ANNALS OF ECONOMICS AND FINANCE **20-2**, 611–631 (2019)

Human Capital Accumulation and Life-cycle Earning

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We investigate optimal time allocation between human capital accumulation and wage earning in dynamic settings. We present closed form solutions (modified MRAP) and find a remarkable difference between deterministic and stochastic exponential horizons. If death date is known, time during the early years is devoted to learning, then allocated between learning and working at a suitable constant ratio, then devoted solely to working until death; if death date is unknown, then, after an early-years period of pure learning, time is allocated between learning and working according to a suitable constant ratio forever.

Key Words: Human capital; Wage earning; Random life; MRAP; HJB equation.
JEL Classification Numbers: C61, E10, I20.

1. INTRODUCTION

Learning is advocated in today's society. To some extent, it has been a consensus that one of the main engines of economic growth is the improvement of human capital. Much has been written on human capital accumulation since an early study by Uzawa (1965), with following works including Lucas (1988), Ortigueira (1998) and Meng and Ye (2009). These papers all use an infinite horizon setting. Our work is most closely related to that of Ben-Porath (1967), who studied the finite horizon setting. However, we do not consider the investment issue, and our objective function is not linear. We also investigate a case with a random life horizon. We consider a representative agent who allocates her time (or effort)[1] between human capital accumulation and wage earning. The time is continuous and we use the conventional dynamic optimization approach[2] to investigate the

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[1]See Becker (1965) for early work in allocation of time. Also see Mincer (1974).

[2]Some of the references are Berkovitz (1974), Kamien and Schwartz (1991), and Locatelli (2001).

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agent's choices. Acemoglu (2009, Chapter 10) provides an insightful survey on human capital accumulation and economic growth.

Our main findings are as follows.

In the deterministic case with no human capital depreciation, we obtain a bang-bang solution rather than a learning-by-doing result. For a typical and not very short lifetime, the optimal strategy is to continue accumulating human capital at full effort from the very beginning until a specific finite time point, and to keep earning a wage at full effort thereafter. With capital depreciation, the solution will not be of a bang-bang type; in a typical case, in the middle period of her life, the individual will allocate her time between learning and working according to a suitable constant ratio, just as in the stochastic case.

In the stochastic case, with human capital depreciation being taken into consideration, the solution will not be of a bang-bang type. Instead, in the most interesting case where the initial human capital is low, the optimal strategy is to first keep accumulating human capital until it reaches a specific level, after which time will be allocated between human capital accumulation and wage earning according to a fixed ratio. This result conforms with the reality to some extent in that everybody would continue studying while working so as to offset knowledge depreciation and keep up with the times.

A remarkable difference exists between the two settings as follows. If the agent knows the date of her death exactly, then she will devote all of her time to learning during her early years. She will then allocate her time between learning and working according to a suitable constant ratio, and, as she draws nearer to her death date, will give up on-the-job learning and devote all of her time to working up to the end of her life. If she does not know her death date exactly, then, after some period of pure leaning in her early years, she will allocate her time between learning and working according to a suitable constant ratio forever. In other words, she will not give up on-the-job learning before her death.

From the view point of mathematics, for a linear optimal control problem in both a finite time horizon setting and an infinite time horizon setting, we find that the optimal strategy is a form of trichotomous solution, rather than the classic dichotomous bang-bang solution. Moreover, the corresponding optimal state trajectory in the finite time horizon setting is a modified MRAP (most rapid approach path), rather than the standard MRAP in the infinite time horizon setting. The modification is applied only in the tail part, where it deviates from the standard MRAP and approaches its destiny as rapidly as possible.

2. DETERMINISTIC CASE

In this section, we consider the deterministic finite time horizon case.

2.1. Model setup

Let us consider an agent facing the problem of planing time allocation between learning and working in the time interval $[0, T]$, where $T > 0$ is given. The time point $t = 0$ is the start time for her to begin making a serious plan, and the time point $t = T$ is the end point of her life. Hence, T is her life-span.

For each time point t, let $x(t)$ be her human capital stock, and $u(t) \in [0, 1]$ be the instant proportion of time devoted to human capital accumulation (in other words, learning). We see $1 - u(t)$ as the instant proportion of time devoted to wage earning (in other words, working). We assume that, if the human capital stock is x, then the earning rate is $f(x) = x^\alpha$, and the human capital accumulation rate is $g(x) = x^\beta$ for a time unit at any time point, where $\alpha > 0$ and $0 < \beta < 1$. We do not consider human capital depreciation here.

The agent's decision on time allocation can be formulated into the following optimal control problem P_T

$$\begin{aligned} \max \quad & \int_0^T e^{-\rho t}(1-u)x^\alpha dt, \\ \text{such that} \quad & \\ & \dot{x} = u x^\beta, \\ & 0 \le u \le 1, \\ & x(0) = x_0, \end{aligned}$$

where $\rho > 0$ is her discounting rate, and $x_0 \geq 0$ is her initial human capital. Clearly, all feasible state paths are increasing.

2.2. Main results

For convenience, we follow the convention that $\ln x = -\infty$ for any $x \le 0$, and for any real numbers x, y, we denote $x \wedge y = \min(x, y)$, $x \vee y = \max(x, y)$, and $x^+ = \max(x, 0)$. We denote $I(\cdot)$ as the indicator function, which takes a value of 1 if the condition in the parentheses is true, and 0 otherwise.

