ANNALS OF ECONOMICS AND FINANCE **23-2**, 223–251 (2022)

A Stochastic Model of Rational Addiction*

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We propose a new model of addictive behavior that takes as a starting point the classic model of Becker and Murphy (1988), but incorporates uncertainty. We model uncertainty through the Brownian motion process. This process is used to capture both random events such as exposure to harmful substances, anxiety, tensions and environmental cues which can precipitate and exacerbate addictions, and those sober and thought-provoking episodes that discourage addictions. We derive closed-form formulas for optimal addictive consumption and capital trajectories and examine their global and local properties. Our theory provides plausible explanations for several typical patterns of addiction and has novel policy implications.

Key Words: Rational Addiction; Uncertainty; Health Issue; Stochastic Optimization; Ito Lemma.

JEL Classification Numbers: C61, D01 D11, I10, I18, K32.

1. INTRODUCTION

Addiction to certain substances such as alcohol, tobacco, cocaine, marijuana, and heroin, or activities like gambling, eating, sex, watching television, playing computer games, smartphones, and internet use, can be

* Both authors have no relevant financial interests tied to the research described in this paper. We are very grateful to this journal's co-editor and its anonymous referees, Migul Ballester, Kim-Sau Chung, Karl Claxton, Vince Crawford, Sanjeev Goyal, Andrew Jones, Gerard van der Laan, Qian Lin, Margaret Meyer, Kexue Pu, Luigi Siciliani, Peter Simmons, Hamid Sabourian, Jacco Thijssen, Dolf Talman, Shuming Wang, and many participants at the University of Cambridge, Chinese Academy of Sciences (Beijing), Chinese University of Hongkong, Chongqing University, University of Oxford, Universite Paris-Dauphine, University of York, and several workshops and conferences for their helpful feedback. Part of this research was done while the second author was visiting the University of York. He thanks for its hospitality and acknowledges the grants of NCET-12-0588 and NSFC No.71071172.

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1529-7373/2022

so powerful that it often has a dire effect on the lives of people involved. According to the 2011 and 2012 World Health Organization reports on the harmful use of alcohol and tobacco, these two addictive substances cause approximately 2.5 million and 5 million deaths respectively each year around the world. The number of people who are dependent on alcohol, tobacco or other substances far exceeds the death toll.[1] Alcohol use is the world's third biggest risk factor for disease and disability and is a causal factor in 60 types of diseases and injuries and a component cause in 200 others. It is also closely associated with many serious social problems such as violence, child neglect and abuse, and absenteeism in the workplace. Tobacco smoking alone kills even more than acquired immune deficiency syndrome/human immunodeficiency virus (AIDS/HIV), malaria and tuberculosis combined.[2] Across the globe, 12% of all deaths amongst people aged 30 years and above have been identified to be attributable to tobacco. In particular, tobacco smoking accounts for 71% of all lung cancer deaths and 42% of all chronic obstructive pulmonary disease. Both alcohol drinking and tobacco smoking contribute to family poverty whereby money spent on them can take away a significant part of total household income that may be necessary for the family's use of other goods and services.

Addiction has long been recognized as a fundamental and intriguing problem for researchers, social workers and governments. The primary goal of studying (harmful) addiction is to try to understand its behavior and ultimately to explore its treatment and methods of prevention. Among many theories and models developed so far stands out Becker and Murphy (1988)'s theory of rational addiction (see e.g., Grossman 1993, West 2006, and Moss and Dyer 2010) as a pioneering economic analysis of addiction.[3] Addictive behavior for drinking, smoking, eating, gambling and others is habit-forming, cannot be static and must be dynamic. The key symptoms

[1]In the USA an epidemic of opioid addiction is getting worse in recent years. The Centers for Disease Control and Prevention states in a 2017 report: "This increase is being driven by overdoses related to illicit fentanyl (i.e., synthetic opioids). Since 2010, overdose deaths involving heroin and illicit opioids including fentanyl have increased more than 200%." See `https://www.cdc.gov/drugoverdose/pubs/index.html`.

[2]Although the number of smokers has been decreasing considerably in developed countries in the last few decades due to governments' legislations on cigarettes and unrelenting campaigns of concerned health groups against tobacco, smoking is still a very serious health problem in some developed countries. A 2014 BBC investigation by Peter Taylor states: "Though the (smoking) habit is slowly declining, about one in five British adults still smoke. Smoking among 20 to 34-year-olds has actually increased in the past few years. And though the tobacco industry insists it does not target children, every year 200,000 of those aged 11-15 start smoking." See `https://www.bbc.co.uk/news/health-27546922`.

[3]Follow-up relevant theoretical models of addiction include Dockner and Feichtinger (1993), Orphanides and Zervos (1995), Gruber and Koszegi (2001), Bernheim and Rangel (2004), O'Donoghue and Rabin (2006), Gul and Pesendorfer (2007), and Crawford (2010).

of such behavior are tolerance, withdrawal and reinforcement. Tolerance means that repeatedly using a substance or doing an activity over time requires more and more of the substance or activity to achieve the same level of satisfaction as the individual previously experienced. Withdrawal is a negative state at which an individual will feel extremely uncomfortable when he reduces or stops the consumption of a substance or an activity. For instance, typical alcohol withdrawal symptoms include agitation, delirium tremens, and seizures. Reinforcement refers to the type of behavior that the more an individual consumes a substance or partakes of an activity today, the more he wants to do in the future. Becker and Murphy formulated a deterministic dynamic model to capture these fundamental features of addiction.

In their model, Becker and Murphy used a utility function to quantify the "benefit" or "pleasure" from consuming an addictive good and a normal good. The utility function depends both on the consumption level of the addictive good and the normal good and on the addictive capital that has been accumulated so far. They imposed first- and second-order conditions on the utility function to characterize tolerance, withdrawal and reinforcement. The individual is assumed to be fully aware of the negative consequence of consuming the addictive good and capable of weighing all options rationally and making a consistent plan to maximize utility over his life time. Becker and Murphy did not give a concrete example of the utility function but instead utilized a quadratic function to approximate the function and examined possible dynamic aspects of addictive consumption. Their modeling of rational addiction has subsequently become an important approach to the study of consumption of addictive goods such as tobacco, alcohol, cocaine, coffee and gambling and stimulated a large number of empirical [4] and theoretical works.

