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On the Interaction between Small Decay, Agent Heterogeneity and Diameter of Minimal Strict Nash Networks in Two-way Flow Model

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This paper studies the roles of value heterogeneity, that is, agents are heterogeneous in terms of information values that they possess, in determining the shapes of two-way flow Strict Nash networks when a small amount of decay is present. I do so by extending the two-way flow network with small decay of De Jaegher and Kamphorst (J ECON BEHAV ORGAN, 2015). Results of this extension show that the effects of value heterogeneity resemble the effects of cost heterogeneity found in the literature. Another surprising finding is that value heterogeneity can extend diameters of Strict Nash networks without changing other properties.

Key Words: Network formation; Strict Nash network; Two-way flow network; Agent heterogeneity.

JEL Classification Numbers: C72, D85.

1. INTRODUCTION

The seminal work of Goyal (2000a) proposes a model of network formation that is noncooperative, in the sense that two agents do not need to cooperate to form a link. This follows from two key assumptions: link formation cost is borne solely by link sender and information possessed by each agent is nonrival. Nash equilibrium and/or Strict Nash equilibrium in pure strategies are then used to characterize properties of equilibrium networks. These equilibrium networks are called Nash networks (NNs) and Strict Nash networks (SNNs) respectively. The simplicity of this seminal model means it can serve as a basis that allows researchers to study ef-

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fects of various assumptions on equilibrium networks in isolation of other effects. Literature in this spirit abounds. Examples are imperfect communication through links (Bala and Goyal (2000b), Haller and Sarangi (2005)), group identity (Dev (2014), Dev (2018)), agent heterogeneity (Galeotti et al. (2006); Billand et al. (2011); Charoensook (2015)) and information confirmation (Billand et al. (2017)).

Of particular interest to this paper is the literature on agent heterogeneity. Within this subbranch of the literature, much have been studied about the roles of agent heterogeneity in terms of link formation cost (Galeotti et al. (2006), Billand et al. (2011), Billand et al. (2012), Charoensook (2015)). However, little is known about the role of heterogeneity in terms of information value. For example, *if we assume that agents differ in terms of information value that they possess and communication via links is not perfect, how would such an assumption influence the shapes and properties of SNN*? This paper seeks to address this question. It does so by introducing various forms of value heterogeneity to another rigorous extension of the seminal work of Bala and Goyal (2000a) described above — the minimal two-way flow with small decay studied by De Jaegher and Kamphorst (2015) (DJK), where the term 'small decay' here refers to the fact that communication through links is not perfect but nearly so, and hence no superfluous links are worth establishing since every agent who chooses to form links for his best interest would find that the benefits from establishing a superfluous link cannot cover the associated link formation cost [1] [2]. More specifically, I state the questions, which have not yet been studied, that this paper seeks to contribute to the literature below:

1. Compared to several models in the literature that study the impacts of cost heterogeneity on the shapes of SNN (e.g., Galeotti et al. (2006); Billand et al. (2011)), what are similarities and differences between the roles of value heterogeneity and cost heterogeneities in SNN?

[1] The small decay assumption will soon be introduced in the next section. See also Lemma 4 of DJK, which is included in Appendix A of this paper.

[2] There are at least two rationales for assuming small decay as opposed to any level of decay. First, as explained in DJK "Information decay has two effects. First, ex-ante homogeneous players become heterogeneous by their position in the network . . . Second, decay may give the individual player an incentive to sponsor links to players he is connected to, but indirectly." My focus on small decay as opposed to all levels of decay, therefore, makes it possible to study the interaction between the first effect mentioned in DJK (2015) with value heterogeneity in isolation of the second effect. Second, small decay implies that SNN is minimal (See Lemma 4 of DJK (2015), which is included in Appendix A of this paper). So does no decay assumption. This in turn allows the results of this paper, which assumes value heterogeneity and small decay, to be compared with the existing literature that characterize SNN by assuming cost heterogeneity with no decay (Galeotti et al. (2006), Billand et al. (2011)).

2. Compared to the model of DJK, which assumes small decay and homogeneity in information value and link formation cost, how does the introduction of value heterogeneity impact the shapes and properties of SNNs?

I elaborate briefly on the findings of this paper, the existing literature and relationship between the two as follows. For the question (1), concerning the roles of cost heterogeneity (denoted by c_{ij}, the link formation cost that i pays if he wishes to form a link with another agent j), there are three major forms: (i) player heterogeneity ($c_{ij} = c_i$ for every agent i), i.e., link formation cost depends only on the identity of link sender, (ii) partner heterogeneity ($c_{ij} = c_j$ for every agent j), i.e., link formation cost depends only on the identity of link recipient, and (iii) general heterogeneity, which is such that c_{ij} depends on the identity of link sender as well as link recipient, without any restriction. The roles these three forms of cost heterogeneity on SNN, assuming no decay, are studied in Galeotti et al. (2006) and Billand et al. (2011), with fine-detail equilibrium characterization given [3]. For heterogeneity in information value, little is known. To the knowledge of the author the only finding in the literature is that, given no information decay and value heterogeneity assumes general form (V_{ij} being the information value of j that i receives without any restriction), every non-empty component of SNN is a center-sponsored star, which is identical to the shape of non-empty SNN when agent homogeneity is assumed (c.f. Proposition 3.1 in Galeotti et al. (2006)) [4]. Put differently, *what is known in the literature is that value heterogeneity has very little to no effect on the shape of SNN.* A simple intuition explains this result: due to the assmumption of no decay every agent in the same component receives the same amount of information. Therefore, from the point of view of link sender all agents are homogeneous in the sense that they are equally attractive as link recipients, which is not different from the original setting of Bala and Goyal (2000a) that assumes value homogeneity. That is, under no decay assumption value heterogeneity cannot cause agents in the same component to be heterogeneous. Of course, if the assumption of no decay is removed this line of reasoning is no longer valid. *This raises the question of whether value heterogeneity can have an impact on the properties and shapes of SNNs if decay is assumed to be present.* This is precisely a primary motivation and the main research question of this paper. Indeed to the knowledge of the author this paper is the first work in the literature that finds that value heterogeneity has substantive impacts on the shapes and properties of SNN. Specifically, the main analysis in section 3 provides equilibrium characterizations of SNN, assuming that value heterogeneity

[3]There are also some other forms, which allows c_{ij} to depend on both link sender i and recipient j. See, e.g., Charoensook (2015) and Billand et al. (2010).

[4]See the definition of center-sponsored star in the next section.

takes three forms, namely player value heterogeneity ($V_{ij} = V_i$), partner value heterogeneity ($V_{ij} = V_j$) and general value heterogeneity (V_{ij} has no restriction). Note that these three forms of value heterogeneity correspond to the three forms of cost heterogeneity mentioned above. This allows for the comparison between existing literature that studies the roles of cost heterogeneity on SNNs mentioned above with the results in this paper that studies the roles of value heterogeneity on SNNs. The last section of this paper elaborates on this comparison.

Next, for the question (2), compared to the model of DJK, which assumes small decay and homogeneity in information value and link formation cost, the most surprising finding is that partner heterogeneity in information value does not change the main properties of SNN, yet it largely extends diameters of SNN (See Proposition 1 and Figure 3 in the main analysis section). Indeed, a line network, which is the network with a maximum diameter, can also be SNN. This is a sharp contrast to a major finding of DJK, which finds that under the assumption of small decay the diameter of SNN is relatively small compared to the population size. Since DJK mention that this feature of SNN resembles small-world networks, a major finding of this paper is that the introduction of partner value heterogeneity into the model of DJK breaks away this resemblance. See the laste two paragraphs of section 3.1 for further elaboration.

Lastly, I remark that this paper also fulfills some missing technical details in terms of equilibrium chracterization in DJK. Specifically, Lemma 1 in DJK states that a middle agent, *if exists*, is always a link recipient for a small level of information decay in an SNN. Therefore, we do not know the identity of the link recipient if a middle agent does not exist. More generally, we also do not know the identity of the link recipient for a small level of decay if value heterogeneity is assumed. In this paper, I fulfill these missing pieces by establishing a new concept called 'positionally optimal agent,' a concept that is closely related yet more general than the concept of middle agent in DJK (see Remark 1 and Example 2 in this paper). I then show that: (i) positionally optimal agent always exists and (ii) as decay becomes very small a positionally optimal agent is always a link recipient. These two key results are major building blocks for the equilibrium chracterization and sufficiency result of SNNs in the main analysis section. The definition of positionally optimal agent is given in Section 2.1. Lemma 1 in Section 3.1, which generalizes Lemma 1 of DJK to the case of partner value heterogeneity, provides the formal statement of these results.

This paper proceeds as follows. I introduce the model and notations in the next section (Section 2). Importantly, I introduce concepts that generalize key concepts of DJK, namely positionally optimal agent and generalized balancing condition in Section 2.1. They generalize the concepts of middle agent and balancing condition in DJK respectively. The third section, which is the main analysis section, is then divided into three subsections. First is the generalization of DJK's model by allowing for partner value heterogeneity. Proposition 1 in this subsection provides a characterization of SNN under this assumption. I also discuss on some interestingly intricate interaction between value heterogeneity, small decay, and the diameter of SNNs. Specifically I point out that: (i) a balanced SNN is not necessarily resilient to the changes in decreasing decay level and (ii) consequently, for the two-way flow model to have properties that resemble small-world network and serves as a micro-foundation of preferential attachment, it has to be the case that heterogeneity among information value possessed by each agent is sufficiently small. The next two subsections then use analyses in the first subsection to establish equilibrium characterizations for the cases of player value heterogeneity and general value heterogeneity. Finally, in the concluding remarks section I compare the effects of cost heterogeneity with no decay, as in Galeotti et al. (2006) and Billand et al. (2011), and the effects of value heterogeneity with small decay, found in this paper, on SNN. A major finding of this comparison is that cost heterogeneity and value heterogeneity appear to have very similar effects on SNN.