In the $t - x$ plane, the curve

$$x = \kappa(t) := \left[\frac{\alpha}{\rho}\left(1 - e^{\rho(t-T)}\right)\right]^{\frac{1}{1-\beta}}$$

is denoted as Γ. We will see below in Lemma 3 that the curve Γ is the boundary crossing which, by the state variable from left to right, the optimal control u will switch abruptly from 1 to 0. It is easy to see that κ is strictly decreasing and strictly concave, and

$$\kappa(0) = \left[\frac{\alpha}{\rho}\left(1 - e^{-\rho T}\right)\right]^{\frac{1}{1-\beta}}, \quad \kappa(T) = 0.$$

For any $t \in [0, T]$ and $x \geq 0$, we define

$$\tau(t, x) = \inf\left\{s \geq 0 | x^{1-\beta} + (1-\beta)s \geq \kappa(t+s)^{1-\beta}\right\}.$$

It is obvious that $\tau(t, x)$ is determined uniquely by

$$G(t, \tau(t, x)) = \left(\kappa(t)^{1-\beta} - x^{1-\beta}\right)^{+},$$

where

$$G(t, s) = (1-\beta)s + \frac{\alpha}{\rho} e^{\rho(t-T)}\left[e^{\rho s} - 1\right],$$

which is strictly increasing with respect to s for any t, and $G(t, 0) = 0$ for any t. Clearly, $\tau(t, x) \in [0, T-t)$, and $\tau(t, x) = 0$ iff $x \geq \kappa(t)$.

Now we can define a function V as follows:

$$V(t, x) = \frac{1}{\rho}\left(e^{-\rho\tau(t,x)} - e^{\rho(t-T))}\right)\left(x^{1-\beta} + (1-\beta)\tau(t, x)\right)^{\frac{\alpha}{1-\beta}}.$$

In the $t - x$ plane, we divide the region $\mathbb{A} = \{(t, x) | t \in [0, T], x \geq 0\}$ into three parts:

$$\mathbb{A}_1 = \left\{(t, x) \in \mathbb{A} | x < \kappa(t)\right\},$$

$$\mathbb{A}_2 = \left\{(t, x) \in \mathbb{A} | x = \kappa(t)\right\},$$

$$\mathbb{A}_3 = \left\{(t, x) \in \mathbb{A} | x > \kappa(t)\right\}.$$

We define the current value Hamiltonian function as

$$H(x, u, \lambda) = (1-u)x^{\alpha} + \lambda u x^{\beta}.$$

Since H is linear in u, we have

$$\arg\max_{0 \leq u \leq 1} H(x, u, \lambda) = \begin{cases} \{1\}, & \text{if } \lambda > x^{\alpha-\beta}, \\ [0, 1], & \text{if } \lambda = x^{\alpha-\beta}, \\ \{0\}, & \text{if } \lambda < x^{\alpha-\beta}. \end{cases}$$

For simplicity, we denote $\tau = \tau(0, x_0)$.

In order to derive our main results, we will require some lemmas. Firstly, it is easy to see that $\tau(t,x)$ is a continuous but not smooth function in $\mathbb{A}$; in fact, it is not smooth only on the curve Γ. But, somewhat surprisingly, V is smooth.

LEMMA 1. *V is smooth in $\mathbb{A}$.*

Proof. We treat the three regions $\mathbb{A}_1, \mathbb{A}_2$ and $\mathbb{A}_3$ separately.

(i) Region $\mathbb{A}_1$

In this case, we have $\tau(t,x) > 0$,

$$x^{1-\beta} + (1-\beta)\tau(t,x) = \frac{\alpha}{\rho}\left(1 - e^{\rho(t+\tau(t,x)-T)}\right) =: U,$$

and

$$V(t,x) = \frac{1}{\rho}\left(e^{-\rho\varepsilon} - e^{\rho(t-T)}\right) U^{\frac{\alpha}{1-\beta}},$$

where, for simplicity, $\varepsilon = \tau(t,x)$. With subscripts for derivatives, we have the following results regarding the partial derivatives of function V

$$\begin{aligned} V_t &= \left(-e^{-\rho\varepsilon}\varepsilon_t - e^{\rho(t-T)}\right) U^{\frac{\alpha}{1-\beta}} + \tfrac{\alpha}{\rho}\left(e^{-\rho\varepsilon} - e^{\rho(t-T)}\right)\left(x^{1-\beta} + (1-\beta)\varepsilon\right)^{\frac{\alpha}{1-\beta}-1}\varepsilon_t \\ &= \left(-e^{-\rho\varepsilon}\varepsilon_t - e^{\rho(t-T)}\right) U^{\frac{\alpha}{1-\beta}} + e^{-\rho\varepsilon} U^{\frac{\alpha}{1-\beta}}\varepsilon_t \\ &= -e^{\rho(t-T)} U^{\frac{\alpha}{1-\beta}}, \end{aligned}$$

$$\begin{aligned} V_x &= -e^{-\rho\varepsilon}\varepsilon_x U^{\frac{\alpha}{1-\beta}} + \tfrac{\alpha}{\rho}\left(e^{-\rho\varepsilon} - e^{\rho(t-T)}\right)\left(x^{1-\beta} + (1-\beta)\varepsilon\right)^{\frac{\alpha}{1-\beta}-1}\left(x^{-\beta} + \varepsilon_x\right) \\ &= -e^{-\rho\varepsilon}\varepsilon_x U^{\frac{\alpha}{1-\beta}} + e^{-\rho\varepsilon} U^{\frac{\alpha}{1-\beta}}\left(x^{-\beta} + \varepsilon_x\right) \\ &= x^{-\beta} e^{-\rho\varepsilon} U^{\frac{\alpha}{1-\beta}}. \end{aligned}$$

