesis that has recently been suggested.[7] Rational inattention hypothesis argue that investors (or economic agents) process important information first and then go and process remaining information if they still have the cognitive capacity. Thus it is very plausible that significant heterogeneity among investors will arise in the latter stage. In our study, we assume that aggregate risk is important information and investors agree to each other on this information. Further, we assume that investors show differences in their information processing with the remaining information (that is, in general, individual information) due to differences in their cognitive capacities. In short, investors have different opinions about the mean of the drift. In doing so, they exercise their evaluation based on individual information (or extraneous risk), not aggregate information (or aggregate risk). Thus the equation (3) can be justified.

Consistency condition from the equation (3) implies that

$$dB_I^{(i)} = dB_I + \left(\frac{\bar{\mu}_\delta - \bar{\mu}_\delta^{(i)}}{\sigma_I}\right) dt. \tag{4}$$

[6]In most extant literature on different beliefs, investors form their different beliefs through aggregate risk (or aggregate information). Our model differs from those studies.

[7]See Sims (2003), Sims (2006), Xiong and Peng (2006).

Thus we have

$$\begin{aligned} dB_I^{(2)} - dB_I^{(1)} &= \left[\left(\frac{\bar{\mu}_\delta - \bar{\mu}_\delta^{(2)}}{\sigma_I} \right) - \left(\frac{\bar{\mu}_\delta - \bar{\mu}_\delta^{(1)}}{\sigma_I} \right) \right] dt \\ &= \left[\frac{\bar{\mu}_\delta^{(1)} - \bar{\mu}_\delta^{(2)}}{\sigma_I} \right] dt \\ &= \bar{\eta}_t dt, \end{aligned} \tag{5}$$

where $\bar{\eta}_t \equiv \left[\frac{\bar{\mu}_\delta^{(1)} - \bar{\mu}_\delta^{(2)}}{\sigma_I} \right]$ is called the belief difference kernel. $\bar{\eta}_t$ plays an important role later in belief difference process.

2.2. Financial Market

We assume a complete financial market. We have two risky assets. One is the market portfolio and the other is any (representative) individual risky asset. Also there is one riskless asset. Without loss of generality, we assume that any individual asset has risk exposures on both systematic risk and extraneous risk. [8]

Due to investors' different beliefs about the mean of the aggregate dividend growth rate, extraneous risk will not be diversified away.In other words, an extraneous risk still exists even after the diversification since investors still have different evaluations on the mean of the growth rate of aggregate dividend process via extraneous risk. Thus the aggregate asset (market portfolio), and an individual asset have risk exposures both on systematic risk and an extraneous risk.

Perceived price processes are given by

$$\begin{aligned} \frac{dP_{m,t} + D_t}{P_{m,t}} &= \mu_{P_m,t}^{(i)} dt + \sigma_{P_m,A} dB_A + \sigma_{P_m,I} dB_I^{(i)}, \\ \frac{dP_{s,t} + D_{s,t}}{P_{s,t}} &= \mu_{P_s,t}^{(i)} dt + \sigma_{P_s,A} dB_A + \sigma_{P_s,I} dB_I^{(i)}, \end{aligned} \tag{6}$$

and the riskless interest rate is given by r_t.

[8] To be consistent with the aggregate dividend modeling, we can also model individual dividend (cash flow) process, $D_{s,t}$ ($\sum_s D_{s,t} \equiv D_t$), where $D_t \equiv e^{\delta_t}$. Individual dividend $D_{s,t}$ can be any individual asset other than aggregate dividend. We assume that any individual asset has two risk exposures. One is systematic and the other is an extraneous risk. For more general modeling of the individual dividend process, we can rely on a share process, $s_t \equiv D_s(t)/D(t)$ that was introduced in Menzley, Santos, Veronesi (2004). With the share process, any individual process $D_{s,t}$ is defined as $s_t \times D_t$ and its diffusion process can easily be specified with two, one systematic and another extraneous, risk exposures by assuming that any individual asset has only one extraneous risk exposure.

Due to the market completeness, a perceived stochastic discount factor process is expressed as

$$\frac{d\xi_t^{(i)}}{\xi_t^{(i)}} = -r_t dt - \kappa_A^{(i)} dB_A + \kappa_I^{(i)} dB_I^{(i)}, \tag{7}$$

where $\kappa_A(i)$ and $\kappa_I^{(i)}$ are perceived market prices of aggregate risk and extraneous risk respectively.