Our goal is to develop a new model of addictive behavior that takes as its starting point the model of Becker and Murphy (1988), but incorporates an important factor –uncertainty– into the model. A person becomes addicted only after he has exposed to a harmful addictive substance or activity. The first time exposure to the substance or activity is often a random event by mistake for most addicts. It is known that anxiety, insecurity and tension can trigger and worsen an addiction; see e.g., Becker and Murphy (1988) and Goldstein (2001). These stressful events typically arise as random events. Exposure to environment cues occurs also at random. As a case in point, a casual sight of a cigarette advertisement can induce an irresistible

[4] Here is a less-than-exhaustive list of empirical references: Chaloupka (1991), Keeler et al. (1993), Becker et al. (1994), Olekalns and Bardsley (1996), Grossman and Chaloupka (1998), Grossman et al. (1998), Suranovic et al. (1999), Fenn et al. (2001), Pudney(2003), Carbone et al. (2005), and Adda and Cornaglia (2006), and Jones et al. (2014).

impulse to smoke. An unfortunate fact is that addicts of harmful substances usually make their lives more unpredictable, because they can lose their jobs more easily, their marriage can be less stable, and they are also more vulnerable to diseases and accidents. Generally speaking, uncertainty exacerbates addiction; addiction reinforces uncertainty. Vicious circles can arise in an addict's life. Events that can precipitate and aggravate addiction will be called harmful events, while events such as compelling campaigns against drugs that can discourage addiction will be called beneficial events. Our model is the first to grasp these random events via a Wiener stochastic process to study addictive behavior.

As a major departure from the deterministic Becker-Murphy model, we adopt a natural stochastic process, i.e., the Brownian motion or Wiener stochastic process, to capture uncertainty; see e.g., Mirrlees (1964), Merton (1969, 1971), Black and Scholes (1973), and Dixit and Pindyck (1994) for their use of this process in finance and other fields. Specifically, we use the Brownian motion process to describe how uncertainty influences the accumulation of addiction capital and the consumption of addictive good. The incorporation of uncertainty entails some simplification of the model. Our analysis therefore focuses on the essential and tractable case where the normal good is ignored or its consumption level is fixed, as Becker and Murphy (1988) did in their analysis. In contrast to Becker and Murphy (1988) who achieved only approximate solution for their deterministic model, we obtain closed-form formulas of optimal addictive consumption and capital trajectories (Theorem 1) for our stochastic model. The explicit formulas (27), (28) and (29) reveal the complex evolutionary nature of addictive behavior, which is determined by a host of factors both the external or environmental factors and the addict's internal or intrinsic factors, such as uncertain events, volatility of shocks, the addict's attitude towards risk, time preference, and addictive capital depreciation rate. They enable us to derive both qualitative and quantitative properties of the dynamics of addictive behavior, thus allowing us to have a clear picture of how internal and external factors affect the addictive behavior and also to infer policy implications.

In our analysis, we explore a class of addictive utility functions that not only capture three basic characteristics of addictive behavior: tolerance, withdrawal and reinforcement, but also are tractable and can thus facilitate the establishment of various properties of the model.[5] These functions are called addictive multivariate power functions and might be reminiscent of the well-known Cobb-Douglas utility function but have quite a different requirement on parameters. They admit meaningful and intuitive interpre-

[5]This function can be seen as a variant of the function introduced by Abel (1990). See Bakshi and Chen (1996) and Carroll et al. (2000) for other variants in their studies of stock market and growth models, respectively.

tations. For instance, the utility value from consuming any amount of a harmful addictive good is always negative and the addictive good has an irresistible power for its addicts. In the analysis of Becker and Murphy (1988), a quadratic function is used as an approximation of the addict's original utility function near a steady state. Doing so makes it difficult to achieve a clear understanding of the dynamic addictive behavior as the steady state is unknown and in fact needs to be found and the approximation can lose valuable information. The addictive multivariate power utility function renders the approximation unnecessary and allows us to derive closed-form solutions and obtain fresh insights into the evolution of addictive behavior, and to establish both global and local properties of dynamic addictive behavior.

We shall also highlight several other results of our paper. For instance, it will be easy to understand why anxiety and tension can precipitate an addiction. This is so because anxiety and tension will make the individual more present-oriented, and the more present-oriented he is, the more he will consume and thus become more addicted today rather than tomorrow. It is shown in Proposition 3 that while on the one hand, the more volatile the situation an individual faces, the more susceptible to change his addictive consumption will be, on the other hand on average his addictive consumption will decrease. We can classify addictions into three typical patterns: benign, malignant and constant, which are consistent with what is widely observed in reality. All three patterns are determined by multiple internal and external factors. On average, a benign addict tends to give up his addiction eventually, a malignant one most likely goes from bad to worse and runs out of control, while a constant one consumes almost a constant amount of addictive good for an entire life. For instance, some people enjoy drinking a glass of wine almost daily. Our results also offer a natural explanation of why cycles of binges and abstention attempts can occur, and why relapse is common. Binges refer to a phenomenon in which an individual sporadically does too much of a particular activity, especially drinking or eating, in a short period of time. One policy implication of our analysis is that instead of harsh treatment like going cold turkey, which can be extremely painful or sometimes even life-threatening, there always exists soft treatment–a gradual and less painful process toward addiction cessation. Another policy implication is that our theory allows broad and flexible treatments to promote abstention.

This paper proceeds as follows. Section 2 introduces the model. Section 3 establishes our major results. Section 4 discusses policy implications. Section 5 concludes. Some proofs are given in the appendix.