2. THE MODEL

Since this paper is an extension of De Jaegher and Kamphorst (2015) model by allowing for value heterogeneity, most notations here will also follow this aforementioned model. Exceptions are the definitions of the three specific forms of heterogeneity — partner heterogeneity, player heterogeneity and general heterogeneity. These definitions follow Galeotti et al. (2006) and Billand et al. (2011).

Link establishment and individual's strategy. Let $N = \{1, \dots, n\}$ be the set of all agents. An agent $i \in N$ can form a link with another agent j without j's consent. ij denotes such a link. The set of all possible links that i can form is $L_i = \{ij : j \in N \backslash \{i\}\}$. Let $L = \cup_{i \in N} L_i$. A strategy of agent i, which is the set of links that he chooses to form, is denoted by g_i. The strategy space of agent i, denoted by G_i, is $G_i \equiv 2^{L_i}$. The strategy space is $G \equiv 2^L$. A strategy profile is, therefore, $g = \cup_{i \in N} g_i$.

Network representation. A strategy profile can be visually represented by a network, where each node represents an agent and an arrow from node i that points towards another node j represents a link $ij \in g_i$ [5].

[5]See Figure 2 in DJK for examples. Readers should also be noted of discrepancy regarding network representation used in the literature. While this paper and DJK use an arrow from i to j to represent the link $ij \in g_i$, Bala and Goyal (2000a), Galeotti et al. (2006) and Billand et al. (2011) uses the opposite, which is such that an arrow from j to i represents the link $ij \in g_i$.

Therefore, following the literature, the term strategy profile g and network g will be used interchangeably. In case that link sponsorship is not related to the analyses, arrows are removed. Such a network is called an undirected network[6].

Information flow. Let $N_i^S\{g\} \equiv \{j \in N : ij \in g\}$ denote the set of all agents i establishes a link with. We write $\bar{ij} \in g$ if and only if $ij \in g$ or $ji \in g$ or both. Information flow is assumed to be two-way, hence the term 'two-way flow model.' Specifically, let a path between i and j be defined as $P_{ij}(g) = \left\{\overline{i_0 i_1}, ..., \overline{i_{k-1} i_k}\right\} \subseteq g$ such that $i_0 = i, i_k = j$. If there is a path between i and j, we say that i and j are connected. A shortest path between i and j is, of course, the path(s) between i and j with the least amount of links. A distance between i and j is defined as the amount of links of the shortest path(s). If $j = i$ then we assume, following the literature, that the distance of between i and himself is 0.

Value heterogeneity and cost heterogeneity. Let c_{ij} denote the link formation that i bears to form a link with j. Let $\mathcal{C} = \{c_{ij}\}_{ij \in N \times N, i \neq j}$ be the cost structure. If $c_{ij} = c$ for every $i, j \in N$, then $\mathcal{C}$ is said to satisfy cost homogeneity. If $c_{ij} = c_j$ for every $i \neq j$ then $\mathcal{C}$ is said to satisfy partner cost heterogeneity. If $c_{ij} = c_i$ for every $i \neq j$ then $\mathcal{C}$ is said to satisfy player cost heterogeneity. If no restriction is imposed on $\mathcal{C}$ then we say that $\mathcal{C}$ satisfies general cost heterogeneity [7] [8].

Similarly, let V_{ij} denote the value of information of j that arrives to i, given that information flow is perfect. Let $\mathcal{V} = \{V_{ij}\}_{ij \in N \times N}$ be the value structure. If $V_{ij} = V$ for every $i, j \in N$, then $\mathcal{V}$ is said to satisfy value homogeneity. If $V_{ij} = V_j$ for every $i \neq j$ then $\mathcal{V}$ is said to satisfy partner value heterogeneity. If $V_{ij} = V_i$ for every $i \neq j$ then $\mathcal{V}$ is said to satisfy player value heterogeneity. If no restriction is imposed on $\mathcal{V}$ then we say that $\mathcal{V}$ satisfies general value heterogeneity.

Information quantity Let $\sigma \in [0, 1]$ denote the decay factor, which represents the assumption that the proportion that a piece of information decays is $(1 - \sigma)$ per each link it traverses. That is, if the distance between i and j is k then the information that i receives from j is $\sigma^k V_{ij}$. Naturally, if $\sigma = 1$ then we say that there is no (information) decay. If $\sigma < 1$ then there is (information) decay.

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[6] See, for instance, Example 1 and Example 3 in section 2.1 and 3.1 of this paper

[7] The definitions of partner cost heterogeneity, player cost heterogenety and general cost heterogeneity here follow Billand et al. (2011), Galeotti et al. (2006) and Galeotti et al. (2006) respectively.

[8] At this point, I anticipate that readers may find these definitions to be awkward, as agent homogeneity is enclosed as a special case of agent heterogeneity rather than being two polarized concepts. However, these definitions have been used in several past studies (see Billand et al. (2011); Galeotti et al. (2006) and Charoensook (2015)). Consequently, to ease the comparison this paper follows suit.

Small decay assumption Following the assumption above and assuming a path between i and j exists, an agent i can improve the flow of information from another agent j by establishing a link that leads to a shorter path between i and j. Of course i has enough incentive to do so only if the benefit from so doing covers the link establishment cost. However, if the decay is sufficiently small, i.e., σ is sufficiently close to 1, then the decay incurred by each path becomes nearly identical and the incentive to establish an extra path disappears. More rigorously, Lemma 4 in DJK (See Appendix A) states that "there exists $\sigma_M < 1$ such that for all $\sigma > \sigma_M$ no player in any possible network wishes to sponsor a non-minimal link." As a result, there is at most only one path between any pair of agents. This small decay assumption, which by Lemma 4 of DJK guarantees that no agent finds an incentive to establish a non-minimal link, is assumed in DJK and will be assumed throughout this paper.

Better-informed agent In a network g, let $N_i(g)$ be the set of agents with whom i is connected via a path including i himself and let N_i^d be the set of all agents whose distance from i is d. Note that $N_i^0 = \{i\}$. Define total ex-post information that i received in the network g as $I_i(g) = \sum_{d=0}^{n-1} \sum_{j \in N_i^d(g)} \sigma^d V_{ij}$ [9]. Consider $M \subseteq N$ that is connected, ie., there is a path between any distinct pair of agents in M. Let $g_M = \{ij \in g : i, j \in M\}$. i is better-informed than j in the set M if $I_i(g_M) \geq I_j(g_M)$ and best-informed in M if $I_i(g_M) \geq I_j(g_M)$ for all $j \in M$.

Network-related notations A subnetwork of g is a network g' such that $g' \subset g$. A subnetwork g' of g is said to be induced by $M \subset N$ if $ij \in g'$ whenever $ij \in g$ and $i, j \in M$. A network is said to be connected if every pair of agents in the network is connected. g', a subnetwork of g, is said to be a component of g if g' is a maximal connected subnetwork of g. An agent who has no link with any other agent is called a singleton. Note that a singleton is also a component. Specifically we call a component that is a singleton an empty component [10]. A non-empty component of a network or a network is minimal if there is at most one path among any pair of agents in the network. A minimally connected network is a rooted directed tree if every agent receives exactly one link except one agent that receives no link. The agent that receives no link is called root. A minimally connected network or a minimal component of a network is a center-sponsored star (CSS) if it is a rooted directed tree such that every agent who receives a link from the root establishes no link. A minimally connected network or

[9]Note that this definifion generalizes that of DJK in the following sense. If $\mathcal{V}$ satisfies value homogeneity then $I_i(g) = \sum_{d=0}^{n-1} \sigma^d |N_i^d(g)|$, which is precisely the same as in DJK. Note further that if $\mathcal{V}$ satisfies player value homogeneity then $I_i(g) = \sum_{d=0}^{n-1} \sigma^d |N_i^d(g)| V_i$ and if $\mathcal{V}$ satisfies partner value homogeneity then $I_i(g) = \sum_{d=0}^{n-1} \sum_{j \in N_i^d(g)} \sigma^d V_j$.

[10]The definitions of singleton and empty component follow Bala and Goyal (2000a).

a minimal component of a network is a periphery-sponsored star (PSS) if there is precisely one agent who receives a link from every other agent, and every other agent receives no links.

Next, I introduce some specific types of agents and links. A non-recipient agent is an agent that receives no links. A multi-recipient agent is a recipient who receives more than one link. Let $A_{ij}(g)$ where $ij \in g$ be the set of agents that i observes exclusively via the link ij. That is, $A_{ij}(g) \cap N_i(g \backslash ij) = \emptyset$ and $A_{ij}(g) \cup N_i(g \backslash ij) = N_i(g)$. In a minimal connected network g, a link ii' is said to point to agent j if $j \in A_{ii'}(g)$ and to point away from agent j if $j \notin A_{ii'}(g)$. Similarly, links $ii' \in g$ and $jj' \in g$ are said to point to each other if $i' \in A_{jj'}(g)$ and $j' \in A_{ii'}(g)$. A link $ii' \in g$ is said to be an end link if $A_{ii'}(g) = \{i'\}$. For a link ii' that is an end link, i' is an end recipient and i is an end sponsor.