The market prices of risks are computed as follows.

$$\begin{pmatrix} \sigma_{P_m,A} & \sigma_{P_m,I} \\ \sigma_{P_s,A} & \sigma_{P_s,I} \end{pmatrix} = \begin{pmatrix} \kappa_A^{(i)} \\ \kappa_I^{(i)} \end{pmatrix} \begin{pmatrix} \mu_{P_m}^{(i)} - r_t \\ \mu_{P_s}^{(i)} - r_t \end{pmatrix}.$$

A simple manipulation shows that

$$\begin{aligned} \mu_{P_m}^{(1)} - \mu_{P_m}^{(2)} &= \sigma_{P_m.I} \bar{\eta}_t, \\ \mu_{P_s}^{(1)} - \mu_{P_s}^{(2)} &= \sigma_{P_s.I} \bar{\eta}_t, \\ \kappa_A^{(1)} - \kappa_A^{(2)} &= 0, \\ \kappa_I^{(1)} - \kappa_I^{(2)} &= \bar{\eta}_t. \end{aligned} \tag{8}$$

As we can expect from the previous assumption on investors' different beliefs, the market price of the systematic risk is the same across investors while the market prices of extraneous risks are different between investors and the difference is given by the belief difference kernel, $\bar{\eta}_t$.

3. EQUILIBRIUM

3.1. Investors' optimization

With given prices processes of two risky assets and one riskless asset, individual investor i's wealth $W_t^{(i)}$ evolves as

$$\begin{aligned} dW_t^{(i)} =& W_t^{(i)} \left[r_t - \tilde{c}_{i,t} + \phi_m^{(i)}(t)(\mu_{P_m}(t) - r_t) + \phi_s^{(i)}(t)(\mu_{P_s}^{(i)} - r_t) \right] dt \\ &+ W_t^{(i)} \left[\phi_m^{(i)}(t)\sigma_{P_m,A} + \phi_s^{(i)}(t)\sigma_{P_s,A} \right] dB_A(t) \\ &+ W_t^{(i)} \left[\phi_s^{(i)}(t)\sigma_{P_s,I} \right], \end{aligned} \tag{9}$$

where $\tilde{c}_{i,t}$ is the consumption fraction of the i-th investor out of his/her wealth, $c_{i,t}/W_t^{(i)}$, and $\phi_m^{(i)}$, $\phi_s^{(i)}$ are the i-th investor's risky investment fractions out of his/her wealth in the market portfolio and an individual

asset. $b_{i,t}$ is the riskless investment of i-th investor and it is defined as $b_{i,t} \equiv 1 - \phi_m^{(i)} - \phi_s^{(i)}$. Following Dvbvig and Huang (1988), a non-negativity constraint is imposed on the wealth process above to rule out arbitrage condition.

Thanks to the market completeness along with the state price densities across investors

$$d\xi_t^{(i)} = -\xi_t^{(i)} \left[r_t dt + \kappa_A dB_A + \kappa_I^{(i)} dB_I^{(i)} \right], \quad i = 1, 2, \tag{10}$$

individual optimization problem can be stated as a static optimization problem via the Martingale method as follows.

$$\begin{aligned} &\max_{c_k} E^{(k)} \left[\int_0^\infty u_k(c_k(t)))dt \right], \quad \text{subject to} \\ &E^{(k)} \left[\int_0^\infty \xi_t^{(k)} c_k(t) dt \right] \leq W^{(k)}(0) \equiv \omega_k P(0), \end{aligned} \tag{11}$$

where P_t is the total wealth held by both investors at time t.[9] Also we assume that investors have the homogeneous preference with a power utility function, i.e., $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$.

First order condition for the optimal consumption is given by

$$c_{i,t} = I_i \left(\frac{\xi_t^{(i)}}{\lambda_i} \right), \tag{12}$$

where $1/\lambda_i$ is the Lagrange multiplier for investor i-th investor's optimal consumption portfolio choice problem, and $I_i(\cdot)$ is the inverse of investor i's utility function. Note that $1/\lambda_i$ satisfies the static budget constraint above with equality.

3.2. Equilibrium

Now we turn to the equilibrium of the model.