(ii) Region $\mathbb{A}_3$

In this case, we have $\tau(t,x) = 0$, and,

$$V(t,x) = \frac{x^{\alpha}}{\rho}\left(1 - e^{-\rho(T-t)}\right).$$

Hence it follows that

$$V_t(t,x) = -x^{\alpha} e^{-\rho(T-t)},$$

$$V_x(t,x) = \frac{\alpha x^{\alpha-1}}{\rho}\left(1 - e^{-\rho(T-t)}\right).$$

(iii) Region $\mathbb{A}_2$

In this case, we have $\tau(t,x)=0$. Notice that the treatment in case (i) can also be applied to the curve Γ to obtain the left partial derivatives:

$$V_{t-}(t,x) = -x^{\alpha} e^{-\rho(T-t)},$$

$$V_{x-}(t,x) = x^{\alpha-\beta}.$$

Similarly, by applying the treatment in (ii), we obtain the right partial derivatives:

$$V_{t+}(t,x) = -x^{\alpha} e^{-\rho(T-t)},$$

$$V_{x+}(t,x) = x^{\alpha-\beta}.$$

We see that on the curve Γ, $V_{t-}=V_{t+}$, $V_{x-}=V_{x+}$, and hence, V_t and V_x exist.

Moreover, it is clear that, in the whole region $\mathbb{A}$,

$$V_t(t,x) = -e^{\rho(t-T)}\kappa^{\alpha}(t+\tau(t,x)),$$

$$\begin{aligned} V_x(t,x) &= e^{-\rho\tau(t,x)}x^{-\beta}\kappa^{\alpha}(t+\tau(t,x)), && \text{if} \quad \tau(t,x)>0, \\ &= x^{\alpha-1}\kappa^{1-\beta}(t), && \text{if} \quad \tau(t,x)=0. \end{aligned}$$

Therefore, V_t and V_x are all continuous, and hence, V is smooth. Thus the lemma is proved. ▮

Lemma 2. *V satisfies the Hamilton-Jacobi-Bellman (HJB) equation*

$$\rho V(t,x) - V_t(t,x) = \max_{0\le u\le 1} H(x,u,V_x(t,x)). \tag{1}$$

Proof. We explore the three regions $\mathbb{A}_1, \mathbb{A}_2$ and $\mathbb{A}_3$ separately. We use the results and the notations in Lemma 1.

(i) Region $\mathbb{A}_1$

In this case, we have $\tau(t,x)>0$, and

$$V(t,x) = \frac{1}{\rho}\left(e^{-\rho\varepsilon} - e^{\rho(t-T)}\right) U^{\frac{\alpha}{1-\beta}},$$

$$V_t = -e^{\rho(t-T)} U^{\frac{\alpha}{1-\beta}},$$

$$V_x = x^{-\beta} e^{-\rho\varepsilon} U^{\frac{\alpha}{1-\beta}},$$

therefore,

$$\rho V(t,x) - V_t(t,x) = V_x(t,x) x^{\beta}.$$

In addition, notice that

$$\begin{aligned} & Ue^{-\frac{\rho(1-\beta)}{\alpha}\varepsilon} \\ > \;& U\left(1-\tfrac{\rho(1-\beta)}{\alpha}\varepsilon\right) \\ = \;& x^{1-\beta}+(1-\beta)\varepsilon-(1-\beta)\varepsilon\left(1-e^{\rho(t+\varepsilon-T)}\right) \\ = \;& x^{1-\beta}+(1-\beta)\varepsilon e^{\rho(t+\varepsilon-T)} \\ > \;& x^{1-\beta}, \end{aligned}$$

and hence,

$$V_x(t,x) > x^{\alpha-\beta}.$$

It follows that

$$\{1\} = \arg\max_{0\le u\le 1} H(x,u,V_x(t,x)).$$

Therefore,

$$\max_{0\le u\le 1} H(x,u,V_x(t,x)) = V_x(t,x)x^{\beta}.$$

Thus the HJB equation (1) holds.

(ii) Region $\mathbb{A}_2$

In this case, we have $\tau(t,x)=0$,

$$V(t,x) = \frac{1}{\rho}\left(1-e^{\rho(t-T)}\right)x^{\alpha},$$

$$V_t(t,x) = -x^{\alpha}e^{-\rho(T-t)},$$

and

$$V_x(t,x) = x^{\alpha-\beta}.$$

It follows that

$$\rho V(t,x) - V_t(t,x) = V_x(t,x)x^{\beta},$$

and

$$[0,1] = \arg\max_{0\le u\le 1} H(x,u,V_x(t,x)).$$

Therefore,

$$\rho V(t,x) - V_t(t,x) = x^{\alpha} = V_x(t,x)x^{\beta} = \max_{0\le u\le 1} H(x,u,V_x(t,x)),$$

which gives the HJB equation (1).