2. THE MODEL

Before getting into our model with uncertainty, we first present the general class of addictive utility functions stemming from Becker and Murphy (1988) which are very different from the familiar utility functions over normal goods. Following Becker and Murphy, an addictive utility function is a function of not only the current consumption of an addictive good but also its accumulated consumption from the past up to the current time, and (possibly including) of the consumption of a normal good and is given by

$$u(t) = u(c(t), A(t), z(t)), \tag{1}$$

where $c(t)$ is the consumption of the addictive good, $A(t)$ is the *addictive capital or stock*, and $z(t)$ is the consumption of the normal good at time t. We use $A(t)$ as a measure to reflect the accumulated effect of the past consumption of the addictive good up to time t. The utility function u is assumed to be strictly concave of c, A, and z and to have second partial derivatives for each of the arguments. The following mild conditions are imposed on the function u

$$u_c > 0, \quad u_{cc} < 0. \tag{2}$$

$$u_A < 0, \quad u_{AA} < 0. \tag{3}$$

$$u_{cA} > 0. \tag{4}$$

$$u_z > 0, \quad u_{zz} < 0. \tag{5}$$

Here (2) and (5) are most familiar, indicating that the marginal utilities of both addictive good and normal good are positive and decreasing. It means that the more of either good the higher utility but the lower marginal utility the individual will get. The inequality (2) describes *withdrawal* effect, implying that the individual's utility would fall should the consumption of the addictive good be reduced. Clearly, the bigger u_c is, the stronger the withdrawal effect will be. The negative marginal utility of addictive capital A given by (3) captures *tolerance*, saying that less cumulative past consumption of the addictive good will enhance current utility. This assumption is markedly different from typical ones in economic theory whose marginal utilities are assumed to be positive. Like addictive good and normal good, the marginal utility of addictive capital is also decreasing. Finally, inequality (4) reflects *reinforcement* between current addictive consumption and addictive capital, stating that past consumption will bolster current consumption. Reinforcement is also called *adjacent complementarity* (see

Ryder and Heal 1973, Iannaccone 1986, Becker and Murphy (1993), and Dockner and Feichtinger 1993).

We can now introduce our general rational addiction problem under uncertainty. A rational individual makes a consistent plan to maximize his expected utility over time when he chooses his consumption bundle every time. His decision problem can be formulated as

$$\begin{aligned} \max_{c(t),z(t)} \quad & E\left\{\int_0^\infty u(c(t),A(t),z(t))\exp(-\rho t)dt\right\} \\ \text{s.t.} \quad & dA(t) = (c(t)-\delta A(t))dt + \sigma A(t)dv(t),\ A(0)=A_0>0 \\ & \dot{W}(t) = rW(t) - (z(t)+c(t)p_c(t)),\ W(0)=W_0 \end{aligned} \tag{6}$$

The parameter ρ is a constant rate of his time preference, σ is an instantaneous volatility rate, δ is the depreciation rate of the addictive stock over time, and A_0 is the initial addictive stock. The first constraint describes the addictive capital accumulation process and $dA(t)$ stands for the rate of change over time in A. The stochastic term $\sigma A(t)dv(t)$ is introduced here and $v(t)$ is a standard Wiener process.[6] The stochastic term is used to capture a host of random events that join forces with the intentional addictive consumption to influence addiction capital accumulation.

The second constraint is the individual's budget constraint and r is the constant interest rate. $W(t)$ is the wealth at time t and $\dot{W}(t)$ is the rate of change over time in W. W_0 is the initial wealth which can be regarded as the discounted present value of a consumer's lifetime income. The price of the normal good is normalized to 1, and $p_c(t)$ is the price of the addictive good at time t. The addict's goal is to select a bundle of the normal good $z(t)$ and the addictive good $c(t)$ each time under the two constraints so as to maximize his accumulation of utility over an infinite lifetime.

Notice that when there is no uncertainty, i.e., $\sigma = 0$, the above problem reduces to the deterministic dynamic model of Becker and Murphy (1988).

In the following we discuss how to determine both the optimal path of the addictive consumption and the optimal path of addictive capital. For the problem (5), the corresponding Hamilton-Jacobi-Bellman (HJB) equation is given by (see Kamien and Schwartz 1991 and Sethi and Thompson 2000)

$$\rho J = \max_{c,z}\{u(c,A,z) + J_A(c-\delta A) + J_W[rW-(z+cp_c)] + \frac{1}{2}\sigma^2 A^2 J_{AA}\}, \tag{7}$$

[6] This process has been widely used to describe natural random events; see e.g., Mirrlees (1964), Merton (1969, 1971), Black and Scholes (1973), and Dixit and Pindyck (1994).

where the time t is omitted when no confusion can arise. The first order conditions of (7) are

$$u_c + J_A - p_c J_W = 0, \text{ and } u_z - J_W = 0. \tag{8}$$

Recall that $u_{cc} = \frac{\partial^2 u}{\partial c^2} < 0$ and $u_{zz} = \frac{\partial^2 u}{\partial z^2} < 0$. Then u_c and u_z are strictly decreasing functions with respect to c and z, respectively. So their inverse functions exist and can be written as

$$c = u_c^{-1}(p_c J_W - J_A), \text{ and } z = u_z^{-1}(J_W). \tag{9}$$

From $u_z - J_W = 0$ of (8) and (5) we know $J_W > 0$.

Now we have the following simple but basic observation saying that the fundamental economics law still holds under uncertainty.

Proposition 1. *For the problem* (5) *the addictive consumption is a decreasing function of its price p_c, ceteris paribus.*

Proof. Notice that because u_c^{-1} is a strictly decreasing function, we have $\frac{\partial c}{\partial p_c}$ is negative. That is to say, the addictive consumption c will decrease if its price p_c increases. ∎

3. MAIN RESULTS

To analyze the effect of uncertainty on addictive behavior and obtain substantial insights into the problem (5), we shall assume that the normal good consumption is fixed, say $z = 0$,[7] and the individual has a sufficient amount of income. Now the general problem (5) becomes

$$\begin{aligned} &\max_{c(t)} \ E\left\{\int_0^\infty u(c(t), A(t)) \exp(-\rho t) dt\right\} \\ &\text{s.t. } \ dA(t) = (c(t) - \delta A(t))dt + \sigma A(t) dv(t), \ A(0) = A_0 > 0 \end{aligned} \tag{10}$$

Because this is an autonomous stochastic optimal control problem with infinite time horizon, we can assume that the value function is independent of time (see Kamien and Schwartz 1991 and Sethi and Thompson 2000).