Definition 3.1. Given preferences, endowments, and belief dispersion between investors, an equilibrium in the economy is a collection of allocations $\left(c_i^*, \phi_m^{(i),*}, \phi_s^{(i),*}, b_i^* \right)$, and a supporting price system $\left(r, \mu_{P_m}^{(i)}, \mu_{P_s}^{(i)}, \sigma_{P_m}, \sigma_{P_s} \right)$ such that $\left(c_i^*, \phi_m^{(i),*}, \phi_s^{(i),*}, b_i^* \right)$ solves i-th investor's optimal consumption portfolio choice problem given his/her perceived price processes. Also security prices are consistent across investors, and all markets clear at $t \geq 0$.

[9] Total wealth P_t is the same as the market portfolio $P_m(t)$.

Good market clearing condition in the definition of the equilibrium says that we can find state price densities of investors from

$$c_1^*(\xi_t^{(1)}/\lambda_1, t) + c_2^*(\xi_t^{(2)}/\lambda_2, t) = D_t \equiv e^{\delta_t}. \tag{13}$$

To solve for equilibrium quantities, we construct a representative agent utility function with stochastic weights.[10]

We define a stochastic ratio as follows.

$$\lambda_t \equiv \frac{\lambda_1 \xi_t^{(2)}}{\lambda_2 \xi_t^{(1)}}, \tag{14}$$

where $\lambda_0 = \frac{\lambda_1}{\lambda_2}$. Ito's lemma applied to λ_t yields

$$\frac{d\lambda_t}{\lambda_t} = \bar{\eta}_t dB_I^{(2)}. \tag{15}$$

We call this process as the belief difference process as λ_t is completely described by the belief difference kernel, $\bar{\eta}_t$.[11] Also note that this defines the Radon-Nikodym derivative between two investors' probability densities.

Then the representative investor's utility function can be stated as follows.

$$U(C, \lambda) = \max_{c_1 + c_2 \leq D} \frac{\lambda_t}{\lambda_1} u(c_{1,t}) + \frac{1}{\lambda_2} u(c_{2,t}), \tag{16}$$

where C is the aggregate consumption (in equilibrium $C \equiv D$).

Proposition 1 formalizes the equilibrium consumption allocations and provides the pricing of extraneous risk via investors' belief difference in terms of primitives.

PROPOSITION 1. *If equilibrium exists, the equilibrium state price density processes of investors are expressed by*

$$\xi_t^{(1)} = U'(D_t; 1/\lambda_1, \lambda_t/\lambda_2), \quad \xi_t^{(2)} = U'(D_t; 1/\lambda_1, \lambda_t/\lambda_2), \tag{17}$$

where the ratio $\lambda_1 \lambda_2$ satisfies either both investors' budget constraint, i.e.,

$$\begin{aligned} & E^{(1)}\left[\int_0^\infty U'(D_t; 1/\lambda_1, \lambda_t/\lambda_2) I_1(\lambda_1, U'(D_t; 1/\lambda_1, \lambda_t/\lambda_2)) dt\right] \\ = \; & E^{(1)}\left[\int_0^\infty U'(D_t; 1/\lambda_1, \lambda_t/\lambda_2) \epsilon_{1,t} dt\right], \end{aligned} \tag{18}$$

[10] This method has been widely used in many asset pricing models since Huang (1987) and Cuoco and He (1994).

[11] See Gallmeyer and Hollifiled (2008) and Bhamra and Uppal (2010).

where $\epsilon_{i,t}$ is i-th investor's endowment at time t, and the stochastic weighting $\lambda_t \equiv \xi_t^{(1)}/\xi_t^{(2)}$ follows

$$\frac{d\lambda_t}{\lambda_t} = \bar{\eta}_t dB_I^{(2)}. \tag{19}$$

Optimal consumption sharing rules are given by

$$\begin{aligned} c_{2,t} &= \omega(\lambda_t) D_t, \\ c_{1,t} &= (1-\omega(\lambda_t)) D_t, \end{aligned} \tag{20}$$

where $\omega(\lambda_t) \equiv \left[1+\lambda_t^{1/\gamma}\right]^{-1}$. The diffusion processes of consumption sharing rules for both investors are given by

$$\frac{dc_{i,t}}{c_{i,t}} = \mu_{c_i}^{(i)} dt + \sigma_{c_i,A} dB_A + \sigma_{c_i,I} dB_I^{(i)} \quad \textit{for } i=1,2, \tag{21}$$