(iii) Region $\mathbb{A}_3$

In this case, we have $\tau(t,x)=0$,

$$V(t,x)=\frac{x^\alpha}{\rho}\left(1-e^{-\rho(T-t)}\right),$$

$$V_t(t,x)=-x^\alpha e^{-\rho(T-t)},$$

and

$$V_x(t,x)=\frac{\alpha x^{\alpha-1}}{\rho}\left(1-e^{-\rho(T-t)}\right).$$

Therefore we get

$$\rho V(t,x)-V_t(t,x)=x^\alpha.$$

Noticing that $x>\kappa(t)$, we have

$$V_x(t,x)<x^{\alpha-\beta},$$

which yields

$$\{0\}=\arg\max_{0\le u\le 1} H(x,u,V_x(t,x)).$$

Therefore,

$$\max_{0\le u\le 1} H(x,u,V_x(t,x))=x^\alpha.$$

It follows that the HJB equation (1) holds. Thus the lemma is proved. ▮

LEMMA 3. *The control path and its corresponding state path, induced by the Markovian strategy*

$$u(t)=I(x(t)<\kappa(t)), \tag{2}$$

are respectively as follows:

$$u(t)=I(t<\tau), \tag{3}$$

$$x(t)=\left(x_0^{1-\beta}+(1-\beta)[t\wedge\tau]\right)^{\frac{1}{1-\beta}}. \tag{4}$$

Proof. We denote the control path and the corresponding state path, induced by the Markovian strategy $u(t)=I(x(t)<\kappa(t))$, as $z(t)$ and $Z(t)$, respectively. Then, we have $z(t) = I(Z(t) < \kappa(t))$ for any t. Noticing that Z is increasing and κ is strictly decreasing, we define

$$\sigma = \inf \{0 \le t \le T | Z(t) \ge \kappa(t)\}.$$

Since $Z(T) > 0$ and $\kappa(T) = 0$, we know that $\{0 \le t \le T | Z(t) \ge \kappa(t)\}$ must be non-empty, and $\sigma \in [0, T)$. By the continuity of Z and κ, we have $I(Z(t) < \kappa(t)) = I(t < \sigma)$, which means that the control path z satisfies

$$z(t) = I(t < \sigma).$$

It follows that the corresponding state path Z satisfies $\dot{Z}(t) = Z(t)^{\beta}$ for all $t < \sigma$, and $\dot{Z}(t) = 0$ for all $t \ge \sigma$. Thus, we obtain

$$Z(t) = \left(x_0^{1-\beta} + (1-\beta)[t \wedge \sigma]\right)^{\frac{1}{1-\beta}}.$$

If $x_0 \ge \kappa(0)$, then, $Z(0) \ge \kappa(0)$, which implies that $\sigma = 0$. By the definition of τ, we have in this case $\tau = 0$, and hence $\sigma = \tau$.

If $x_0 < \kappa(0)$, then, $\sigma > 0$ and $\tau > 0$. By the continuity of Z and κ, we have $Z(\sigma) = \kappa(\sigma)$, which implies that

$$x_0^{1-\beta} + (1-\beta)\sigma = \frac{\alpha}{\rho}\left(1 - e^{\rho(\sigma - T)}\right).$$

Again, by the definition of τ, we obtain $\sigma = \tau$. Therefore, the lemma is proved. ∎

LEMMA 4. *For any $t \in [0, T]$, the control path (3) and its corresponding state path (4) satisfy*

$$H(x(t), u(t), V_x(t, x(t))) = \max_{0 \le u \le 1} H(x(t), u, V_x(t, x(t))). \tag{5}$$

Proof. It is equivalent to prove that along the state path (4),

$$1 \in \arg\max_{0 \le u \le 1} H(x(t), u, V_x(t, x(t))), \quad \text{if} \quad t < \tau,$$

$$0 \in \arg\max_{0 \le u \le 1} H(x(t), u, V_x(t, x(t))), \quad \text{if} \quad t \ge \tau,$$

for which, according to the proof of Lemma 2, it suffices to show that along the state path (4),

$$(t, x(t)) \in \mathbb{A}_1, \quad \text{if and only if} \quad t < \tau,$$

or, equivalently,

$$x(t) < \kappa(t), \quad \text{if and only if} \quad t < \tau,$$

that is,

$$x_0^{1-\beta} + (1-\beta)[t \wedge \tau] < \kappa(t)^{1-\beta}, \quad \text{if and only if} \quad t < \tau,$$

which is straightforward by the definition of τ,

$$\tau = \inf \left\{ s \geq 0 | x_0^{1-\beta} + (1-\beta)s \geq \kappa(s)^{1-\beta} \right\}.$$

Hence, the lemma is proved. ∎

THEOREM 1. *For the problem P_T, the unique Markovian optimal strategy is given by Equation (2), and $V(0, x_0)$ is its value function.*

Proof. Firstly, we notice that for any control path v and its corresponding state path y, we have

$$\begin{aligned} & \int_0^T e^{-\rho t}(1-v)y^\alpha dt \\ = & \int_0^T e^{-\rho t}\left(H\left(y, v, V_x(t,y)\right) - V_x(t,y)vy^\beta\right) dt \\ \leq & \int_0^T e^{-\rho t}\left(\rho V(t,y) - V_t(t,y) - V_x(t,y)vy^\beta\right) dt \\ = & -\int_0^T \frac{d}{dt}\left(e^{-\rho t}V(t,y)\right) dt \\ = & V(0, x_0). \end{aligned}$$

The inequality turns out to be equality for the control path in Equation (3) and its corresponding state path in Equation (4). Thus we reach the conclusions that the control path in (3) is really optimal for the problem P_T, that the Markovian strategy (2) is optimal, and that $V(0, x_0)$ is its value function.