Lastly, I introduce some notations concerning information flow. Let $M \subset N$ be a set of agents and g' be a network induced by M. Let g' be minimally connected. Due to the fact that there is only one path between every pair of agents in g', a removal of the link $\bar{ij} \in g'$ further splits g' into two disconnected subnetworks - one containing i and the other one containing j. Let $D^i(g' - ij)$ and $D^j(g' - ij)$ denote these two subnetworks respectively. Furthermore, let $N\left(D^i(g' - ij)\right)$ and $N\left(D^j(g' - \bar{ij})\right)$ be the sets of agents in these two sets respectively [11]. We say that information of k flows to i via j if $k \in N\left(D^j(g' - \bar{ij})\right)$. Similarly, we say that information of k flows to j via i if $k \in N\left(D^i(g' - \bar{ij})\right)$.

The payoffs. Let $V_i(g) = f(I_i(g))$ where $f' > 0$. The payoff of i is:

$$U_i(g) = V_i(g) - \sum_{j \in N_i^S} c_{ij}.$$

A special case of this is the so-called linear payoff, which is:

$$U_i(g) = I_i(g) - \sum_{j \in N_i^S} c_{ij}$$

Nash Networks and Strict Nash Networks. Consider a network g^* such that a strategy of i is $g_i^* \subset g^*$. Let $g_{-i}^* = g^* \backslash g_i^*$ so that $g^* = g_i^* \sqcup g_{-i}^*$. g_i^* is said to be a best response of i if $U_i(g^*) \geq U_i\left(g_i \cup g_{-i}^*\right)$ for every $g_i \neq g_i^*$. If the inequality is strict, then g_i^* is a unique best response of i. g is said to be a NN if every agent chooses his best response. A SNN is a network such that every agent chooses his unique best response. In the main analysis section, SNN is used for equilibrium characterization rather than NN in order to allow for the comparison between our models,

[11] These notations follow Charoensook (2019).

which assume value heterogeneity, and existing models in the literature that assumes cost heterogeneity such as Billand et al. (2011) and Galeotti et al. (2006).

2.1. Positionally Optimal Agent and Generalized Balancing Condition: Definitions

This subsection establishes two new concepts - Positionally Optimal Agent and Generalized Balancing Condition. They generalize two concepts in DJK, namely middle agent and the balancing condition respectively by allowing for partner value heterogeneity. These two new concepts will then be used to generalize key results of DJK in the main analysis section, which provide characterization and sufficiency result for SNN given that partner value heterogeneity is assumed [12]. In Sections 3.2 and 3.3, which are extensions of the main analysis section, these concepts will also be further generalized to allow for other forms of value heterogeneity. Before doing so, I first recall the definitions of middle agent and balancing condition from DJK below.

Definition 2.1. (Middle Agent (DJK, Definition 2, p.224)) Consider a minimal connected subset of players M, $M \subset N$. We say that player j is in the middle of set M in network g if for each neighbor k of j in the network g the following holds: in g more than half of the players in M (including k and j) are closer to j than k [13].

Definition 2.2. (Balancing Condition and Balanced Network (DJK, Definition 3, p.224)) A minimal network g satisfies the balancing condition if for any $ij \in g$ we have that j is in the middle of $A_{ij}(g)$. In that case, we say that the network g is balanced [14].

Intuitively, a middle agent is an agent whose position or location is superior to every other agents in the same minimal component, in the sense that due to his location he suffers from information decay less than every other agent does, given that the decay is sufficiently small. This causes him to be a best-informed agent, which in turn makes him attractive as a link recipient. This is the intuition behind Lemma 1 and Proposition 2 of DJK (readers can refer to Lemma 1 and Proposition 2 of DJK in Appendix A of this paper).

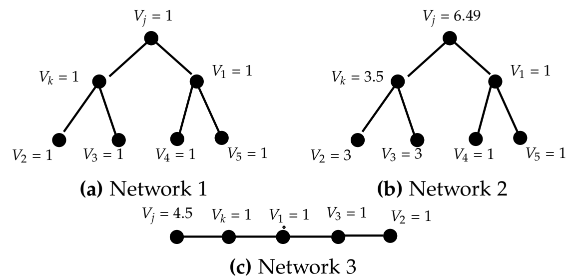
[12]This subsection is inspired by a comment from Christophe Bravard, to whom I wish to express my gratitude.
[13]Figure 6 in DJK provide examples of middle agents.
[14]Figure 6 in DJK provide examples of balanced networks.

At this point, two natural questions arise: (i) if $\mathcal{V}$ does not satisfy homogeneity (recall that DJK's model assumes such), will a middle agent remain attractive as a link recipient?; and (ii) regardless of whether $\mathcal{V}$ satisfies homogeneity or not, in case that a middle agent does not exist, which agent is most attractive as a link recipient? I establish Lemma 1 in the main analysis section to answer these questions. The first steps towards the establishment of this Lemma, though, is to build a concept that generalizes the concept of middle agent of DJK, which I do so below by allowing for partner value heterogeneity [15].

DEFINITION 2.3. (Positional Superiority) In a network g, consider a minimal connected subset of players $M \subset N$ and let $i, j \in M$. Let $X_{i,M}(i,j;g)$ and $X_{j,M}(i,j;g)$ be the set of agents that are closer to i than to j (including i himself) and the set of agents that are closer to j than to i (including j himself) respectively. Let $\mathcal{V}$ satisfies partner value heterogeneity. i is positionally superior to j in the set M if $\sum_{k\in X_{i,M}(i,j;g)} V_k \geq \sum_{k\in X_{j,M}(i,j;g)} V_k$. We write $i \succsim^M j$ for short. If the inequality is strict, we write $i \succ^M j$ for short.

That is, to consider whether i is positionally superior to j, we first identify the set of all agents that are closer to i than to j (including i himself) and the set of all agents that are closer to j than to i (including j himself), we then identify the sum of information value of each of these two sets. If the former is at least as much as the latter, then it is concluded that i is positionally superior to j. Example 1 below illustrates this concept.

FIG. 1. Three undirected networks for Example 1.



[15]Note that the concepts of positional superiority and generalized balancing condition here assume partner value heterogeneity, since they are intended to be used in the first part of the main analysis section, which also assumes partner value heterogeneity. Afterwards, these concepts as well as the results in the main analysis section will be further extended to accommodate other forms of heterogeneity in the second part of the main analysis section.

Example 2.1. Consider the three (undirected) networks in Figure 1. Let $M = N$. Note that $\mathcal{V}$ for the network 1 satisfies homogeneity while $\mathcal{V}$ for the network 2 and 3 satisfies partner heterogeneity. In the network 1 , j is positionally superior to k because $\sum_{l\in X_{j,M}(j,k;g)} V_l = 1+1+1+1 = 4 \geq \sum_{l\in X_{k,M}(j,k;g)} V_l = 1+1+1 = 3$. In the network 2 , k is positionally superior to j because $\sum_{l\in X_{j,M}(j,k;g)} V_l = 6.49+1+1+1 = 9.49 < \sum_{l\in X_{k,M}(j,k;g)} V_l = 3.5+3+3 = 9.5$. In the network 3, j is positionally superior to k because $\sum_{l\in X_{j,M}(j,k;g)} V_l = 4.5 \geq \sum_{l\in X_{k,M}(j,k;g)} V_l = 1+1+1+1 = 4$.

Intuitions related to the above examples are elaborated here. First, if value homogeneity - as in the network 1 in Figure 1 - is assumed then the middle agent (if exists) is always strictly positionally superior to every other agent. This is because value homogeneity necessitates that i is positionally superior to j if the amount of agents that are closer to i than j is more than the amount of agents that are closer to j than i, which is precisely the definition of middle agent. Of course, if partner value heterogeneity is assumed then this line of reasoning breaks down. Indeed, note that in the second network in Figure 1 there are only three agents who are closer to k than j while there are four agents who are closer to j than k. But k is positionally superior to j because those three agents who are closer to k possess relatively higher information values. This line of reasoning is further illustrated in the third network in Figure 1. Even though an agent (agent k in this case) is located at an extreme end of the network he is still positionally superior to every other agent because the information that he possesses has a value that is relatively much higher than those of others. In conclusion, whether an agent is positionally superior to another agent depends not only on the relative locations of two agents but also information value of other agents in the network. This intuition is used to generalize the concept of middle agent in DJK as follows:

Definition 2.4. (Positionally Optimal Agent) In a network g, consider a minimal connected subset of agents $M \subset N$. Let $\mathcal{V}$ satisfies partner value heterogeneity. An agent $i \in N(g)$ is said to be a <u>positionally optimal agent</u> in the set M if $i \succsim^M j$ for every $j \in M$ and $j \neq i$.

That is, an agent is positionally optimal in the set M if he is positionally superior to every other agent in the set M.

Example 2.2. Consider the three networks in Figure 1. It is straightforward to verify that the middle agent j is positionally optimal in network 1, k is positionally optimal in network 2, and j is positionally optimal in the line network 3.

Remark 2.1. In a network g, consider a minimal connected subset of players $M \subset N$. a middle agent (if exists) is a unique positionally optimal agent in M given the specific assumption that value homogeneity is assumed [16].

Finally, I generalize the balancing condition in DJK. Recall that the balancing condition in DJK requires that every link recipient is a middle agent, it naturally follows that the generalized balancing condition below simply replaces the term middle agent by the term positionally optimal agent, which is a more general term.