[7] In their deterministic model Becker and Murphy (1988) first set up a general model of having a normal good and an addictive good and then analyzed their model by ignoring the normal good. Here we follow them by ignoring the normal good. Doing so is certainly not ideal but still grasps the essence of the problem as the consumption of the addictive substance plays a much more significant role in affecting addictive behavior than the normal good does.

Thus by setting $J = J(A)$, we have the HJB equation as follows

$$\rho J = \max_{c}\{u(c,A) + J_A(c - \delta A) + \frac{1}{2}\sigma^2 A^2 J_{AA}\}. \tag{11}$$

The first order condition is

$$u_c + J_A = 0 \tag{12}$$

Because $\frac{\partial^2 u}{\partial c^2} < 0$ and thus u_c is a strictly decreasing function, the inverse function of u_c exists and can be given as

$$c = u_c^{-1}(-J_A). \tag{13}$$

Using (13) to substitute for c in (11) gives the HJB equation

$$\rho J = u(u_c^{-1}(-J_A), A) + J_A(u_c^{-1}(-J_A) - \delta A) + \frac{1}{2}\sigma^2 A^2 J_{AA}. \tag{14}$$

In order to derive a closed-form formula for both the optimal path of the addictive consumption and the optimal path of the addictive capital, we shall explore the following multivariate power utility function which is well-defined for $A \geq 0$ and $c > 0$

$$u(c,A) = -\frac{A^\beta}{c^\alpha},\ \beta \geq \alpha + 1,\ \alpha > 0. \tag{15}$$

We call this function *the addictive multivariate power utility function*, which appears somewhat related to the well-known Cobb-Douglas function and the utility function of Abel (1990)[8] but has a different requirement on parameters.

The following proposition shows that the addictive multivariate power utility function satisfies all the conditions (2), (3), and (4) required for addictive utility functions. More importantly, this utility function is tractable, dispenses with the use of any approximation such as the quadratic function used by Becker and Murphy (1988) and enables us to obtain closed-form solutions.

[8]For a macro model Abel introduces the function $u(c_t, v_t) = \frac{1}{1-\alpha}(\frac{c_t}{v_t})^{1-\alpha}$ with $\alpha > 0$ to reflect the idea of catching up with Joneses, where c_t is an individual's consumption at time t and v_t may be an average or aggregated consumption of the society at time t. Abel (1990) dealt with obviously a totally different problem from the addiction problem of Becker and Murphy (1988).

Proposition 2. *If $\beta \geq \alpha + 1 > 1$, the function $u(c, A) = -\frac{A^\beta}{c^\alpha}$ is a well-defined concave function for $A \geq 0$ and $c > 0$ and possesses the withdrawal, tolerance and reinforcement properties* (2), (3), *and* (4).[9]

Proof. For the function $u(c, A) = -\frac{A^\beta}{c^\alpha}$ in the domain of $A > 0$ and $c > 0$, it is easy to calculate its 1st and 2nd derivatives as follows:

$$u_c = \alpha \frac{A^\beta}{c^{(\alpha+1)}} > 0, \quad u_{cc} = -\alpha(\alpha+1)\frac{A^\beta}{c^{(\alpha+2)}} < 0, \tag{16}$$

$$u_A = -\beta \frac{A^{\beta-1}}{c^\alpha} < 0, \quad u_{AA} = -\beta(\beta-1)\frac{A^{\beta-2}}{c^\alpha} < 0, \tag{17}$$

$$u_{cA} = \alpha\beta \frac{A^{\beta-1}}{c^{(\alpha+1)}} > 0 \tag{18}$$

And the Hessian matrix is

$$H = \begin{pmatrix} u_{AA} & u_{Ac} \\ u_{cA} & u_{cc} \end{pmatrix} = \begin{pmatrix} -\beta(\beta-1)\frac{A^{\beta-2}}{c^\alpha} & \alpha\beta\frac{A^{\beta-1}}{c^{(\alpha+1)}} \\ \alpha\beta\frac{A^{\beta-1}}{c^{(\alpha+1)}} & -\alpha(\alpha+1)\frac{A^\beta}{c^{(\alpha+2)}} \end{pmatrix} \tag{19}$$

Then we have $u_{AA} < 0$, $u_{cc} < 0$ and

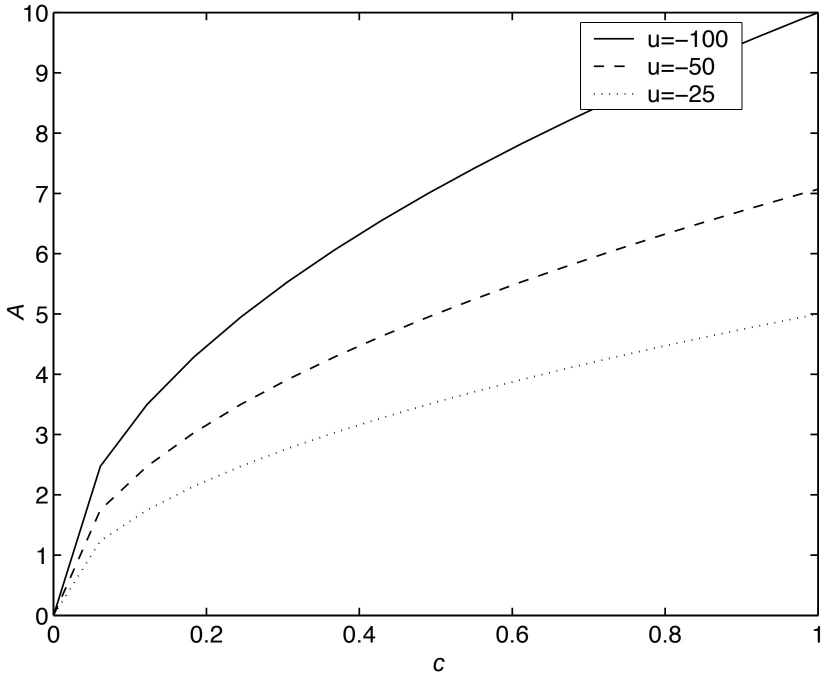
$$\begin{aligned} u_{cc}u_{AA} - (u_{cA})^2 &= \left[-\alpha(\alpha+1)\frac{A^\beta}{c^{(\alpha+2)}}\right]\left[-\beta(\beta-1)\frac{A^{\beta-2}}{c^\alpha}\right] - \left[\alpha\beta\frac{A^{\beta-1}}{c^{-(\alpha+1)}}\right]^2 \\ &= \alpha(\alpha+1)\beta(\beta-1)\frac{A^{2\beta-2}}{c^{(2\alpha+2)}} - \alpha^2\beta^2\frac{A^{2\beta-2}}{c^{(2\alpha+2)}} \\ &= [\beta-(\alpha+1)]\alpha\beta\frac{A^{2\beta-2}}{c^{(2\alpha+2)}} \geq 0 \end{aligned} \tag{20}$$