Now, we address the uniqueness of the solution. We restrict our consideration to only Markovian strategies. We know that the Markovian optimal strategy must be the solution of the static optimization problem in the right-hand side of the HJB equation (1). By using the same method as in Lemma 3, we can show that the unique solution for the static optimization problem in the right-hand side of the HJB equation (1) is simply equation (2). The theorem is proved. ∎

Note that τ is the point of switching from full-time learning to full-time working prescribed by the optimal strategy (see equation (2)). It can be viewed as schooling time. We explore its properties in the next theorems.

THEOREM 2. *Along the state path in (4), $\tau(t, x(t))$ is decreasing in $[0, T]$, and for any $0 \le s \le t \le T$,*

$$t \wedge \tau + \tau(t, x(t)) = \tau; \tag{6}$$

$$\tau(s, x(s)) - \tau(t, x(t)) = (t - s) \wedge \tau(s, x(s)). \tag{7}$$

Proof. In fact, if $x(t) \ge \kappa(t)$, then, by the definition of τ, we have $t \ge \tau$, and

$$\tau(t, x(t)) = \inf \left\{0 \le s \le T | x(t)^{1-\beta} + (1-\beta)s \ge \kappa(t+s)^{1-\beta}\right\} = 0.$$

Hence Equation (6) holds. If $x(t) < \kappa(t)$, then, by the definition of τ, we have $t < \tau$, and

$$\begin{aligned}\tau(t, x(t)) &= \inf \left\{0 \le s \le T | x(t)^{1-\beta} + (1-\beta)s \ge \kappa(t+s)^{1-\beta}\right\} \\ &= \inf \left\{0 \le s \le T | x_0^{1-\beta} + (1-\beta)(t+s) \ge \kappa(t+s)^{1-\beta}\right\} \\ &= \inf \left\{0 \le m \le T | x_0^{1-\beta} + (1-\beta)m \ge \kappa(m)^{1-\beta}\right\} - t \\ &= \tau - t,\end{aligned}$$

which yields (6). Equation (7) can be proved similarly. The theorem is proved. ▮

The following theorem states that, provided the initial human capital is sufficiently low and life span is sufficiently large, the agent must study. That is, one must accumulate human capital in one's early years before entering into society to earn a wage.

THEOREM 3. *$\tau > 0$ if and only if*

$$e^{-\rho T} + \frac{\rho}{\alpha} x_0^{1-\beta} < 1. \tag{8}$$

And if $x_0 \in [0, x^*)$, then, $\tau < T^*$, and $\tau \uparrow T^*$ as $T \to \infty$, where

$$x^* = \left(\frac{\alpha}{\rho}\right)^{\frac{1}{1-\beta}},$$

$$T^* = \frac{1}{1-\beta}\left(\frac{\alpha}{\rho} - x_0^{1-\beta}\right).$$

Proof. The first part of this theorem is straightforward, as a direct consequence of the definition of τ.

Next, fix $x_0 \in [0, x^*)$ arbitrarily. For sufficiently large T, we have $x_0 < \kappa(0)$, and

$$x_0^{1-\beta} + (1-\beta)\tau = \frac{\alpha}{\rho}\left(1 - e^{\rho(\tau - T)}\right) < \frac{\alpha}{\rho},$$

and hence, $\tau < T^*$, and $\tau \uparrow T^*$ as $T \to \infty$. Therefore, the theorem is proved. ▮

From Theorem 3 we know that the time of pure human capital accumulation without working is uniformly bounded regardless of the agent's life span. Generally speaking, if her initial human capital is large, or she discounts the future sharply, or her life-time is short, or the sensitivity of wage earning to human capital is low, then the agent will not wish to learn and will devote all of her time to working. The following theorem also supports this concept.

THEOREM 4. *τ is strictly decreasing with respect to x_0 and ρ, and strictly increasing with respect to α and T.*

Proof. One can easily verify that

$$\frac{\partial \tau}{\partial x_0} < 0; \quad \frac{\partial \tau}{\partial \rho} < 0; \quad \frac{\partial \tau}{\partial \alpha} > 0; \quad \frac{\partial \tau}{\partial T} > 0,$$

which yield the results. ▮

The time span of studying or accumulating human capital is decreasing with respect to the agent's initial human capital and her discount rate. This appears to coincide with the reality. The less she knows, the longer she should keep studying; the less patient she is, the less time she would spend studying and the earlier she would start working to earn an income.

The time span of studying or human capital accumulation is increasing with respect to her lifetime and the degrees of contribution (or elasticity) of human capital stock to wage earning. In general, the longer she lives, the longer she would wish to study; the higher the value of human capital to wage earning, the longer she would wish to study.

In the sequel, we analyze the monotonicity of τ with respect to β. We assume that $x_0 < \kappa(0) = \frac{\alpha}{\rho}(1 - e^{-\rho T})$, otherwise, $\tau = 0$. Therefore, τ is uniquely determined by

$$x_0^{1-\beta} + (1-\beta)\tau = \frac{\alpha}{\rho}\left(1 - e^{\rho(\tau - T)}\right),$$

and $\tau \in (0, T)$. It's easy to see that

$$\left(1 - \beta + \alpha e^{\rho(\tau - T)}\right) \frac{\partial \tau}{\partial \beta} = \tau + x_0^{1-\beta} \ln x_0.$$

Therefore $\frac{\partial \tau}{\partial \beta}$ has the same sign as $\tau - x_0^{1-\beta} \ln x_0^{-1}$. Now, we define

$$\Phi(t) = x_0^{1-\beta} + (1-\beta)t + \frac{\alpha}{\rho}\left(e^{\rho(t-T)} - 1\right).$$

Clearly, Φ is strictly increasing, and $\Phi(\tau) = 0$. We denote

$$\Delta = \Phi(x_0^{1-\beta} \ln x_0^{-1}).$$

Then, we obtain the following results immediately:

THEOREM 5. *$\frac{\partial \tau}{\partial \beta} > 0$ (< 0) if and only if $\tau + x_0^{1-\beta} \ln x_0 > 0$ (< 0), which holds if and only if $\Delta < 0$ (> 0).*

The proof is straightforward, and hence is omitted.