where

$$\begin{aligned} \mu_{c_2}^{(2)} &= \mu_D - \frac{1}{2}\frac{1}{\gamma}\left(\frac{1}{\gamma}-1\right)\left(\frac{\lambda_t^{1/\gamma}}{1+\lambda_t^{1/\gamma}}\right)\bar{\eta}_t^2 + \frac{1}{\gamma^2}\left(\frac{\lambda_t^{1/\gamma}}{1+\lambda_t^{1/\gamma}}\right)^2 \bar{\eta}_t^2, \\ \mu_{c_1}^{(1)} &= \left(1+\frac{c_2(t)}{c_1(t)}\right)\mu_D - \frac{c_2(t)}{c_1(t)}\mu_{c_2}^{(2)} - \frac{c_2(t)}{c_1(t)}\bar{\eta}_t \sigma_{c_2,I}(t), \\ \sigma_{c_1,A} &= \sigma_{c_2,A} = \sigma_D, \\ \sigma_{c_2,I}(t) &= -\left(\frac{\lambda_t^{1/\gamma}}{1+\lambda_t^{1/\gamma}}\right)\left(\frac{1}{\gamma}\right)\bar{\eta}_t, \\ \sigma_{c_1,I}(t) &= -\left(\frac{c_2}{c_1}\right)\sigma_{c_2,I} = -\left(\frac{1}{\lambda_t^{1/\gamma}}\right)\sigma_{c_2,I}(t), \end{aligned} \tag{22}$$

and $\mu_D = \mu_{D,t} \equiv \mu_{\delta,t} + \frac{1}{2}\sigma_{\delta,A}^2$, $\sigma_D \equiv \sigma_{\delta,A}$.

Proof. See appendix. ∎

As is shown in Proposition 1, investors' equilibrium consumption allocations are explicitly affected by extraneous risk via investors' belief difference. Qualitatively, this result is consistent with existing studies that indicate individual consumptions are influenced by (uninsurable) idiosyncratic (labor income) risk. Our model shows that even under a complete market economy, extraneous (or idiosyncratic) risk affects equilibrium consumptions via investors' belief differences. Also equation (1) shows that

volatility of consumptions is affected by both aggregate and extraneous risk exposures. Note that first, in an economy without investors' belief difference, there is no effect of extraneous risk in equilibrium as extraneous risk shows up in equilibrium only via belief difference. Second, even in a model with investors' belief difference, we might not be able to incorporate extraneous risk into equilibrium if we assume conventional belief difference assumption by which investors have different opinions only through aggregate risk or information.

Remark 3.1. [Existence of equilibrium] We can show the existence of equilibrium with some additional technical conditions. We omit the details by relying on Basak (2000) and Basak (2005) that provide the argument on this issue.

We now turn to the security dynamics. Following Propositions 2, 3, and 4 summarize equilibrium security price dynamics.

PROPOSITION 2. *The equilibrium interest rate, r_t, is given by*

$$\begin{aligned} r(t) &= A(t)\left[\sum_{k=1}^{2} c_k(t)\mu_{c_k}^{(k)} - \frac{1}{2}\sum_{k=1}^{2} P_k(t)c_k(t)^2\left(\sigma_{c_k,A}^2 + \sigma_{c_k,I}^2\right)\right] \\ &= \gamma\mu_D - \frac{1}{2}\gamma(1+\gamma)\sigma_{D,A}^2 - \frac{1}{2}\left(\frac{1+\gamma}{\gamma}\right)\left(\frac{\lambda_t^{1/\gamma}}{1+\lambda_t^{1/\gamma}}\right)^2\bar{\eta}_t^2, \end{aligned} \tag{23}$$

where P_k is the prudence parameter of investor k, and A is the aggregate absolute risk aversion parameter.

Proof. See appendix. ∎

Some interesting features arise in equilibrium riskless rate of return. First two terms in riskless interest rate are conventional ones that can be found from a standard asset pricing model. When we have high dividend growth from aggregate dividend, the economy gets better. Thus riskless interest rates increases. On the other hand, investors would like to save to compensate the volatility from the aggregate dividend. Thus the second term is induced by investors' precautionary savings motives. Thus it decreases riskless interest rate. Unlike conventional asset pricing model, we have an additional term. As was shown in the previous Proposition 1, we have additional volatility from extraneous risk via investors' belief difference. Thus It induces more precautionary savings motif. As is shown in the equation, third term uniformly decreases riskless interest rate. According to somewhat conventional belief difference literature such as Basak (2005),

Gallmeyer and Hollifiled (2008), equilibrium riskless rate is not uniformly affected by belief difference. However in this model with power utility form along with belief difference via extraneous risk, riskless interest rate is uniformly lower than the conventional counterpart.