Definition 2.5. (Generalized Balancing Condition) Let $\mathcal{V}$ satisfies partner value heterogeneity. A minimal network g satisfies the generalized balancing condition if for any $ij \in g$ we have that j is a positionally optimal agent in the set $A_{ij}(g)$ [17].

3. MAIN RESULTS

3.1. Generalization: Partner Heterogeneity and Small Decay

In this subsection, I generalize two key results of DJK - characterization of all non-empty minimal Nash networks (Proposition 1) and sufficiency (Proposition 2). I do so by allowing for *both* partner heterogeneity in information value and small decay rather than small decay and value homogeneity as in DJK. This subsection will first provide a generalization of Lemma 1 of DJK. I also elaborate on how this generalization fulfills some technical details left unmentioned in DJK. Proposition 1 in this subsection then uses this generalized Lemma 1 of DJK to provide characterization and sufficiency results for SNN for the case of partner value heterogeneity. Finally, this subsection concludes with some remarks on how partner value heterogeneity impacts SNN compared to the case of value homogeneity in DJK. All proofs of propositions and lemmata in this paper and the original results of DJK, to which readers can refer for the sake of comparison with my results, are relegated to the Appendix.

Lemma 1. (Generalization of Lemma 1 [De Jaegher and Kamphorst (2015)]) *Consider the case of small decay and $\mathcal{V}$ satisfying partner value*

[16] This follows directly from the definition of middle agent, which is an such that more than half of the agents is closer to him than to every other agent.

[17] Recall, from the previous subsection, that $A_{ij}(g)$ is the set of all agents that i observes exclusively via the a link ij.

heterogeneity. There exists a $\sigma_B < 1$ such that the following property holds for any minimal Nash network g. Consider any $ij \in g$:

1.In $A_{ij}(g)$, a positionally optimal agent always exists .

2.Let k be a positionally optimal agent in $A_{ij}(g)$, then $j = k$ for all $\sigma \in [\sigma_B, 1)$

Importantly, not only does Lemma 1 above generalize Lemma 1 in DJK but also fulfills a missing piece not mentioned in DJK, which is the existence of link recipient in a NN[18]. Specifically Lemma 1 in DJK only guarantees that a link recipient is a middle agent, 'if exists'. Lemma 1 in DJK, therefore, does not answer who the link recipient is if a middle agent does not exist. My Lemma above fulfills this missing piece by guaranteeing that a positionally optimal agent always exists, and that he is always a link recipient in a NN given that information decay is sufficiently small [19].

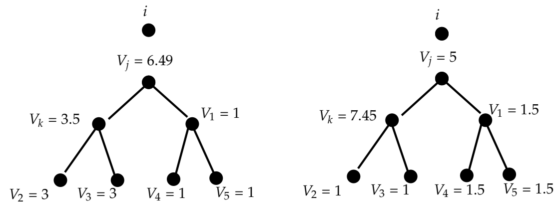
In addition to the above technical differences, worth mentioning is also some intuitive comparison between Lemma 1 in DJK and my generalized Lemma 1 of DJK above. First, consider Lemma 1 of DJK. Since the scope of Lemma 1 of DJK, which assumes value homogeneity, limits to the fact that a middle agent is a link recipient, an impression is that a middle agent is an agent who is 'resilient' to the changes in decreasing decay level. That is, so long as $\sigma > \sigma_B$ a middle agent is always a link recipient. A corollary of this line of reasoning is that results of DJK also give an impression that a balanced network, in which every link recipient is a middle agent, is also resilient to the changes in decreasing decay level. On the contrary, my generalized Lemma 1 above does not give such an impression. What my generalized Lemma 1 above suggests is that it is *not* sufficient to consider merely an external appearance of a network if our objective is to find out whether a link recipient is resilient to the changes in decreasing decay level, since the external appearance of a network alone cannot determine whether a link recipient is also a positionally optimal agent as required in Lemma 1. Rather, one has to consider an intricate interplay between network structure, information value possessed by each agent and location of each agent, relative to one another, to determine such. Example 1 below illustrates this intuition.

EXAMPLE 3.1. Consider the (undirected) network g_1 in Figure 2. Which agent would i want to form a link with? For $\sigma = 0.99$ it is straightforward to show that the middle agent j is best-informed in the set of all

[18] for the sake of reference, see the original Lemma 1 of DJK in Appendix A.

[19] This results from the fact that an positionally optimal agent is always the best-informed agent so long as the decay is sufficiently small. Indeed this is the key intuition of the proof of this Lemma (see Appendix B).

FIG. 2. Networks with six agents



(a) network g_1 **(b)** network g_2

agents in this network, hence for a sufficiently low c agent i's best response is to establish a link with j. However, for $\sigma \geq 0.997$ we have that the agent k replaces agent j as the best-informed agent. Note that k is also a positionally optimal agent among the set of all agents except i. This fact is in line with the generalized Lemma 1 of DJK above.

Next, consider the (undirected) network g_2 in Figure 2, which looks identical to the network g_1 above except that information value of each agent is changed. Which agent would i want to form a link with? We now have a surprising result that is opposite of that in network g_1 above. For $\sigma = 0.99$, k is the best-informed agent. But for $\sigma \geq 0.997$ the agent j is the best-informed agent and also a positionally optimal agent [20].

At this point, a natural question that arises is how a positionally optimal agent can be identified in a network. Remark 3.1 below answers this question.

Remark 3.1. Let $\mathcal{V}$ satisfy value partner heterogeneity and g be a minimal network. Consider a minimally connected set of agents $M \subset N$ and the (sub-)network g' induced by M. A positionally optimal agent denoted by i^* in M is an agent such that, assuming no decay, the information that flows to j via i^* is at least as much as the information that flows to i^* via j for every agent j that is one-link away from i^* [21]. Specifically, i^* is a positionally optimal agent in the set M if for every $j \in N^1_{i^*}(g')$ it holds true that $\sum_{l \in N\left(D^{i^*}(g'-i^*\bar{j})\right)} V_l \geq \sum_{l \in N(D^j(g'-i^*\bar{j}))} V_l$ [22].

This remark suggests that to identify whether an agent i is positionally optimal is rather simple: we only need to verify that information that flows

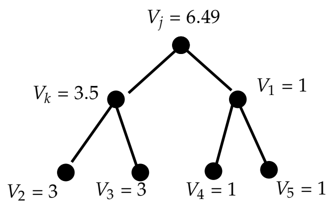
[20] This example is inspired by Example 1 in DJK.

[21] This Remark 2 benefits from an editorial comment of an anonymous referee, whom the author would like to thank.

[22] This remark is a straightforward corollary of Prelemma 1 (See Appendix B). Hence, the proof is omitted.

from i to j is never less than the information that flows from j to i for every j that has a link with i. I illustrate this through an example below.

FIG. 3. Example 3.2



Example 3.2. Consider the (undirected) network g in Figure 3. I claim that k is a positionally optimal agent. To do so first note that k has three links - $\bar{kj}, \bar{k3}$ and $\bar{k2}$. For the link $\bar{kj}$ we have $\sum_{l \in N\left(D^k\left(g-\bar{kj}\right)\right)} V_l = 3.5+3+3 = 9.5 > \sum_{l \in N\left(D^j\left(g-\bar{kj}\right)\right)} V_l = 6.49+1+1+1 = 9.49$. For the link $\bar{k3}$ and $\bar{k2}$ we have $\sum_{l \in N\left(D^k\left(g-\bar{k3}\right)\right)} V_l = 3.5+3+6.49+1+1+1 = 15.99 > \sum_{l \in N\left(D^3\left(g-\bar{k3}\right)\right)} V_l = 3$ and $\sum_{l \in N\left(D^k\left(g-\bar{k2}\right)\right)} V_l = 3.5+3+6.49+1+1+1 = 15.99 > \sum_{l \in N\left(D^2\left(g-\bar{k2}\right)\right)} V_l = 3$. Thus, information that flows from k to each of his direct neighbor is bigger than information that flows from each of his direct neighbor to k. It is concluded, therefore, that k is a positionally optimal agent.

In the same way, it is straightforward to prove that j is not a positionally optimal agent. This is because for the link $\bar{kj}$ we have $\sum_{l \in N\left(D^j\left(g-\bar{kj}\right)\right)} V_l = 6.49 + 1 + 1 + 1 = 9.49 \ngeq \sum_{l \in N\left(D^k\left(g-\bar{kj}\right)\right)} V_l = 3.5 + 3 + 3 = 9.50$.

Two important implications of the concept of positionally optimal agent and Lemma 1 are as follows. First, in a minimally connected subnetwork $g' \subset g$ *any* agent can become a positionally optimal agent if the information value of his own is set to be sufficiently high. This is due to the fact that an agent is always closest to himself than any other agent in the network. Since by Lemma 1 we know that for a range of decay sufficiently small a positionally optimal agent is always a link recipient, then it follows that we can cause any agent to be a link recipient by setting the information value of this agent to be high enough that he becomes a positionally optimal agent. Second, consider three agents i, i' and j in a SNN g. Let $ij \in g$ but $i'j \notin g$. Then suppose we fictitiously remove the link ij and further ask i' 'with whom do you want to choose, among agents in $A_{ij}(g)$, do you wish to establish a link with?' The answer from i' would also be j since, by Lemma 1, $ij \in g$ necessitates that j is a positionally optimal agent and hence a best-informed agent in $A_{ij}(g)$. Note that this line of reasoning is what partner value heterogeneity case and value homogeneity have in common. These two implications prompt us to establish the sufficiency and characterization result below.