So H is negative semi-definite under the condition of $\beta - (\alpha+1) \geq 0$. In other words, the utility function $-\frac{A^\beta}{c^\alpha}$ is concave for $\beta \geq \alpha + 1 > 1$. ∎

It is worthy of note about the function $u(c, A)$ at $A = 0$ and $c > 0$. This point is not quite as innocent as it appears. It implies, for instance, that the addictive good has an irresistible power for a potential addict who has not yet had any addictive stock to consume some amount. Furthermore, when the person has stored up some addictive capital $A > 0$, the addictive good will become more irresistible and seduce him into consuming more.

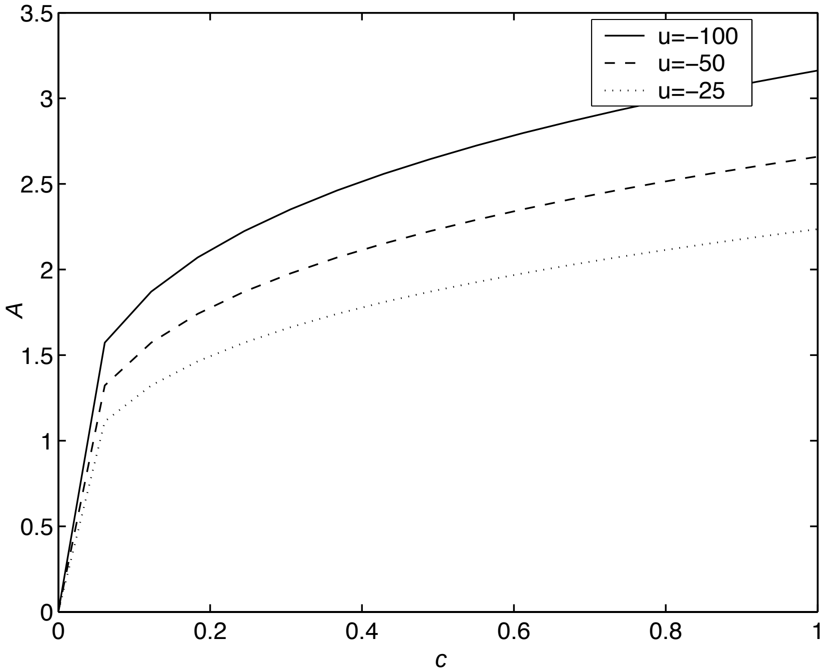
[9] As far as we know, this is the first class of utility functions that are known to exhibit the properties of tolerance, withdrawal and reinforcement, since Becker and Murphy (1988) formulated these mathematical properties.

FIG. 1. The indifference curves for the case of $\alpha = 1$ and $\beta = 2$.



We will use the indifference curves $IC(u) = \{(c, A) \mid u(c, A) = u\}$ of the utility function $u(c, A) = -\frac{A^\beta}{c^\alpha}$ to demonstrate some other properties of this class of addictive utility functions. Figures 1 and 2 illustrate the indifference curves at several different levels of utility. Both figures show that for each given utility level, the addictive consumption is an increasing function of the addictive capital and vice versa. This means that if the addictive capital is rising, the addict needs to increase his addictive consumption in order to maintain the same level of utility, and that the rise of addictive consumption will also increase the addictive capital. Both figures indicate a salient feature that when the addictive capital is small, the indifference curve is approximately linear but very steep; however, when the addictive capital exceeds a certain level, the indifference curve is still approximately linear but becomes almost flat, slightly upwards. This means that when the level of addictive capital is low, the addict will only need to respond with a small increase of addictive consumption to a small increase of addictive capital in order to maintain the same level of utility, but when the addictive capital becomes relatively high, a small increase of addictive capital will require a huge increase of addictive consumption in order to have the same level of utility. An immediate policy implication is that it would be more effective to stop or control addiction earlier than later.

FIG. 2. The indifference curves for the case of $\alpha = 1$ and $\beta = 4$.



Both figures also demonstrate another important feature that the bigger the parameter β is, the flatter of the indifference curve becomes when the addictive capital exceeds a certain level, and the more severe the addiction is. Generally speaking, α and β are closely related to addictive goods and addicts themselves. The stronger the addiction, the greater the β will be.