COROLLARY 1. *If $x_0 \geq 1$, then, τ is strictly increasing with respect to β.*

The proof is easy, hence omitted.

COROLLARY 2. *For any $\epsilon \in (0, 1)$, τ is strictly increasing with respect to $\beta \in (0, 1-\epsilon)$ if x_0 is sufficiently small.*

Proof. Fix $\epsilon \in (0, 1)$ arbitrarily. It is easy to see that along with $x_0 \to 0$, we have $x_0^{1-\beta} \to 0$ and $x_0^{1-\beta} \ln x_0 \to 0$ uniformly for all $\beta \in (0, 1-\epsilon)$, and hence, $\Delta < 0$ for sufficiently small x_0, which yields the result immediately. ▌

COROLLARY 3. *If $\kappa(0) < 1$, then, τ is strictly decreasing with respect to β, provided $|\kappa(0) - x_0|$ is sufficiently small.*

Proof. In fact, along with $x_0 \to \kappa(0)$, we have $\tau \to 0$ uniformly for all $\beta \in (0, 1)$, and $x_0^{1-\beta} \ln x_0 \to \kappa(0)^{1-\beta} \ln \kappa(0) < 0$, and hence, $\tau + x_0^{1-\beta} \ln x_0 \to \kappa(0)^{1-\beta} \ln \kappa(0) < 0$, which yields the result immediately. ▌

2.3. Some remarks

The relationship between the time span of human capital accumulation and the degrees of contribution of human capital stock to human capital growth is complicated and ambiguous. In general, when the agent's initial human capital is sufficiently low, the higher the human capital stock elasticity of human capital growth rate (or in other words, the greater sensitivity of the speed of knowledge growth to the knowledge stock), the more time the agent would wish to study; in another typical case, where the agent discounts the future so sharply or the human capital stock elasticity of wage earning is so small that the initial human capital upper bound is low, then, an opposite phenomenon occurs, such that the greater sensitivity of the speed of knowledge growth to the knowledge stock, the less time the agent would wish to study.

The solution of the problem P_T can also be called a MRAP (most rapid approach path) in the sense that the path is just the one which most rapidly approaches the critical curve Γ. For the extreme case, where $T = \infty$, the problem P_T becomes the standard linear infinite time horizon optimal control problem

$$\begin{aligned} \max \quad & \int_0^\infty e^{-\rho t}(1-u)x^\alpha dt, \\ \text{such that} \quad & \dot{x} = ux^\beta, \\ & 0 \le u \le 1, \\ & x(0) = x_0, \end{aligned}$$

for which the solution is just the standard MRAP, which most rapidly approaches the horizontal line $x \equiv x^*$, and x^* is its unique steady state. In addition, we notice that the critical curve Γ reduces to this horizontal line as $T \to \infty$.

Let us discuss the meaning of $H = f + \lambda g$. We know that $\lambda = V_x$ is the shadow or real value of human capital, and hence, in any state x, if we choose u as the time of learning, then, f can be interpreted as the explicit (surface, direct) income from working, and λg can be seen as the implicit (latent, indirect) income from human capital accumulation, H therefore is just the total symbolic income at any moment. Moreover, Lemma 2 tells us that, at any time, we should choose u so as to maximize H, rather than f itself. If we only maximize the explicit income and ignore the implicit income, then the total real income of our whole life will not be maximized. In other words, if we win at every time, we will lose as a whole. This is the essence of the Pontryagin maximum principle.

If, in the above model, the assumption $0 < \beta < 1$ is replaced by $\beta = 1$, then similar results still hold and the treatment will be much easier.

In contrast to this finite time horizon problem, where the Markovian optimal strategy is non-stationary, the infinite time horizon homogeneous problem, which is considered in the next section, has a stationary Markovian optimal strategy.

2.4. Extension to the case with depreciation

If human capital depreciation is considered, then, the problem becomes more complicated.

Consider the problem (P)

$$\begin{aligned} \max \quad & \int_0^T e^{-\rho t}(1-u)x^\alpha dt, \\ \text{such that} \quad & \dot{x} = ux^\beta - \delta x, \\ & 0 \le u \le 1, \\ & x(0) = x_0, \end{aligned}$$

where $\rho > 0$ is the agent's discounting rate, $\delta > 0$ is the human capital depreciation rate, and $x_0 \geq 0$ is the agent's initial human capital.

To solve this problem, we first consider its corresponding infinite time horizon problem (P_∞):

$$\begin{aligned} \max \quad & \int_0^\infty e^{-\rho t}(1-u)x^\alpha dt, \\ \text{such that} \quad & \dot{x} = ux^\beta - \delta x, \\ & 0 \le u \le 1, \\ & x(0) = x_0. \end{aligned}$$

We denote the optimal control path and the corresponding optimal state path of problem (P_∞) as w and W respectively.

