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Concavity-Preserving Integration and Its Application in Principal-Agent Problems

Jia Xie*

This paper finds the necessary and sufficient condition for an integration to be concavity preserving. Using this condition, we can, for the first time in the literature of the principal-agent problems, justify the first-order approach without requiring the contract to be monotonic.

Key Words: Concavity-preserving integration; First-order approach; Principal-agent problems.
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1. INTRODUCTION

Assume a random variable $\tilde{x}$, whose probability density function is denoted by $f(x|a)$ indexed by a parameter a. We find the necessary and sufficient condition for $\varphi^T(a) \equiv \int \varphi(x)f(x|a)dx$ to be concavity preserving, that is, $\varphi^T(a)$ is concave in a for any concave function $\varphi(x)$.

Jewitt (1988) and Conlon (2009) have found sufficient conditions for $\varphi^T(a)$ to be mono-tonicity and concavity preserving, i.e., $\varphi^T(a)$ is nondecreasing and concave in a for any nondecreasing and concave function $\varphi(x)$. Conlon (2009) has also found sufficient conditions for $\varphi^T(a)$ to be monotonicity preserving, i.e., $\varphi^T(a)$ is nondecreasing in a for any nondecreasing function $\varphi(x)$. However, there has been no study on concavity preservation of $\varphi^T(a)$ as we are aware of, and this paper tends to fill in this blank.

The concavity-preserving integration has important implications in the principal-agent problem. Particularly, we are able to justify the first-order approach, as long as the optimal contract is concave. This is the first time

* Department of Finance, Mihaylo College of Business and Economics, California State University Fullerton. 2550 Nutwood Ave. Fullerton, CA 9283, USA. Email: jxie@fullerton.edu.

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in relevant literature that the first-order approach can be justified without requiring the contract to be monotonic.[1]

2. CONCAVITY-PRESERVING INTEGRATION

The condition is best represented in the state-space formulation. Assume that $\tilde{x}$ is generated by a parameter a and another random variable $\tilde{s}$ with probability density function $f(s)$, i.e., $\tilde{x} \equiv x(\tilde{s}, a)$. Then $\varphi^T(a) = \int \varphi(x(s,a))f(s)ds$.

PROPOSITION 1. *$\varphi^T(a)$ is concavity preserving if and only if the following two conditions hold for all a and all constant α:*

$$\int x_{aa}(s,a)f(s)ds = 0 \tag{1}$$

$$\int_{S(\alpha,a)} x_{aa}(s,a)f(s)ds \leq 0, \tag{2}$$

where $S(\alpha, a) \equiv \{s|x(s,a) \leq \alpha\}$. In particular, $\varphi^T(a)$ is concavity preserving if $x(s,a)$ is linear in a for almost all s.

Proof. We prove the sufficiency of (1) and (2) first. $\varphi^T(a)$ is concave if and only if $\frac{\partial^2 \varphi^T(a)}{\partial a^2} \leq 0$, $\forall a$. We have:

$$\begin{aligned}\frac{\partial^2 \varphi^T(a)}{\partial a^2} &= \frac{\partial^2 \int \varphi(x(s,a))f(s)ds}{\partial a^2} \\ &= \int [\varphi_{xx}(x(s,a))x_a^2(s,a) + \varphi_x(x(s,a))x_{aa}(s,a)]f(s)ds \\ &= \int \varphi_{xx}(x(s,a))x_a^2(s,a)f(s)ds + \int \varphi_x(x(s,a))x_{aa}(s,a)f(s)ds \end{aligned} \tag{3}$$

The first integration in (3) is always non-positive, as $\varphi_{xx}(x) \leq 0$ due to $\varphi(x)$ being concave. Then it suffices to prove that the second integration is non-positive as well, that is, $\int \varphi_x(x(s,a))x_{aa}(s,a)ds \leq 0$ if (1) and (2) hold. First, define $I(x \leq \gamma) = 1$ if $x \leq \gamma$, and zero otherwise. $\varphi_x(x)$ is non-increasing due to $\varphi(x)$ being concave. Therefore we can define its inverse function as follows: $\varphi_x^{-1}(\beta) \equiv \sup\{x|\varphi_x(x) \geq \beta\}$. Define:

$$\varphi_x^{\beta,n}(x) = \beta + \sum_{i=1}^{\infty} \frac{1}{n} I\left(x \leq \varphi_x^{-1}\left(\beta + \frac{i}{n}\right)\right). \tag{4}$$

[1]For the literature of the first-order approach in principal-agent problems, see Rogerson, 1985; Jewitt, 1988; Sinclaire-Desgagne, 1994; Mirrlees, 1999; Conlon, 2009 a,b; Jung and Kim, 2015; and Xie, 2017.