PROPOSITION 3. *The parametric expressions of the aggregate market price of risk and market prices of the idiosyncratic risks across investors are given by*

$$\begin{aligned}
&\theta_A(t) = \gamma\sigma_{D,A},\\
&\theta_I^{(2)}(t) = -\bar{\eta}_t\left(\frac{A(t)}{A_1(t)}\right) = -\bar{\eta}_t\frac{c_1(t)}{D_t} = -\bar{\eta}_t(1-\omega(\lambda_t)),\\
&\theta_I^{(1)}(t) = \bar{\eta}_t\left(\frac{A(t)}{A_2(t)}\right) = \bar{\eta}_t\frac{c_2(t)}{D_t} = \bar{\eta}_t\omega(\lambda_t),
\end{aligned} \tag{24}$$

where A_k is the investor k's absolute risk aversion parameter.

Proof. See appendix. ∎

Note that in the model, we assume that investors have different beliefs through extraneous risk. This implies that investors agree to each other on aggregate risk aspect. Thus the market price of aggregate risk is the same across investors, which is confirmed above. On the other hand, investors have different beliefs via extraneous risk, their market prices of extraneous riskshould be different and they are linked via belief different kernel, $\bar{\eta}_t$ and the signs are opposite as one is optimistic and the other is pessimistic. Opposite sign is important in interpretation. Compared to homogeneous model, more optimistic investors have more risk since risk is transferred from more pessimistic to more optimistic. This can be seen clearly from the opposite sign of two market prices of extraneous risks.

Perceived equilibrium price of aggregate market portfolio is given by

$$P_{m,t} = \frac{D_t}{\left(1+\lambda_t^{1/\gamma}\right)^{\gamma}} E_t^{(2)}\left[\int_t^{\infty}\left(1+\lambda_t^{1/\gamma}\right)^{\gamma} D_t^{1-\gamma}\right]. \tag{25}$$

Unfortunately, this representation, in general, does not allow a closed form solution except a special case.[12] Instead of deriving a closed form solution of

[12]If we can approximate the diffusion process of $\left(1+\lambda_t^{1/\gamma}\right)^{\gamma}$ with a fixed drift and diffusion parameter, then a closed form solution can be obtained. However, as will be discussed later, an approximation does not readily exist so that only a numerical might be carried out to analyze stock price dynamics such as stock volatility and trading volume.

the equilibrium price of market portfolio, we rely on a symbolic expression of the equilibrium expected excess return of the market portfolio. Note that the expected excess return of an asset is represented by the cross product of market prices of risks and their corresponding diffusion coefficients of the asset's price process. Following Proposition summarizes the expression of the equilibrium expected excess return of the market portfolio.

PROPOSITION 4. *Aggregate expected excess return is represented as*

$$\begin{aligned}\mu^{(2)}_{P_m} - r &= \kappa_A \sigma_{P_m,A} + \kappa^{(2)}_I \sigma_{P_m,I} \\ &= \gamma \sigma_{D,A} \sigma_{P_m,A} - \bar{\eta}_t (1 - \omega(\lambda_t)) \sigma_{P_m,I} \\ &= A(t) Cov\left(\frac{dP_{m,t} + D_t}{P_{m,t}}, dD_t \right) - \frac{A_1(t)}{A(t)} Cov\left(\frac{dP_{m,t} + D_t}{P_{m,t}}, \frac{d\lambda_t}{\lambda_t} \right)\end{aligned} \tag{26}$$

$$\begin{aligned}\mu^{(1)}_{P_m} - r &= \kappa_A \sigma_{P_m,A} + \kappa^{(1)}_I \sigma_{P_m,I} \\ &= \gamma \sigma_{D,A} \sigma_{P_m,A} + \bar{\eta}_t \omega(\lambda_t) \sigma_{P_m,I} \\ &= A(t) Cov\left(\frac{dP_{m,t} + D_t}{P_{m,t}}, dD_t \right) + \frac{A_2(t)}{A(t)} Cov\left(\frac{dP_{m,t} + D_t}{P_{m,t}}, \frac{d\lambda_t}{\lambda_t} \right)\end{aligned} \tag{27}$$