Clearly, problem (P_∞) is equivalent to problem (P'_∞):

$$\begin{aligned} \max \quad & \int_0^\infty e^{-\rho t}\left((x^\alpha - \delta x^{\alpha+1-\beta}) - x^{\alpha-\beta}\dot{x}\right) dt, \\ \text{such that} \quad & -\delta x \le \dot{x} \le x^\beta - \delta x, \\ & x(0) = x_0. \end{aligned}$$

It is well known that the unique solution of the problem (P'_∞) is the standard MRAP, that is, the optimal state path is simply the path that approaches the unique steady state x_s most rapidly from the initial state x_0 among all of the feasible paths. That is, the optimal state path of problem

(P'_∞) (and problem (P_∞)) is

$$\begin{aligned} W(t) &= x_s \vee \left[x_0 e^{-\delta t}\right], && \text{if} \quad x_0 \geq x_s, \\ &= x_s \wedge \left[\tfrac{1}{\delta} - \left(\tfrac{1}{\delta} - x_0^{1-\beta}\right) e^{-\delta(1-\beta)t}\right]^{\frac{1}{1-\beta}}, && \text{if} \quad x_0 < x_s. \end{aligned}$$

and the corresponding optimal control path for problem (P) is

$$w(t) = I(x_0 < x_s, t < K) + u_s I(t \geq K),$$

where

$$x_s = \left[\frac{\alpha}{\delta(\alpha + 1 - \beta) + \rho}\right]^{\frac{1}{1-\beta}},$$

$$u_s = \frac{\alpha\delta}{\delta(\alpha + 1 - \beta) + \rho},$$

and K is the first time at which the optimal state path W touches the point x_s. More precisely,

$$\begin{aligned} K &= \tfrac{1}{\delta(1-\beta)} \ln \tfrac{\delta^{-1} - x_0^{1-\beta}}{\delta^{-1} - x_s^{1-\beta}}, && \text{if} \quad x_0 < x_s, \\ &= \tfrac{1}{\delta} \ln \tfrac{x_0}{x_s}, && \text{if} \quad x_0 \geq x_s. \end{aligned}$$

And, clearly, the corresponding Markovian strategy for the problem (P_∞) is

$$u(t) = \begin{cases} 1, & x(t) < x_s, \\ u_s, & x(t) = x_s, \\ 0, & x(t) > x_s. \end{cases}$$

We define

$$\kappa(t) = x_s \wedge \left[\frac{\alpha}{\alpha\delta + \rho}\left(1 - e^{(\alpha\delta+\rho)(t-T)}\right)\right]^{\frac{1}{1-\beta}},$$

and denote

$$A = \{t \in [0, T] | W(t) = \kappa(t)\}.$$

Clearly, the set A is either empty or a closed interval. We denote

$$\tau = \sup A,$$

with the convention $\sup A = 0$ if A is empty.

By the same dynamic programming method (Bellman equation), which was used for the case without capital depreciation in the last subsection, we can prove the following result.

THEOREM 6. *For problem* (P), *the unique Markovian optimal strategy is*

$$u(t) = \begin{cases} 1, & x(t) < \kappa(t), \\ u_s, & x(t) = \kappa(t), \\ 0, & x(t) > \kappa(t). \end{cases} \tag{9}$$

The control path and the corresponding state path, induced by the Markovian strategy (9), are respectively

$$u(t) = w(t)I(t < \tau),$$

and

$$x(t) = W(t \wedge \tau)e^{-\delta(t \vee \tau - \tau)}.$$

COROLLARY 4. *If* $x_0 \geq \kappa(0)$, *then, the optimal state path is decreasing. If* $x_0 < \kappa(0)$, *then, the optimal state path is increasing in* $[0, \tau]$ *and decreasing in* $[\tau, T]$.

We can see that, in general, the finite time horizon case is similar to the infinite time horizon case. The main difference is as follows. For the infinite time horizon case, there are only two phases: in the first phase, the state approaches the steady state at the most rapid speed; in the second phase, it stays in the steady state forever. In comparison, in the finite time horizon case, there are in general three phases: in the first phase, the state approaches the steady state at the most rapid speed; in the second phase, it stays in the steady state for some time; in the third phase, the state decreases at the most rapid speed. This third phase always exists, while the first two may not for certain parameters in a specific range.

In our human capital accumulation context, this can be explained as follows. In general, the steady state is the best, the human capital stock is kept at a suitable level, and the time is allocated between learning and working according to a suitable fixed ratio. In this case, the total benefit is kept sustainable at the highest level, where the total benefit comprises two parts: (i) instant income (the direct earning from working); and (ii) latent income (the implicit shadow value added from the increment of human capital through learning). However, if the agent knows her lifetime exactly, then, for some period before death, learning is not needed. Learning is fruitless, since the essential object of learning is to change her fate of future. Before the time of her death, which is known to her, however, she considers that she does not have future, and hence, gives up learning and devotes all of her time to working and earning money.

Of course, this is not consistent with the reality, and this inconsistence relates, in part, to the fact that we have ignored leisure and recreation. If we modified our model to include recreation, this defect could be overcome. Such a modification is left for subsequent works.

3. A STOCHASTIC CASE

In this section, we consider the stochastic case: the agent's lifetime T is random and exponentially distributed with parameter $\lambda > 0$, and hence, the expected lifetime is $1/\lambda$. We assume that the depreciation rate of human capital is a constant $\delta > 0$.

The agent will find the optimal path of time allocation between learning and working, that is, the problem the agent will try to solve is the following stochastic optimal control problem $(\mathbb{P})$:

$$\begin{aligned} \max \quad & E\int_0^T e^{-\rho t}(1-u)x^\alpha dt, \\ \text{such that} \quad & \dot{x} = ux^\beta - \delta x, \\ & 0 \le u \le 1, \\ & x(0) = x_0, \end{aligned}$$

where $\rho > 0$ is the agent's discounting rate, and $x_0 > 0$ is the agent's initial human capital.