PROPOSITION 1. (Partner Value Heterogeneity: Characterization and Sufficiency of SNN)
Let $\mathcal{V}$ satisfy partner value heterogeneity and let g be a non-empty minimal SNN of the two-way flow model with decay. Then g has similar characteristics to those of SNNs under the assumption of value homogeneity as in Proposition 1 of De Jaegher and Kamphorst (2015), which we quote below:

1.g has one of the following two configurations

(1a)g is a rooted directed tree with all links pointing away from its root: the unique non-recipient player. Each best-informed player in g is either the root player, or receives a link from the root player;

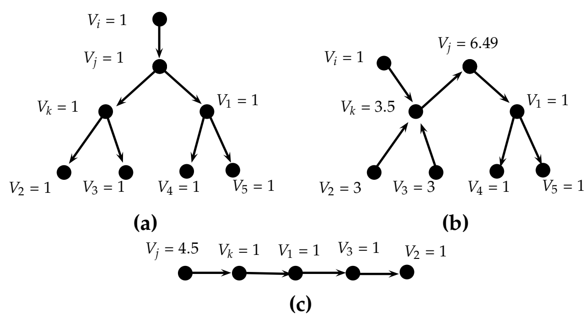
(1b)g is a directed tree with a unique multi-recipient player. Any link not received by this player points away from him. Moreover, this player is the unique best-informed player in g.

2.for all $\sigma \in (\sigma_M, 1)$, any non-empty SNN of the two-way flow model with decay has one of the two configurations above.

Conversely, any minimally connected network whose properties are as described in (1.a) or (1.b) above can be supported as SNN for a range of σ, c and $\mathcal{V}$ that satisfies partner value heterogeneity.

Figure 4 below illustrates three networks that can be supported as SNNs according to the above proposition.

FIG. 4. Examples of SNNs that can be supported by $\mathcal{V}$ that satisfies partner value heterogeneity, $c = 0.98$, $\sigma \geq 0.997$ and a linear payoff.



In conclusion, compared to Proposition 1 and 2 of DJK, which are respectively characterization and sufficiency of SNN for the case of value homogeneity and small decay, my Proposition 1 above shows that partner value heterogeneity influences SNN in a surprisingly specific way: partner value heterogeneity extends diameters of SNNs without changing any other properties. I formally state this comparison as a remark below.

Remark 3.2. [Comparison between the roles of partner value heterogeneity and value homogeneity on SNN with small decay] The introduction of partner value heterogeneity into the two-way flow model with small decay of DJK, which asssumes value homogeneity, does not change the main properties of SNN. Specifically properties 1 and 2 of the characterization part above are identical to those of Proposition 1 in DJK. However, partner value heterogeneity allows SNN to have a diameter that is longer than value homogeneity does. That is, while Proposition 2 of DJK only guarantees that any balanced network can be supported as SNN and Proposition 6 of DJK shows that the maximal diameter of a balanced network d is relatively smaller than the population size n, the last part of Proposition 1 above guarantees that even a line network whose diameter is $d = n - 1$ can be supported as SNN if partner value heterogeneity is assumed.

Finally, I end this main analysis section by relating my results to discussions on two network concepts - small-world network and preferential attachment - mentioned in DJK. First, recall that DJK mentions that the role of small decay in their model provides a micro-foundation to the small-world properties of network because "preferential attachment is key to understanding small world networks. Preferential attachment means that *'new' players are more likely to form links with players who have many links than with players who have few links.* The two-way flow model with decay offers a micro-foundation for preferential attachment. Given that you care about the distance to other players, it is typically more attractive to *sponsor a link to a player with many links, than to a player with few links.*" In contrast, my result shows that once small decay interacts with partner heterogeneity in information value then it is not necessarily the case that a link sender finds an agent that has many links to be more attractive than agents that have few inks. Indeed, network 3 in Example 1 shows that an agent with just one link can also be attractive as a link recipient. This is given that he possesses information whose value is much larger than those of other agents. In conclusion, my results show that for the role of small decay to serve as a micro-foundation to the small-world network the value of information possessed by each agent is required to be identical or nearly so [23].

Lastly, it is worth mentioning that while SNN given the presence of parter heterogeneity in information value may no longer resemble small-world network, each link recipient still serves the same pivotal function in the network. Specifically, each link recipient is an agent upon which other agents rely for the sake of information flow in the following senses. First, it is the agent that, from the point of view of a link sender, minimizes the information loss due to small decay. This holds true whether the link recipient has many links or few links. In case that the link recipient has few links, the information loss is minimized because - roughly speaking - those few links are links that a large amount of information flows through. In case that the link recipient has a many links, the information loss is minimized because the link recipient is in the middle and hence the distance from most agents are also minimized. In addition, consider Remark 3.1. It says that a positionally optimal agent denoted by i^* is an agent such that information flows via i^* to each of his direct neighbor more than the other way round. Therefore, if a link between i^* and a direct neighbor is removed so that the communication between the two agents is disabled, his neighbor will suffer more information loss than i^* does. As such, each link recipient - be it the case of value homogeneity and or partner value heterogeneity - is a pivotal agent in the sense that he serves as an agent through which a relatively large amount of information flows to other agents.

3.2. Further Extension 1: Player Value Heterogeneity and SNN

In the previous section, I generalize the two-way flow model with small decay of DJK by allowing for partner value heterogeneity. Key results of this generalization rests upon the definitions of positionally optimal agent and generalized balancing condition that I established in section 2.1, which generalize the definition of middle agent and balancing condition in DJK. Quite interestingly, by further generalizing these two concepts, characterization and sufficiency result of SNNs can be achieved for other forms of value heterogeneity, namely player value heterogeneity and general value heterogeneity. This subsection, therefore, begins with an introduction of some new notations.

[23]'Nearly so' in the sense that if heterogeneity in information value is present yet the extent to which values are different across agents is very small, then the effects of value heterogeneity become negligible. Consequently a link sender finds that an agent with many links is more attractive as a link recipient than agents with few links, like the case of value homogeneity.

First, define the total ex-post information that i' receives from the perspective of i in g as:

$$I_{i \to i'}(g) = \sum_{d=0}^{n-1} \sum_{k \in N_{i'}^{d}(g'), k \neq i} \sigma^{d} V_{ik}$$

Note that, suppose that i and i' are disconnected and i forms a link with i', then i receives $\sigma I_{i \to i'}(g) = \sum_{d=0}^{n-1} \sum_{k \in N_{i'}^{d}(g'), k \neq i} \sigma^{d+1} V_{ik}$ [24]. Consider a minimally connected set of agents $M \subset N$ and the network g' induced by M. i' is said to be better-informed than j' from the perspective of i (such that $i \neq i', j'$) in the set M if $I_{i \to i'}(g') \geq I_{i \to j'}(g')$ and i' is best informed from the perspective of i in the set M if $I_{i \to i'}(g') \geq I_{i \to j'}(g')$ for every agent $j' \in M$ [25].

Remark 3.3. Let g be a minimal network. Consider a minimally connected set of agents $M \subset N$ and the network g' induced by M. For any $j, j' \in M$ j is better informed than j' in M, assuming $\mathcal{V}$ satisfying value homogeneity, if and only if j is better-informed than j' in M from the perspective of i for every $i \neq j, j'$, assuming $\mathcal{V}$ satisfying player value heterogeneity. Consequently, Lemma 1 in DJK, which assumes $\mathcal{V}$ satisfying value homogeneity, also holds true if $\mathcal{V}$ is assumed to satisfy player value heterogeneity.

That is, there is almost no difference between the case of value homogeneity and the case of player value heterogeneity in SNN with small decay. This is because for player value heterogeneity, which is such that $V_{ij} = V_i$ for every $j \neq i$, from the perspective of i every other agent is homogeneous in terms of information value [26]. Thus, similar to the case of value homogeneity, what matters from the perspective of a link sender is to make sure that the link that is established shortens the overall distance, and hence minimize information decay, between him and other agents. This in turn guarantees that a middle agent is always a best-informed agent. However, there is still a difference between the case of value homogeneity and the case of player value heterogeneity. While in the case of value homogeneity

[24] The term 'from the perspective' here reflects the fact that $I_{i \to i'}(g)$ can be different from $I_{j \to i'}(g)$ for any i, j that are not in the same component as i', which in turn is due to the fact that each V_{xy} depends on the identity of the agent x and y.

[25] Intuitively, $I_{i \to i'}(g)$ is the information that i receives if he were fictitiously located where i' is in the network.

[26] Another way to observe that there is almost no difference between these two cases is to observe the term $I_i(g)$. Suppose that g is minimally connected, then $I_i(g) = \sum_{d=0}^{n-1} \sigma^d |N_i^d(g)|$ for the case of value homogeneity and $I_i(g) = \sum_{d=0}^{n-1} \sigma^d |N_i^d(g)| V_i$ for the case of value player heterogeneity. That is, the only difference is the term V_i found the lattter case.

as in DJK SNN is minimally connected, in the case of player value homogeneity SNN can contain singletons. The intuition is very straightforward: if a singleton i has V_i that is very small, say almost zero, then it is possible that i finds that the benefits from link establishment cannot cover the associated link formation cost. In addition, consider an end link and an end recipient in a non empty component. If this end link is removed then an end recipient becomes a singleton. Thus, an end sponsor is always indifferent between the existing end recipient and a singleton. It follows that an SNN in the case of player value heterogeneity and small decay contains no singleton unless the unique non-empty component is PSS, which has no end link. This leads to the following characterization and sufficiency results for the case of player value homogeneity.