Under objective (or given) probability measure,

$$\begin{aligned}\mu_{P_m} - r &= \gamma \sigma_{P_m,A} \sigma_{D,A} + (1 - \omega(\lambda_t)) \left[\frac{\bar{\mu}_\delta - \bar{\mu}^{(1)}_\delta}{\sigma_I} \right] \sigma_{P_m,I} + \omega(\lambda_t) \left[\frac{\bar{\mu}_\delta - \bar{\mu}^{(2)}_\delta}{\sigma_I} \right] \sigma_{P_m,I} \\ &= A(t) Cov\left(\frac{dP_{m,t} + D_t}{P_{m,t}}, dD_t \right) \\ &\quad + \frac{A(t)}{A_1(t)} \left[\frac{\bar{\mu}_\delta - \bar{\mu}^{(1)}_\delta}{\sigma_I} \right] \sigma_{P_m,I} + \frac{A(t)}{A_2(t)} \left[\frac{\bar{\mu}_\delta - \bar{\mu}^{(2)}_\delta}{\sigma_I} \right] \sigma_{P_m,I}\end{aligned} \tag{28}$$

Proof. See appendix. ∎

First two equations are perceived equilibrium expected excess return of the market portfolio. As was shown in the signs of market prices of extraneous risks in the previous Proposition 3, the extraneous risk component of two perceived expected excess return is different. For simplicity, we assume that the second investor is more optimistic. Then the belief difference

kernel, $\bar{\eta}_t$ is negative. Thus the expected excess return (or required excess rate of return) on the market portfolio for the second (first) investor is higher (lower) than that of homogeneous counterpart. As was mentioned before, some risk is transferred from more pessimistic investors to more optimistic investors. Thus optimistic (pessimistic) investors require more (less) expected return on the market portfolio. When we manipulate more, we reach the equation (28). The first term is still the homogeneous case. But there are two additional terms that stem from extraneous risk via investors' belief differences. If economists (or econometricians) ought to see this equation (28), then conventional homogeneous case might be severely biased depending on the magnitude of the investors' belief differences. If the conventional equity premium puzzle arises from the homogeneous case, then we can expect that additional terms of the equation (28) might be positive enough to match the observed expected excess return. There are several possibilities for the positivity of two additional terms. If both perceived mean growth are lower than the true one, then obviously it is positive. If one of $\bar{\mu}_\delta - \bar{\mu}_\delta^{(i)}$ is positive and the other is negative, then relative wealth between two investors will determine the positivity. For instance, if the optimistic investor has higher perceived value of the aggregate mean dividend growth rate, and the pessimistic investor has lower perceived value of the aggregate mean dividend growth rate, then relative wealth should be adjusted to make the two additional terms positive.

Remark 3.2. As was mentioned before, a closed form solution for the equilibrium price or price-dividend ratio of the market portfolio is not obtainable under the current setting. If we could approximate the process of $\left(1+\lambda_t^{1/\gamma}\right)^\gamma$, then we might be able to get a specific form of the equilibrium price diffusion process of the market portfolio. In this case, it would be possible to have more specific form of the equilibrium return diffusion process so that we can simulate the level of the return and further investigate the equity premium at the aggregate. When the return process is simulated, the level of the return, hence the average return is not affected by either subjective or objective probability measure due to observational consistency. The importance of obtaining more specific form of diffusion process (regardless it is a fully closed form or an approximate form) is because we can easily investigate the quantitative effect of the extraneous risk via investors' belief difference.

4. CONCLUSION

In this paper, we explore a simple continuous-time exchange economy asset pricing model. We incorporate investors' belief heterogeneity via extraneous risk or idiosyncratic information along with investors' confusion in evaluating the mean growth rate of the aggregate dividend process. With these assumptions, the model explicitly prices extraneous risk in equilibrium via investors' different beliefs. Thus our model provides a way of pricing extraneous risk in an complete financial market model without assuming incomplete market with uninsured income risk or under-diversification of portfolios. Major equilibrium quantities are impacted by extraneous risk which is consistent with recent literature indicating investors' consumption and portfolio returns seem to be affected by extraneous or idiosyncratic risk. It is worth to investigate quantitative aspects of the model's implications in the future research.