Then $\varphi_x^{\beta,n}(x)$ approximates $\varphi_x^{\beta}(x) \equiv \max(\beta, \varphi_x(x))$ uniformly from above. Figure 1 shows $\varphi_x^{\beta,n}(x)$ and $\varphi_x(x)$, as functions of x.

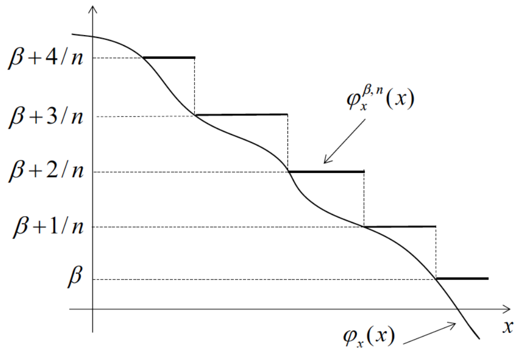

Figure 1: $\varphi_x^{\beta,n}(x)$ and $\varphi_x(x)$ as functions of x

Then we have:

$$
\begin{aligned}
& \int \varphi_x^{\beta,x}(x(s,a))x_{aa}(s,a)f(s)ds \\
= & \int \left[\beta + \sum_{i=1}^{\infty} I\left(x \leq \varphi_x^{-1}\left(\beta + \frac{i}{n}\right)\right)\right] x_{aa}(s,a)f(s)ds \\
= & \ \beta \int x_{aa}(s,a)f(s)ds + \frac{1}{n}\sum_{i=1}^{\infty} \int I\left(x \leq \varphi_x^{-1}\left(\beta + \frac{i}{n}\right)\right) x_{aa}(s,a)f(s)ds \\
= & \ \frac{1}{n}\sum_{i=0}^{\infty} \int I\left(x \leq \varphi_x^{-1}\left(\beta + \frac{i}{n}\right)\right) x_{aa}(s,a)f(s)ds \\
= & \ \frac{1}{n}\sum_{i=0}^{\infty} \int_{S(\varphi_x^{-1}(\beta+\frac{i}{n}),a)} x_{aa}(s,a)f(s)ds \leq 0,
\end{aligned} \tag{5}
$$

where the first equality follows from substituting (4) in, the third equality follows from (1), and the inequality at the end follows from (2). Then we have:

$$\int \varphi_x^{\beta}(x(s,a))x_{aa}(s,a)f(s)ds = \lim_{n\to\infty} \int \varphi_x^{\beta,n}(x(s,a))x_{aa}(s,a)f(s)ds \leq 0, \tag{6}$$

where the inequality follows from (5). Then according to the monotone convergence theorem, we have:

$$\int \varphi_x(x(s,a))x_{aa}(s,a)f(s)ds = \lim_{\beta\to-\infty} \int \varphi_x^{\beta}(x(s,a))x_{aa}(s,a)f(s)ds \leq 0,$$

where the inequality follows from (6). This completes the proof of sufficiency.

To proof the necessity of (1) and (2), we need to construct concave functions $\varphi(x)$ such that, if (1) or (2) does not hold, then $\varphi^T(a)$ is not concave in a. First, assume (1) does not hold, i.e., there is a constant $\lambda \neq 0$ such that $\int x_{aa}(s,a)f(s)ds = \lambda$. Let $\varphi(x) = \lambda x$. Then $\varphi(x)$ is concave in x, with $\varphi_x(x) = \lambda$ and $\varphi_{xx}(x) = 0$. Then by substituting the last two equalities into (3), we have $\partial^2\varphi^T(a)/a^2 = \lambda \int x_{aa}(s,a)f(s)ds = \lambda^2 > 0$. Therefore, $\varphi^T(a)$ is not concave in a.