3.1. Closed-Form Solutions

We shall derive an optimal solution to the problem (10) where the utility function $u(c, A)$ takes the form of (15). Now the maximization problem can be transformed into the minimization problem:

$$\begin{aligned} \min_{c(t)} \quad & E\left\{\int_0^\infty \frac{(A(t))^\beta}{(c(t))^\alpha}\exp(-\rho t)dt\right\} \\ \text{s.t.} \quad & dA(t) = (c(t) - \delta A(t))dt + \sigma A(t)dv(t),\ A(0) = A_0 > 0. \end{aligned}$$

Let $\tilde{J}$ be the value function of the minimization problem. It is clear that $J = -\tilde{J}$. Then the HJB equation becomes

$$\rho\tilde{J} = \min_{c}\left\{\frac{A^{\beta}}{c^{\alpha}} + \tilde{J}_A(c-\delta A) + \frac{1}{2}\sigma^2A^2\tilde{J}_{AA}\right\}. \tag{21}$$

By first order condition, we have

$$c = \left(\frac{\alpha A^{\beta}}{\tilde{J}_A}\right)^{\frac{1}{1+\alpha}}. \tag{22}$$

Using (22) to substitute for c in (21) yields

$$\rho\tilde{J} = A^{\beta}\left(\left(\frac{\alpha A^{\beta}}{\tilde{J}_A}\right)^{\frac{1}{1+\alpha}}\right)^{-\alpha} + \tilde{J}_A\left(\left(\frac{\alpha A^{\beta}}{\tilde{J}_A}\right)^{\frac{1}{1+\alpha}} - \delta A\right) + \frac{1}{2}\sigma^2A^2\tilde{J}_{AA}. \tag{23}$$

Let us guess that the value function is of the following form

$$\tilde{J}(A) = aA^{\gamma},\ \gamma \neq 0.$$

Then we have

$$\tilde{J}_A = a\gamma A^{\gamma-1}, \quad \tilde{J}_{AA} = a\gamma(\gamma-1)A^{\gamma-2}.$$

Using these formulas in (23) results in

$$\rho aA^{\gamma} = A^{\beta}\left(\left(\frac{\alpha A^{\beta}}{a\gamma A^{\gamma-1}}\right)^{\frac{1}{1+\alpha}}\right)^{-\alpha} + a\gamma A^{\gamma-1}\left(\left(\frac{\alpha A^{\beta}}{a\gamma A^{\gamma-1}}\right)^{\frac{1}{1+\alpha}} - \delta A\right) + \frac{1}{2}\sigma^2A^2a\gamma(\gamma-1)A^{\gamma-2}$$

which can be rewritten as

$$\rho aA^{\gamma} = \left[\left(\frac{a\gamma}{\alpha}\right)^{\frac{\alpha}{1+\alpha}} + a\gamma\left(\frac{\alpha}{a\gamma}\right)^{\frac{1}{1+\alpha}}\right]A^{\frac{\beta+\alpha\gamma-\alpha}{1+\alpha}} - \delta a\gamma A^{\gamma} + \frac{1}{2}\sigma^2a\gamma(\gamma-1)A^{\gamma}. \tag{24}$$

We select parameter γ to make A's power equal in every term of (24). Then we have

$$\frac{\beta+\alpha\gamma-\alpha}{1+\alpha} = \gamma.$$

Solving this equation gives $\gamma = \beta - \alpha$. Replacing γ by $\beta - \alpha$ in (24) leads to

$$\rho a = \left(\frac{a(\beta-\alpha)}{\alpha}\right)^{\frac{\alpha}{1+\alpha}} + a(\beta-\alpha)\left(\frac{\alpha}{a(\beta-\alpha)}\right)^{\frac{1}{1+\alpha}} - \delta a(\beta-\alpha) + \frac{1}{2}\sigma^2a(\beta-\alpha)(\beta-\alpha-1).$$

It follows that

$$a = \left(\frac{(\frac{\beta-\alpha}{\alpha})^{\frac{\alpha}{1+\alpha}} + (\beta-\alpha)^{\frac{\alpha}{1+\alpha}} \alpha^{\frac{1}{1+\alpha}}}{\rho + \delta(\beta-\alpha) - \frac{1}{2}\sigma^2(\beta-\alpha)(\beta-\alpha-1)} \right)^{1+\alpha}.$$

Replacing γ by $\beta - \alpha$ and the constant a by the above formula in $\tilde{J}(A) = aA^{\gamma}$ produces an explicit value function[10]

$$\tilde{J}(A) = \left(\frac{(\frac{\beta-\alpha}{\alpha})^{\frac{\alpha}{1+\alpha}} + (\beta-\alpha)^{\frac{\alpha}{1+\alpha}} \alpha^{\frac{1}{1+\alpha}}}{\rho + \delta(\beta-\alpha) - \frac{1}{2}\sigma^2(\beta-\alpha)(\beta-\alpha-1)} \right)^{1+\alpha} A^{\beta-\alpha}$$

Differentiating both sides of this equation with respect to A gives

$$\tilde{J}_A = (\beta-\alpha) \left(\frac{(\frac{\beta-\alpha}{\alpha})^{\frac{\alpha}{1+\alpha}} + (\beta-\alpha)^{\frac{\alpha}{1+\alpha}} \alpha^{\frac{1}{1+\alpha}}}{\rho + \delta(\beta-\alpha) - \frac{1}{2}\sigma^2(\beta-\alpha)(\beta-\alpha-1)} \right)^{1+\alpha} A^{\beta-\alpha-1}. \quad (25)$$

$-\tilde{J}_A$ is the shadow price of addictive capital. Replacing $\tilde{J}_A$ by the above formula in (22), we can obtain the expression of the optimal addictive consumption[11]

$$c(t) = \frac{\alpha}{1+\alpha} \left[\frac{\rho}{\beta-\alpha} + \delta - \frac{1}{2}\sigma^2(\beta-\alpha-1) \right] A(t). \quad (26)$$

Using this formula to substitute for $c(t)$ in $dA(t) = (c(t) - \delta A(t))dt + \sigma A(t)dv(t)$ yields

$$dA(t) = \left\{ \frac{\alpha}{1+\alpha} [\frac{\rho}{\beta-\alpha} - \frac{1}{2}\sigma^2(\beta-\alpha-1)] - \frac{\delta}{1+\alpha} \right\} A(t)dt + \sigma A(t)dv(t),\ A(0) = A_0.$$

Solving this stochastic differential equation gives

$$\begin{aligned} A(t) &= A_0 \exp\{[\tfrac{\alpha}{1+\alpha}(\tfrac{\rho}{\beta-\alpha} - \tfrac{1}{2}\sigma^2(\beta-\alpha-1)) - \tfrac{\delta}{1+\alpha} - \tfrac{\sigma^2}{2}]t + \sigma v(t)\} \\ &= A_0 \exp((\eta - \delta - \tfrac{\sigma^2}{2})t + \sigma v(t)) \end{aligned} \quad (27)$$

where $\eta = \frac{\alpha}{1+\alpha}(\frac{\rho}{\beta-\alpha} + \delta - \frac{1}{2}\sigma^2(\beta-\alpha-1))$. Using $A(t)$ in (27) we obtain the optimal addictive consumption and its expectation

$$c(t) = A_0 \eta \exp((\eta - \delta - \frac{\sigma^2}{2})t + \sigma v(t)) \quad (28)$$

[10] Recall that $J(A) = -\tilde{J}(A)$. So we can obtain the analytical results about the effect of different factors on the total utility.