For convenience, we denote

$$x_* = \left(\frac{1}{\delta}\right)^{\frac{1}{1-\beta}},$$

$$x_s = \left(\frac{\alpha}{\delta(\alpha+1-\beta)+\rho+\lambda}\right)^{\frac{1}{1-\beta}},$$

and

$$u_s = \frac{\alpha\delta}{\delta(\alpha+1-\beta)+\rho+\lambda}.$$

Clearly,

$$x_s < x_*,$$

and x_* is the unique steady state for the dynamical system

$$\dot{x} = x^\beta - \delta x$$

on the strictly positive half line $(0, \infty)$, and it is globally and asymptotically stable.

THEOREM 7. *For the problem* $(\mathbb{P})$*, the unique Markovian optimal strategy is*

$$u(t) = \begin{cases} 1, & x(t) < x_s, \\ u_s, & x(t) = x_s, \\ 0, & x(t) > x_s. \end{cases} \tag{10}$$

Proof. It is well known that the problem $(\mathbb{P})$ is equivalent to

$$\begin{aligned} \max \quad & \int_0^\infty e^{-(\rho+\lambda)t}(1-u)x^\alpha dt, \\ \text{such that} \quad & \dot{x} = ux^\beta - \delta x, \\ & 0 \le u \le 1, \\ & x(0) = x_0, \end{aligned}$$

which, in turn, is equivalent to the following dynamic programming problem $(\mathbb{P}')$:

$$\begin{aligned} \max \quad & \int_0^\infty e^{-(\rho+\lambda)t} \left((x^\alpha - \delta x^{\alpha+1-\beta}) - x^{\alpha-\beta}\dot{x} \right) dt, \\ \text{such that} \quad & -\delta x \le \dot{x} \le x^\beta - \delta x, \\ & x(0) = x_0. \end{aligned}$$

It is easy to see that the unique solution of the above problem is the standard MRAP, that is, the optimal path of x is just the one that approaches x_s most rapidly from the initial state among all the feasible paths. And, obviously, the corresponding Markovian strategy is just equation (10). The theorem is proved. ∎

In the typical case, where the agent's initial capital is low, the optimal strategy for her is to continue devoting all of her time to human capital accumulation until this reaches a specific level through a finite time span, and thereafter, to allocate her time between learning and working at a fixed ratio in order to offset the effect of human capital depreciation. In other words, in reality, a person's knowledge is continuously depreciating. Hence, she should continue studying and updating her knowledge all of her life in order to keep up.

For the problem $(\mathbb{P}')$, denote the MRAP as x, and let

$$\tau = \inf \{ t \ge 0 | x(t) = x_s \},$$

that is, τ is the least time needed to approach the steady state from the initial state. We term it the schooling time. And, for the problem ($\mathbb{P}$), the approaching process could be interrupted by the death of the individual, and hence, we call $T \wedge \tau$ the stopped schooling time.

It is easy to find that

$$\tau = \begin{cases} \frac{1}{\delta(1-\beta)} \ln \frac{x_*^{1-\beta} - x_0^{1-\beta}}{x_*^{1-\beta} - x_s^{1-\beta}}, & x_0 \in (0, x_s), \\ \frac{1}{\delta} \ln \frac{x_0}{x_s}, & x_0 \in [x_s, \infty) \end{cases}$$

$$E[T \wedge \tau] = \frac{1}{\lambda}\left(1 - \exp\{-\lambda\tau\}\right).$$

The next theorem follows immediately.

THEOREM 8. *x_s is increasing in α and β, and is decreasing in ρ, λ and δ. $E(T \wedge \tau)$ is decreasing with respect to $\rho, \lambda, \delta, x_0 \in (0, x_s)$ and increasing with respect to $\alpha, x_0 \in (x_s, \infty)$.*

Recall that in a typical case, where the initial human capital is low, the initial time period is fully devoted to learning without working. The expected stopped schooling time is decreasing with respect to small initial capital, the impatience degree represented by the discount rate, and the human capital depreciation rate. It is increasing with respect to the expected lifetime, the elasticity of human capital to wage, and the initial human capital if it is sufficiently large. All of these points coincide with our intuition.

At the end of this paper, we comment that, in Chapter 10 (Human Capital and Economic Growth) of his book, *Introduction to Modern Economic Growth*, Acemoglu (2009) presented a simplified Ben-Porath model as follows:

$$\begin{aligned} \max \quad & \int_0^\infty e^{-\rho t}(1-u)x\,dt, \\ \text{such that} \quad & \dot{x} = f(ux) - \delta x, \\ & 0 \le u \le 1, \\ & x(0) = x_0 \end{aligned}$$

where f satisfies $f(0) = 0$, $f' > 0, f'' < 0$, $f'(0) = \infty$, and $f'(\infty) = 0$. We can prove that it has a unique steady state (x_s, u_s), a saddle point, and a unique perfect Markovian strategy

$$u(t) = \begin{cases} 1, & x(t) < u_s, \\ u_s, & x(t) \ge u_s, \end{cases}$$

and the corresponding trajectory in the $x - u$ plain is simply an MRAP.

Acknowledgements. The first author acknowledges the support of Key Laboratory of Mathematical Economics and Quantitative Finance, Peking University.

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