Proposition 2. *(Player Value Heterogeneity: Characterization and Sufficiency of SNN)*

1.(Characterization) Given the small decay assumption, $\sigma \in (\sigma_M, 1)$, and $\mathcal{V}$ satisfying player value heterogeneity, a non-empty SNN g is minimal and consists of a unique non-empty component, which has similar characteristics as those of SNN when $\mathcal{V}$ satisfies value homogeneity as in Proposition 1 of De Jaegher and Kamphorst (2015), which is quoted below:

(1a)a rooted directed tree with all links pointing away from its root: the unique non-recipient player.

(1b)a directed tree with a unique multi-recipient player. Any link not received by this player points away from him.

Moreover, if the unique non-empty component of SNN is not a PSS, then g is connected.

2. (Sufficiency) Any minimally connected network g that is balanced and satisfies the properties (1a) or (1b) stated above can be supported by a range of σ, c and $\mathcal{V}$ that satisfies player value heterogeneity.

3.3. Further Extension 2: General Value Heterogeneity and Small Decay

Finally, I turn to establish the characterization and sufficiency result for the case of general value heterogeneity, which is the most general case. To do so, I extend key definitions - positional superiority and positionally optimal agent - as well as Lemma 1 in the main analysis section to allow for the case of general value heterogeneity below.

Definition 3.1. (Positional superiority for general value heterogeneity) In a network g, consider a minimal connected subset of players $M \subset N$. Let $\mathcal{V}$ satisfies general value heterogeneity. Consider agents $i, j \in M$

and $x \notin M$. i is positionally superior to j from the perspective of x if $\sum_{k \in X_{i,M}(i,j;g)} V_{xk} \geq \sum_{k \in X_{j,M}(i,j;g)} V_{xk}$. We write $i \succsim_x^M j$ for short. If the inequality is strict, we write $i \succ_x^M j$ for short.

Definition 3.2. (Positionally optimal agent for general value heterogeneity) In a network g, consider a minimal connected subset of players $M \subset N$. Let $\mathcal{V}$ satisfies general value heterogeneity. Let $i \in M$ and $x \notin M$. i is said to be a positionally optimal agent from the perspective of x in M if $i \succsim_x^M j$ for every $j \in M$ and $j \neq i$.

Definition 3.3. (Generalized Balancing Condition for general value heterogeneity) Let $\mathcal{V}$ satisfies general value heterogeneity. A minimal network g satisfies the generalized balancing condition if for any $ij \in g$ we have that j is a positionally optimal agent from the perspective of i in the set $A_{ij}(g)$.

Note that to generalize the definitions for partner value heterogeneity case to the case of general value heterogeneity, the only words added to each definition above are 'from the perspective of...' This follows from the assumption of general value heterogeneity, which allows for $V_{ij} \neq V_{i'j}$ for any $i \neq i'$. Thus, from the perspective of i an agent j could be positionally optimal but not so from the perspective of i'. This prompts us to generalize Lemma 1 in the main analysis section as follows.

Lemma 2. *(Generalization of Lemma 1 (De Jaegher and Kamphorst (2015)) for the case of general value heterogeneity) Consider the case of small decay and $\mathcal{V}$ satisfying general value heterogeneity. There exists a $\sigma_B < 1$ such that the following property holds for any minimal NN g. Consider any $ij \in g$:*

1.In $A_{ij}(g)$, a positionally optimal agent from the perspective of i always exists .

2.Let k be a positionally optimal agent from the perspective of i in $A_{ij}(g)$, then $j = k$ for all $\sigma \in [\sigma_B, 1)$.

Since this lemma is a straightforward extension of Lemma 1, the proof for this lemma is omitted. A major consequence of this Lemma is as follows. In a minimal non-empty component g' of a network g, consider a link ij in this component. By setting V_{ij} to be sufficiently high above $V_{ij'}$ for any $j' \neq j$ in g', then j becomes a positionally optimal agent from the perspective of i. In turn, by Lemma 2 above j is guaranteed to be the link recipient if the agent i wishes to form a link with an agent in $A_{ij}(g')$, given that c is sufficiently low and σ is sufficiently high. By the same analogy if V_{ij} and V_{ji} are set to be sufficiently low then i and j will have no incentive to establish a link with each other, which follows that by doing so we can make sure that any pair of agents i and j are not in the same component. As a result, we have that: (i) in an SNN *any* agent can become a link recipient and (ii) minimal SNN can have multiple non-empty components. More formally,

Proposition 3. *Any minimal network can be supported as SNN by a range of* c*,* σ *and* $\mathcal{V}$ *that satisfies general value heterogeneity.*

The proof is omitted since it straightforwardly follows from the intuition above [27].

4. CONCLUDING REMARKS

In this paper, I generalize the two-way flow model of network with small decay studied by DJK by allowing for heterogeneity in information value. I provide characterization of SNNs by assuming three forms of value heterogeneity - partner heterogeneity, player heterogeneity and general heterogeneity. My results show that (i) partner heterogeneity extends diameters of SNNs without changing any other properties, (ii) player heterogeneity ha almost no effect on SNN except that SNN can contain singletons if the unique non-empty component is a periphery-sponsored star and (iii) general heterogeneity allows any minimal network to be SNN. In the main analysis section, I relate the assumption of partner value heterogeneity to the concepts of preferential attachment and small-world network discussed in DJK. The conclusion is that the two-way flow with small decay provides a micro-foundation for preferential attachment, and hence resulting in SNNs that resemble small-world network, only if value is assumed to be homogeneous or nearly so. This is due to the fact that once partner value heterogeneity is introduced then it is no longer the case that an agent that has many links is more attractive as a link recipient than agents that have fewer links.

[27] Note that the existence of σ and c that supports a minimal network to be SNN follows precisely the same analogy as Generalized Lemma 1 of DJK and Proof of Proposition 1 in Section 3.1. Intuitively, if σ is set to be sufficiently close to 1 then in an SNN every link ij is such that j is a positionally optimal agent from the perspective of i. Also if c is set to be sufficiently low then every agent who wishes to form a link does find that the benefits from link formation covers the cost c.

In addition, since these three forms of heterogeneity studied in this paper are also studied in the existing literature in terms of link formation cost, it is worth comparing the effects of value heterogeneity on SNN found in this paper with the effects of cost heterogeneity on SNN found in the existing literature. I do so as follows.

Player heterogeneity To see the effects of player cost heterogeneity (no decay) on SNN, we compare Proposition 4.2 of Bala and Goyal (2000a) (homogeneity in value and cost with no decay) against Proposition 3.1 of Galeotti et al. (2006) (player cost heterogeneity with no decay). In both cases SNN has precisely the same shape, which is a connected center-sponsored star. Therefore, player cost heterogeneity has no effects on the shape of SNN. Next, to observe the effects of player heterogeneity with small decay, we compare Proposition 2 in this paper (player value heterogeneity with small decay) against Proposition 1 and 2 in DJK (value homogeneity with small decay). SNNs in both cases have precisely the same shape, except that in the case of player value heterogeneity singletons can exist if the unique non-empty component of SNN is PSS. In conclusion, player heterogeneity - be they in terms of cost or value - has little to no effects on SNN.

Partner heterogeneity To see the effects of partner cost heterogeneity (no decay) on SNN, we compare Proposition 4.2 of Bala and Goyal (2000a) (homogeneity in value and cost with no decay) against Proposition 1 of Billand et al. (2011) (partner cost heterogeneity with no decay). The only non-empty SNN in Proposition 4.2 of Bala and Goyal (2000a) is center-sponsored star whose diameter is 2, while the class of non-empty SNNs in Proposition 1 of Billand et al. (2011) is a large class that includes line networks whose diameter is $n - 1$. We conclude that partner cost heterogeneity extends diameters of SNN to be as long as the population size. Similarly, to see the effects of partner value heterogeneity (with decay) on SNN, we compare Proposition 1 and 2 of DJK (homogeneity in value and cost with decay) against Proposition 1 in this paper (partner value heterogeneity with decay). As mention in Remark 3 partner value heterogeneity also allows SNN to have a diameter that is as long as the population size. Therefore, we can conclude that partner heterogeneity - be they in terms of value or cost - extends diameters of SNN [28].

General heterogeneity Both general cost heterogeneity with no decay (Proposition 3.2 in Galeotti et al. (2006)) and general value heterogeneity with small decay (Proposition 3 in this paper) have precisely the same result: any minimal network can be supported as SNN.

[28]Indeed, the set of all possible SNNs in Proposition 1 of Billand et al. (2011) and Proposition 1 in this paper are identical. This is albeit the fact that the terminologies used in these two propositions are different. Lemma 3 in Charoensook (2019) reconciles the terminological discrepancy through an equivalent statement.

Finally, while this paper contributes to the literature in agent heterogeneity by allowing for the comparisons between the roles of cost heterogeneity (with no decay) and the roles of value heterogeneity (with small decay) in SNN, we still do not know how value heterogeneity impacts properties of efficient networks. Such a study could be useful, since it would allow for the comparison between the effects of value heterogeneity and the effects of cost heterogeneity on efficient networks, which has recently been studied by Unlu (2018). This is a future matter that this author plans to explore.