[11] See the derivation of (26) in the Appendix.

$$E\{c(t)\} = A_0\eta\exp((\eta-\delta)t) \tag{29}$$

Obviously, in order for the solution $c(t)$ to be meaningful (i.e., $c(t)\geq 0$), η needs to be positive. Observe that (27) and (28) have identical structures and (28) and (29) have rather similar structures. (28) gives the optimal addictive consumption trajectory which is a geometric Brownian motion, while (29) provides the expected optimal addictive consumption trajectory which is a constant with respect to t. These two formulas will be used to analyze both short-run and long-run addictive behavior. We now summarize this discussion in the theorem below which establishes optimality of the solution.

THEOREM 1. *The formulas* (27) *and* (28) *are an optimal addictive capital trajectory and an optimal addictive consumption trajectory to the problem* (10)*, respectively.*

Proof. For $A>0$ and $T>0$, let $(A(t),c(t))$ be an arbitrary admissible addictive capital and consumption. Let $V(\cdot)$ denote an arbitrary solution of the HJB equation (21). Applying the well-known Ito Lemma (see e.g., Oksendal 2000) to the function $w(t,A)=e^{-\rho t}V(A)$ yields

$$\begin{aligned}
dw &= w_t dt + w_A dA + \tfrac{1}{2}w_{AA}(dA)^2\\
&= -\rho e^{-\rho t}V(A)dt + e^{-\rho t}(V_A dA + \tfrac{1}{2}V_{AA}dA\cdot dA)\\
&= -\rho e^{-\rho t}V(A)dt + e^{-\rho t}\{V_A[(c(t)-\delta A(t))dt + \delta A(t)dv(t)]\\
&\quad +\tfrac{1}{2}V_{AA}[(c(t)-\delta A(t))dt + \sigma A(t)dv(t)]^2\}\\
&= -\rho e^{-\rho t}V(A)dt + e^{-\rho t}\{V_A[(c(t)-\delta A(t))dt + \delta A(t)dv(t)] + \tfrac{1}{2}V_{AA}\sigma^2(A(t))^2dt\}
\end{aligned}$$

where we use the rules $(dt)^2 = dt\cdot dv(t) = dv(t)\cdot dt = 0$ and $dv(t)\cdot dv(t) = dt$. Then we have

$$\begin{aligned}
&E\{w(T,A(T)) - w(0,A(0))\} = E\{\int_0^T e^{-\rho t}[-\rho V(A(t)) + V_A(c(t)-\delta A(t)) + \tfrac{1}{2}\sigma^2(A(t))^2V_{AA}]dt\}\\
&= E\{\int_0^T e^{-\rho t}[-\rho V(A(t)) + V_A(c(t)-\delta A(t)) + \tfrac{1}{2}\sigma^2(A(t))^2V_{AA} + \tfrac{(A(t))^\beta}{(c(t))^\alpha} - \tfrac{(A(t))^\beta}{(c(t))^\alpha}]dt\}\\
&= E\{\int_0^T e^{-\rho t}[-\rho V(A(t)) + V_A(c(t)-\delta A(t)) + \tfrac{1}{2}\sigma^2(A(t))^2V_{AA} + \tfrac{(A(t))^\beta}{(c(t))^\alpha}]dt\} - E\{\int_0^T e^{-\rho t}\tfrac{(A(t))^\beta}{(c(t))^\alpha}dt\}
\end{aligned}$$

where we use the equality $E\{\int_0^T \delta A(t)dv(t)\} = 0$, a well-known property of the Ito Integral; see e.g., Oksendal (2000, Theorem 3.2.1, p.30). Furthermore, we obtain

$$\begin{aligned} w(0, A(0)) \;=\; & E\{w(T, A(T))\} \\ & -E\{\int_0^T e^{-\rho t}[-\rho V(A(t)) + V_A(c(t) - \delta A(t)) + \tfrac{1}{2}\sigma^2 (A(t))^2 V_{AA} + \tfrac{(A(t))^\beta}{(c(t))^\alpha}]dt\} \\ & +E\{\int_0^T e^{-\rho t} \tfrac{(A(t))^\beta}{(c(t))^\alpha} dt\} \end{aligned}$$

From the HJB equation (21), we know that

$$-\rho V(A(t)) + V_A(c(t) - \delta A(t)) + \frac{1}{2}\sigma^2 (A(t))^2 V_{AA} + \frac{(A(t))^\beta}{(c(t))^\alpha} \geq 0.$$

Then

$$V(A_0) = w(0, A(0)) \leq E\{w(T, A(T))\} + E\{\int_0^T e^{-\rho t} \frac{(A(t))^\beta}{(c(t))^\alpha} dt\} \tag{30}$$

By using the transversality condition

$$\lim_{T\to\infty} E\{w(T, A(T))\} = 0$$

and

$$\lim_{T\to\infty} E\{\int_0^T e^{-\rho t} \frac{(A(t))^\beta}{(c(t))^\alpha} dt\} = E\{\int_0^\infty e^{-\rho t} \frac{(A(t))^\beta}{(c(t))^\alpha} dt\}$$

we can derive from the inequality (30) that

$$V(A_0) \leq E\{\int_0^\infty e^{-\rho t} \frac{(A(t))^\beta}{(c(t))^\alpha} dt\} \tag{31}$$