Second, assume (2) does not hold, i.e, there is a constant α such that $\int_{S(\alpha,a)} x_{aa}(s,a)f(s)ds = \mu > 0$. Let $S^c(\alpha,a)$ be the complement set of $S(\alpha,a)$, i.e., $S^c(\alpha,a) \equiv \{s|x(s,a) > \alpha\}$. Then according to (1), $\int_{S^c(\alpha,a)} x_{aa}(s,a)f(s)ds = -\int_{S(\alpha,a)} x_{aa}(s,a)f(s)ds = -\mu < 0$. Then we construct $\varphi(x)$ as follows:

$$\varphi(x) = \begin{cases} \mu x & \text{if } x \leq \alpha, \\ -\mu x & \text{if } x > \alpha. \end{cases}$$

Then $\varphi(x)$ is concave in x, with $\varphi_{xx}(x) = 0$ and

$$\varphi_x(x) = \begin{cases} \mu & \text{if } x \leq \alpha, \\ -\mu & \text{if } x > \alpha. \end{cases} \tag{7}$$

Equation (7) is equivalent to

$$\varphi_x(x(s,a)) = \begin{cases} \mu & \text{if } s \in S(\alpha,a), \\ -\mu & \text{if } s \in S^c(\alpha,a). \end{cases} \tag{8}$$

Then according to (3), we have

$$\begin{aligned} \frac{\partial^2\varphi^T(a)}{a^2} &= \int \varphi_x(x(s,a))x_{aa}(s,a)f(s)ds \\ &= \int_{S(\alpha,a)} \varphi_x(x(s,a))x_{aa}(s,a)f(s)ds + \int_{S^c(\alpha,a)} \varphi_x(x(s,a))x_{aa}(s,a)f(s)ds \\ &= \mu \int_{S(\alpha,a)} x_{aa}(s,a)f(s)ds - \mu \int_{S^c(\alpha,a)} x_{aa}(s,a)f(s)ds. \\ &= \mu^2 + \mu^2 = 2\mu^2 > 0, \end{aligned} \tag{9}$$

where the second equality follows from the fact that $S(\alpha,a)$ and $S^c(\alpha,a)$ are complement sets, the third equality follows from (8), the fourth equality follows from the assumption that $\int_{S(\alpha,a)} x_{aa}(s,a)f(s)ds = \mu$ and that $\int_{S^c(\alpha,a)} x_{aa}(s,a)f(s)ds = -\mu$. (9) implies $\varphi^T(a)$ is not concave in a. We have thus proven that (2) is necessary for φ^T to be concavity preserving.

Finally, if $x(s,a)$ is linear in a, then $x_{aa}(s,a) = 0$ and thus both (1) and (2) hold. ▮

3. THE FIRST-ORDER APPROACH IN PRINCIPAL-AGENT MODEL

The principal-agent model can be described as follows. There is a principal and an agent. The agent exerts an effort $a \in R^+$ that stochastically generates a signal, denoted by a random variable $\tilde{x}$. Let $f(x|a)$ denote the density function of $\tilde{x}$. The outcome is determined by the function $\pi(x)$, and the payment to the agent is specified by $s(x)$. The principal is risk neutral, with her welfare being $\pi(x) - s(x)$, while the agent is risk averse, with a utility function $u(s(x))$. In addition, the agent has an increasing and convex cost function $c(a)$. The principal's problem is to choose a target action a^* and a contract $s^*(\cdot)$ that solve the follow program:

$$\max_{a^*, s^*(\cdot)} \int [\pi(x) - s^*(x)] f(x|a^*) dx \tag{10}$$

$$\text{s.t.} \quad \int u(s^*(x)) f(x|a^*) dx - c(a^*) \geq 0, \text{ and} \tag{11}$$

$$a^* = \text{argmax}_a \int u(s^*(x)) f(x|a) dx - c(a). \tag{12}$$

Constraint (11) is the participation constraint, which states that the agent's expected utility must be no less than his outside reservation utility, which is normalized to zero. Constraint (12) is the incentive compatibility constraint, which states that, given the payment schedule s^*, a^* maximizes the agent's expected utility.

A technical challenge is the infinite number of constraints imposed by (12). A common solution is to use the first-order approach, which replaces (12) with the first-order necessary condition that

$$\int u(s^*(x)) f_a(x|a^*) dx - c_a(a^*) = 0, \tag{13}$$

where the subscript denotes a partial derivative in a. The first-order approach is valid if

$$\int u(s^*(x)) f(x|a) dx \text{ is concave in } a. \tag{14}$$

Proposition 2. *If $s^*(x)$ is concave and Conditions (1) and (2) are satisfied, then (14) holds and the first-order approach is valid.*

Proof. Since the utility function $u(\cdot)$ is increasing and concave, if $s^*(x)$ is concave in x, so is $u(s^*(x))$. Then according to Proposition 1, $u(s^*(x))f(x|a)dx$ is concave in a. An important implication of Proposition

2 is that, for the first time in the principal-agent model literature, the contract does not have to be nondecreasing for the first-order approach to be valid. ∎

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