APPENDIX A

Preliminary Lemmata and Propositions from DJK (2015)

In this paper, lemmata and propositions are generalization of those in DJK. Hence, for the sake of reference and ease of comparison this subsection contains lemmata and propositions in DJK that are mentioned in this paper.

Lemma 1 (from DJK). Consider the case of small decay. There exists a $\sigma_B < 1$ such that the following property holds for any minimal Nash network g. Consider any $ij \in g$ such that $A_{ij}(g)$ has a middle player, say k. Then $j = k$ for all $\sigma \in [\sigma_B, 1)$.

Lemma 3 (from DJK).[1] In the absence of decay, no player in any network g prefers to sponsor non-minimal links

Lemma 4 (from DJK). For any $c > 0$, $n \geq 4$, and $f(I)$, there exists $\sigma_M < 1$ such that for all $\sigma > \sigma_M$ no player in any possible network wishes to sponsor a non-minimal link.

Lemma 5 (from DJK). Let network g be a non-empty Nash network. Then g has no singleton component.

Lemma 6 (from DJK). Let network g be a non-empty Nash network. Then g is connected.

Corollary 1 (from DJK). Let $c < f(1+\sigma) - f(1)$. Then any Nash network is connected.

Lemma 7 (from DJK). Let g be a minimal connected Nash network. For all $ij \in g$, it must be the case that, among the players in set $M = A_{ij}(g)$, j is a best-informed player in network g_M, and that j is the unique best-

[1]Note that Lemma 2 of DJK is not included here, since it pertains to the analysis of stochastically stable network rather than SNN.

informed player in g_M if g is a SNN. Moreover, among the players in set M, player j is the unique best-informed player in g.

Lemma 8 (from DJK). Let g be a minimal connected Nash network. If g contains links ii' and jj' which point towards each other, then $i' = j'$

Corollary 2 (from DJK). Let g be a minimal connected Nash network. Then g is either a PSS or g contains at least one end link.

Lemma 9 (from DJK). Let network g be a non-empty minimal Nash network. Then g contains either a unique non-recipient player and no multi-recipient players, or a unique multi-recipient player and multiple non-recipient players.

Proposition 1 (from DJK). Let g be a non-empty minimal Nash network of Two-way flow model with decay. Then

1. g has one of the following two configurations:

(i) g is a rooted directed tree with all links pointing away from its root: the unique non-recipient player. Each best-informed player in g is either the root player, or receives a link from the root player;

(ii) g is a directed tree with a unique multi-recipient player. Any link not received by this player points away from him. Moreover, this player is the unique best-informed player in g;

2. In both configurations of part 1, for any path of links in g pointing in the same direction, say $\{j_0j_1, j_1j_2, \ldots, j_{k-1}j_k,\}$, we have $I_{j_1}(g) > I_{j_2}(g) >, \ldots, > I_{k-1}(g) > I_k(g)$

3. In both configurations of part 1, for $ij \in g$ if $|A_{ij(g)}| = 2$, then g is not a SNN; if $|A_{ij(g)}| \geq 3$, then there exists $j\bar{k}, j\bar{k}' \in g$ such that $k \neq k'$;

4. for all $\sigma \in (\sigma_M, 1)$, any non-empty Nash network of the two-way flow model with decay satisfies Parts 1-3.

Proposition 2 (from DJK). Consider any balanced, minimal network g that satisfies the properties of Proposition 1. Then network g is a SNN for a range of σ and c

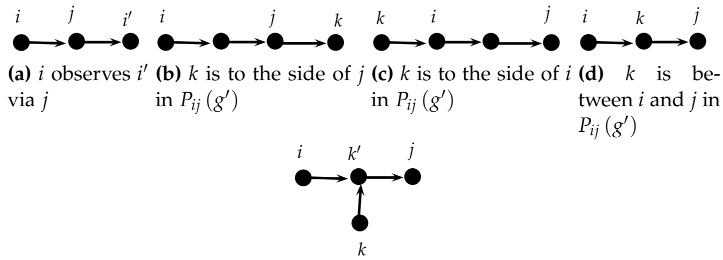
APPENDIX B

Proof of Lemma 1 in this paper

In this subsection, I prove Lemma 1 in this paper, which is a generalization of Lemma 1 in DJK (see Appendix A). This proof rests upon a prelemma, which I establish below. Before so doing, the following notations regarding locations of agents in a network are introduced.

Locations of agents. Consider a minimally connected subnetwork $g' \subseteq g$ and three agents i, j, i' in g'. Consider a path $P_{ii'}(g')$. Let $N(P_{ii'}(g'))$ be the set of all agents in $P_{ii'}(g')$. We say that i observes i' via j if $j \in N(P_{ii'}(g'))$. We say that k is to the side of j in $P_{ij}(g')$ if i observes k via j. We say that k is to the side of i in $P_{ij}(g')$ if j observes k via i. We say that k is between i and j in $P_{ij}(g')$ if $k \in N(P_{ij}(g'))$. Lastly, we say that k is within i and j in $P_{ij}(g')$ if there exists another agent k' that is between i and j such that i and j observe k via k'.

FIG. 1. Locations of agents.



(a) i observes i' via j **(b)** k is to the side of j in $P_{ij}(g')$ **(c)** k is to the side of i in $P_{ij}(g')$ **(d)** k is between i and j in $P_{ij}(g')$

(e) k is within i and j in $P_{ij}(g')$

Prelemma. In a network g, let $M \subset N$ be a minimal connected subset of agents. Consider $g_M = \{kl \in g; k, l \in M\}$ and let $\bar{ij} \in g_M$. If $i \succsim^M j$ then $i \succsim^M j'$ for every $j' \in M$ that is to the side of j in the path $P_{ij}(g) = \bar{ij}$.

Proof of Prelemma 1. First, recall that $X_{j,M}(i, j; g)$ is the set of agents, including j himself, that is closer to j than to i. I introduce three disjointed subsets of $X_{j,M}(i, j; g)$ as follows. $X^{side}_{j,M}(i, j; g)$ is the set of agents that are closer to j than to i and are to the side of j in $P_{ij}(g)$, $X^{between}_{j,M}(i, j; g)$ is the set of agents that are closer to j than to i and are between i and j in $P_{ij}(g')$, $X^{within}_{j,M}(i, j; g)$ is the set of agents that are closer to j than to i and are within i and j in $P_{ij}(g')$. Hence, we have $X_{j,M}(i, j; g) = X^{side}_{j,M}(i, j; g) \sqcup X^{between}_{j,M}(i, j; g) \sqcup X^{within}_{j,M}(i, j; g) \sqcup \{j\}$. Next, recall that $P_{ij}(g') = \bar{ij}$ so that $X^{within}_{j,M}(i, j; g) = X^{between}_{j,M}(i, j; g) = \emptyset$. Consequently, $X_{j,M}(i, j; g) = X^{side}_{j,M}(i, j; g) \sqcup \{j\}$ and:

$$\sum_{k \in X_j(i,j;g)} V_k = \sum_{k \in X^{side}_j(i,j;g)} V_k + V_j \tag{B.1}$$

Next, consider an agent j' that is to the side of j in $P_{ij}(g)$. Since the sequence of agents in $P_{ij'}(g)$ is $i, j, \ldots, j'$ we have $X^{side}_{j',M}(i, j'; g), X^{within}_{j',M}(i, j; g)$, $X^{between}_{j',M}(i, j; g), \{j'\} \subset X^{side}_j(i, j; g) \sqcup \{j\}$. Therefore, we have:

$$\begin{aligned}\sum_{k\in X_{j,M}(i,j;g)} V_k &= \sum_{k\in X^{side}_{j,M}(i,j;g)} V_k + V_j \\ &\geq \sum_{k\in X^{side}_{j',M}(i,j';g)} V_k + \sum_{k\in X^{within}_{j',M}(i,j';g)} V_k + \sum_{k\in X^{between}_{j',M}(i,j';g)} V_k + V_{j'}\end{aligned}$$

Note that the equality on the left is due to Eq.B.1. Note further that the right-hand side of the inequality is nothing else but $\sum_{k\in X_{j',M}(i,j';g)} V_k$. Therefore, the above expression is reduced to:

$$\sum_{k\in X_{j,M}(i,j;g)} V_k \geq \sum_{k\in X_{j',M}(i,j';g)} V_k \tag{B.2}$$

Next, we turn to describe $X_{i,M}(i,j;g)$. By the same analogy as the above we have $X_{i,M}(i,j;g) = X^{side}_{i,M}(i,j;g) \sqcup X^{within}_{i,M}(i,j;g) \sqcup X^{between}_{i,M}(i,j;g) \sqcup \{i\} = X^{side}_{i,M}(i,j;g) \sqcup \{i\}$ because $P_{ij}(g) = \overline{ij}$ so that $X^{within}_{i,M}(i,j;g) = X^{between}_{i,M}(i,j;g) = \emptyset$. Recall that j' is to the side of j in $P_{ij}(g)$ so that the sequence of $P_{ij'}(g')$ is $i, j, ..., j'$. Hence, $X^{side}_{i,M}(i,j;g) = X^{side}_{i,M}(P_{ij'};g')$ so that $X_{i,M}(i,j;g) = X^{side}_{i,M}(i,j;g) \sqcup \{i\} \subsetneq X_{i,M}(i,j';g)$. It follows that:

$$\sum_{k\in X_{i,M}(i,j';g)} V_k \geq \sum_{k\in X_{i,M}(i,j;g)} V_k \tag{B.3}$$

Finally, recall that $i \succsim^M j$ if and only if $\sum_{k\in X_{i,M}(i,j;g)} V_k \geq \sum_{k\in X_{j,M}(i,j;g)} V_k$. Thus, we if we suppose that $i \succsim^M j$ then a combination of Equations B.2 and B.3 leads to:

$$\begin{aligned}\sum_{k\in X_{i,M}(i,j';g)} V_k &\geq \sum_{k\in X_{i,M}(i,j;g)} V_k \\ &\geq \sum_{k\in X_{j,M}(i,j;g)} V_k \\ &\geq \sum_{k\in X_{j',M}(i,j';g)} V_k\end{aligned}$$

Note that the first inequality from the left is precisely Equation B.3 and the first inequality from the right is precisely Equation B.2. The inequality in the middle is due to the definition of $i \succsim^M j$. If we consider only the first term and the last term from the left, the above inequality is reduced to:

$$\sum_{k\in X_{i,M}(i,j';g)} V_k \geq \sum_{k\in X_{j',M}(i,j';g)} V_k$$

Note that the above inequality is precisely the definition of $i \succsim^M j'$, which is what we intend to prove. ▮

We use this prelemma to prove Lemma 1 in this paper below.