APPENDIX A

Proof (Proof of Proposition 1). First two expressions are results from a variation Karatzas, Lehoczky, and Shreve (1990) by incorporating heterogenous stochastic discount factors across investors. The belief difference diffusion process is just the result of Ito's Lemma and the definition of λ_t. With the consumption goods market clearing equation, we have the following;

$$\left[\left(\frac{1}{\lambda_2\lambda_t}\right)^{-\frac{1}{\gamma}} + \left(\frac{1}{\lambda_2}\right)^{-\frac{1}{\gamma}}\right] \xi_t^{(2)^{-\frac{1}{\gamma}}} = D_t. \tag{A.1}$$

By arranging this equation we get $\xi_t^{(2)}$. Plugging this back into the equation (12) yields the required expression for the consumption sharing rules. Finally by applying Ito lemma to optimal consumption sharing rules, we get the required diffusion expressions. ∎

Proof (Proof of Proposition 2). First we define some quantities below for the proof. These are parameters of individual investor's absolute risk aversion and prudence, A_i and P_i and of representative investor's absolute

risk aversion and prudence, A and P.

$$\begin{aligned}
A_k(t) &= -\frac{u^{''}(c_k(t)))}{u^{'}(c_k(t))} = \frac{\gamma}{c_k(t)},\\
P_k(t) &\equiv -\frac{u^{'''}(c_k(t))}{u^{''}(c_k(t))} = \frac{1+\gamma}{c_k(t)},\\
A(t) &\equiv -\frac{U^{''}(C(t))}{U^{'}(C(t))} = \frac{1}{1/A_1(t)+1/A_2(t)} = \frac{\gamma}{c_1(t)+c_2(t)} = \frac{\gamma}{D_t},\\
P(t) &\equiv -\frac{U^{'''}(C(t))}{U^{''}(C(t))} = \sum_{k=1}^{2}\left(\frac{A(t)}{A_k(t)}\right)^2 P_k(t) = \frac{1+\gamma}{D_t}.
\end{aligned} \tag{A.2}$$

By applying Ito lemma to the first order condition of the individual optimization problem, $u'(c_k) = \frac{\xi^{(k)}}{\lambda_k}$, we have the following conditions

$$\begin{aligned}
&-\frac{\xi_t^{(k)}}{\lambda_k} r_t = c_k(t)u^{''}(c_k(t))\mu_{c_k}^{(k)} + \frac{1}{2}c_k(t)^2 u^{'''}(c_k(t))(\sigma_{c_k,A}^2 + \sigma_{c_k,I}^2),\\
&-\frac{\xi_t^{(k)}}{\lambda_k}\theta_A = c_k(t)u^{''}(c_k(t))\sigma_{c_k,A},\\
&-\frac{\xi_t^{(k)}}{\lambda_k}\theta_I^{(k)} = c_k(t)u^{''}(c_k(t))\sigma_{c_k,I},
\end{aligned} \tag{A.3}$$

by matching coefficients of dt, dB_A and $dB_I^{(k)}$ respectively. Also by applying Ito lemma to the consumption goods market clearing condition with true idiosyncratic Brownian motion, $B_I(t)$, we get the following conditions

$$\begin{aligned}
&c_1\mu_{c_1}^{(1)} + \mu_{c_2}^{(2)} = D\mu_D,\\
&D\sigma_{D,A} = c_1\sigma_{c_1,A} + c_2\sigma_{c_2,A},\\
&c_1\sigma_{c_1,I} + c_2\sigma_{c_2,I} = 0,
\end{aligned} \tag{A.4}$$

by matching coefficients of dt, dB_A, and dB_I(instead of $B_I^{(k)}$) respectively. Thanks to conditions above and the first order condition of the individual optimization problem, the first equation in (A.3) becomes

$$\frac{r_t}{A_k} = c_k\mu_{c_k}^{(k)} - \frac{1}{2}P_k c_k^2\left(\sigma_{c_k,A}^2 + \sigma_{c_k,I}^2\right). \tag{A.5}$$

By summing this equation across investors, we get the required equilibrium interest rate in the first equality. For the second equality, we can use definitions of A_t, A_k, P_k, P_t and the diffusion coefficient of equilibrium consumption sharing rule to get the required results. ∎

Proof (Proof of Proposition 3). Parametric expressions follow from definitions and the optimal consumption sharing rules. ▌

Proof (Proof of Proposition 4). Definition of the expected excess return of an asset in equilibrium. See Duffie (2001). ▌

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