Proof (Proof of Lemma 1).

Part 1: a positionally optimal agent always exists. First, note that a corollary of Prelemma 1 is that if an agent k is such that $k \succsim^{A_{ij}(g)} j'$ for every j' that is one-link away from k, i.e., $j' \in N_k^1(g), A_{ij}(g)$, then k is a positionally optimal agent in $A_{ij}(g)$. Thus, to prove by contradiction we assume that every agent k is such that there exists $j' \in N_k^1(g), A_{ij}(g)$ such that $j' \succ^{A_{ij}(g)} k$. Now let us consider any agent and call him i_1. Let another agent $i_2 \in N_{i_1}^1(g), A_{ij}(g)$ be such that $i_2 \succ^{A_{ij}(g)} i_1$. Again, we know that there exists $i_3 \in N_{i_2}^1(g), A_{ij}(g)$ such that $i_3 \succ^{A_{ij}(g)} i_2$. Consequently, we can construct a path whose sequence is $i_1, i_2, i_3, ...$ such that $i_1 \succ^{A_{ij}(g)} i_2, i_2 \succ^{A_{ij}(g)} i_3, ...$ Note that this line of reasoning can be repeated infinitely. Hence, the amount of agents in this path is infinite. A contradiction.

Part 2: Let k be a positionally optimal agent in $A_{ij}(g)$, then $j = k$. The proof follows precisely that of Lemma 1 of DJK, except that a few terms have to be replaced. First, replace $|N_i^d(g, j^1)|$ by $\sum_{k\in N_i^d(g,j^1)} V_k$ and $|N_i^d(g, j^2)|$ by $\sum_{k\in N_i^d(g,j^2)} V_k$ in Equation (4) and (5) in DJK. Note that these replacements are intuitive. In DJK $|N_i^d(g, j)|$ is the sum of information that i receives from observing j since every agent's information value is 1. Here this term becomes $\sum_{k\in N_i^d(g,j)} V_k$ due to the assumption of partner value heterogeneity.

By the same analogy, in the last paragraph of proof of Lemma 1 in DJK we replace $|N_i^d(g', j^p)|$ by $\sum_{k\in N_i^d(g',j^p)} V_k$ and $|N_i^d(g', j^{p+1})|$ by $\sum_{k\in N_i^d(g',j^{p+1})} V_k$. Lastly, we replace the word 'middle player' by the word 'positionally optimal agent'. ▮

APPENDIX C

Proofs of Propositions

Proof (Proof of Proposition 1). **Part 1: Characterization.** The proof follows precisely Lemma 1 to 9 and Proof of Proposition 1 of DJK, specifically part 1a, 1b and 4, without any modification.

Part 2: Sufficiency. First, I establish the following remark: in a minimal network g, let $g' \subseteq g$ be a minimally connected subnetwork of g

and M be the set of agents in g'. Let g' be supported by (any arbitrary) $\mathcal{V}$ that satisfies partner value heterogeneity. For any agent $j \in M$ there exists $\bar{V}_j$ such that whenever V_j is replaced by $V'_j > \bar{V}_j$ while holding V_k constant for every $k \neq j$, then j becomes a positionally optimal agent in M [1].

Next, let us first assign an arbitrary $\mathcal{V}$ such that $V_j = V > 0$ for every agent j in this network g. Onward I divide the proof into two cases: rooted directed tree case and directed tree with a unique multi recipient case. For the rooted directed tree case, let agent i^* be the root. Let i^d denote an agent whose distance from i^* is d. Let $\bar{d}$ denote the longest distance between i^* and an agent in this network. Note that since this network is a directed tree rooted at i^* we know that i^d receives exactly one link from i^{d-1} for any $d \geq 1$.Thus, beginning from the distance $\bar{d}-1$, consdider by the remark in the above paragraph it suffices that we change $V_{i^{\bar{d}-1}} = V$ to be $V_{i^{\bar{d}-1}} >> V_l = V$ for every agent $l \in A_{i^{\bar{d}-2}i^{\bar{d}-1}}(g)$. Similarly, for the distance $\bar{d}-2$, by the remark in the above paragraph it suffices that we change $V_{i^{\bar{d}-2}} = V$ to be $V_{i^{\bar{d}-2}} >> V_l$ for every agent $l \in A_{i^{\bar{d}-3}i^{\bar{d}-2}}(g)$, which in turn guarantees that $i^{\bar{d}-2}$ is a positionally optimal agent in $A_{i^{\bar{d}-2}i^{\bar{d}-1}}(g)$.Thus by repeating precisely this process until the distance $d = 1$ we are able to identify $\mathcal{V}$ satisfying partner value heterogeneity that guarantees that every link recipient is a positionally optimal agent, which is a necessary condition for a minimal network to be Nash for a range of σ sufficiently close to 1. Note that this proof is for the rooted directed tree case. For the case of directed tree with a unique multi recipient, the proof is nearly identical. The only additional step is that we need to set $V_{i^*} >> V_j$ where i^* is the unique multi recipient and for every $j \neq i^*$. Again, in so doing by the remark in the above paragraph and by Lemma 1 i^* is guaranteed to be a positionally optimal agent and hence a link recipient for a range of σ sufficiently close to 1.

Finally, we need to prove for the existence of the range of c and σ that support the network g is SNN. The proof here follows precisely the proof of Proposition 2 in DJK (pp.232-233). The only modification is that the term 'balancing condition' has to be replaced by the term 'generalized balancing condition.' ▮

Proof (Proof of Proposition 2).

Part 1: Characterization. Let $g' \subset g$ be a minimally connected component in g. All proofs follow the proofs of Lemma 3, 4, 6 - 9 and Proposition 1 part 1a and 1b, except that the following modifications are needed. First, the statement of Lemma 6 of DJK becomes 'Let g be a non-empty

[1]The proof of this remark follows from the fact that j is positionally superior to j' in M if $\sum_{k\in X_{j,M}(j,j';g)} V_k \geq \sum_{k\in X_{j',M}(j,j';g)} V_k$, which implies that the left-hand side of inequality, $\sum_{k\in X_{j,M}(j,j';g)} V_k$, can be increased by increasing V_j .

Nash network, then g has exactly one non-empty component,' which follows that the first line of the proof in DJK can be removed. Next, replace the inequalities in the fourth line by $I_{i\to i'}(g\backslash ii') > I_{i\to j'}(g) > I_{i\to j'}(g\backslash jj')$ and the inequalities in the fifth line by $I_{j\to j'}(g\backslash jj') > I_{j\to i'}(g) > I_{j\to i'}(g\backslash ii') \iff \epsilon I_{j\to j'}(g\backslash jj') > \epsilon I_{j\to i'}(g) > \epsilon I_{j\to i'}(g\backslash ii') \iff I_{i\to j'}(g\backslash jj') > \epsilon I_{j\to i'}(g) > I_{i\to i'}(g\backslash ii')$.

For Lemma 7 and 8, in the first sentence of each of these two lemmata replace the term 'let g be a minimal connected Nash network' by 'let g be a non-empty minimal Nash network with $g' \subset g$ being the unique non-empty component in g.' For every other sentences replace every g by g'. In addition, note that the proofs of these two lemmata also hold for the case of player value heterogeneity because of Remark 4 (see section 3.2)

Next, as a corollary of Lemma 8 in DJK, we have that either g' has at least one end link or g' is a PSS. In case that g' has at least one end link, then there can be no singleton in $g \supseteq g'$. Otherwise, g is not an SNN since the agent that sponsors an end link is indifferent between the recipient of the end link and a singleton. Consequently, unless g' is PSS then g' is a minimally connected component that contains all agents in the network, ie., $g' = g$.

For Lemma 9, in the first sentence the term 'let g be a minimal connected Nash network' by 'let g be a non-empty minimal Nash network with $g' \subset g$ being the unique non-empty component in g.' Also replace the term $(n-1)$ by $(|N(g')| - 1)$.

Finally, for the Proof for Proposition 1 part 1a and part 1b, simply note that the second paragraph of each part is not related to the proof of my Proposition 2 since my Proposition 2 mentions nothing about the identity of best-informed agent.

Part 2: Sufficiency. The proof follows Proposition 2 of DJK without any modification. ∎

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