Since (31) holds for any admissible addictive capital $A(t)$ and consumption $c(t)$, we have

$$V(A_0) \leq \min_{c(t)} E\{\int_0^\infty e^{-\rho t} \frac{(A(t))^\beta}{(c(t))^\alpha} dt\} = \tilde{J}$$

On the other hand, let $A^*(t)$ and $c^*(t)$ be given respectively by (27) and (28). Recall that to derive $A^*(t)$ and $c^*(t)$, we have used the form of $V(A) = aA^\gamma$. Observe that

$$-\rho V(A^*(t)) + V_A(c^*(t) - \delta A^*(t)) + \frac{1}{2}\sigma^2 (A^*(t))^2 V_{AA} + \frac{(A^*(t))^\beta}{(c^*(t))^\alpha} = 0$$

This means that the HJB equation is satisfied by $(A(t), c(t))$. Therefore,

$$\begin{aligned}
V(A_0) &= w(0, A(0)) \\
&= E\{w(T, A^*(T))\} + E\{\int_0^T e^{-\rho t} \tfrac{(A^*(t))^\beta}{(c^*(t))^\alpha} dt\} \\
&= E\{e^{-\rho T} V(A^*(T))\} + E\{\int_0^T e^{-\rho t} \tfrac{(A^*(t))^\beta}{(c^*(t))^\alpha} dt\}
\end{aligned}$$

It remains to show that $\lim_{T\to\infty} E\{w(T, A^*(T))\} = 0$. Notice that

$$\begin{aligned}
\lim_{T\to\infty} E\{w(T, A^*(T))\} &= \lim_{T\to\infty} E\{e^{-\rho T} a A^*(T)^{\beta-\alpha}\} \\
&= \lim_{T\to\infty} E\{\exp(-\rho T) a \{A_0 \exp[(\eta - \delta - \tfrac{\sigma^2}{2})T + \sigma v(T)]\}^{\beta-\alpha}\} \\
&= \lim_{T\to\infty} a A_0^{\beta-\alpha} E\{\exp\{-\rho T + (\beta-\alpha)[(\eta - \delta - \tfrac{\sigma^2}{2})T + \sigma v(T)]\}\} \\
&= \lim_{T\to\infty} a A_0^{\beta-\alpha} \exp\{[-\rho + (\beta-\alpha)(\eta - \delta)]T\} \\
&= \lim_{T\to\infty} a A_0^{\beta-\alpha} \exp\{\{-\rho + (\beta-\alpha)\{\tfrac{\alpha}{1+\alpha}[\tfrac{\rho}{\beta-\alpha} + \\
&\quad \delta - \tfrac{1}{2}\sigma^2(\beta-\alpha-1)] - \delta\}\}T\} \\
&= \lim_{T\to\infty} a A_0^{\beta-\alpha} \exp\{-\{\tfrac{\rho}{1+\alpha} + \tfrac{\beta-\alpha}{1+\alpha}[\tfrac{1}{2}\alpha\sigma^2(\beta-\alpha-1) + \delta]\}T\} \\
&= 0
\end{aligned}$$

where we have used the fact of $\beta - \alpha - 1 \geq 0$. Then we have

$$\begin{aligned}
V(A_0) &= \lim_{T\to\infty} E\{\int_0^T e^{-\rho t} \tfrac{(A^*(t))^\beta}{(c^*(t))^\alpha} dt\} \\
&= E\{\int_0^\infty e^{-\rho t} \tfrac{(A^*(t))^\beta}{(c^*(t))^\alpha} dt\} \\
&\geq \min E\{\int_0^\infty e^{-\rho t} \tfrac{(A(t))^\beta}{(c(t))^\alpha} dt\} \\
&= \tilde{J}
\end{aligned} \tag{32}$$

By (31) and (32), $V(A_0) = \tilde{J}$. ∎

In the following sections we will derive various properties of the solution (27) and (28) and explore their? implications.

3.2. Cycles of Binges and Abstention Attempts

Binges are very common in alcohol drinking, cigarette smoking, eating and some other kinds of addiction. By binge we mean a short period of excessive indulgence in a good or an activity. Knowing the harmful effect of addiction, addicts also often attempt to reduce or quit their addictive consumption. We call this phenomenon abstention attempt. In fact cycles of binges and abstention attempts, such as overeating and dieting, are a familiar addictive behavioral pattern. Such cycles are usually irregular and are triggered by random events. Our model is capable of capturing this important feature of dynamic addictive behavior. In our model (10), random events are described by the standard Wiener process $v(t)$, $A(t)$ is a state variable, and $c(t)$ is a control variable. $v(t)$ is a random variable and directly affects the level of addictive capital $A(t)$. The individual cannot directly control his addictive capital $A(t)$ but can influence it by choosing an appropriate addictive consumption $c(t)$. In the model, beneficial events plunge the addictive capital $A(t)$ to (local) lows while harmful events compel $A(t)$ to jump to (local) highs. Marcus and Siedler (2015) offer an empirical study on binge alcohol drinking.

Harmful random events include marital breakup, job loss, death of a loved one, other stressful events, and environmental cues. Siegel et al. (1982) indicated that some cues could be fatal for addicts. Laibson (2001) and Bernheim and Rangel (2004) investigated how cues may affect addiction in different contexts. Certain harmful events can be happy occasions; for instance, friends gathering can create binge drinking or smoking or eating. Harmful events can induce powerful or overwhelming cravings for addictive consumption. In our model, this means that such events will instantly spur the addictive capital to reach a peak. Because of adjacent complementarity, i.e., reinforcement, between addictive capital and addictive consumption, the addict will immediately respond with a large increase of addictive consumption in order to sustain his current utility.

Beneficial events can be the death of a friend caused by addiction, a lesson of good counseling, compelling campaigns against drugs, reading of a good book on addiction control, a piece of horrific news on addiction, and etc. Such events usually appear to be sober or thought-provoking episodes. They can prompt addicts to have a strong desire to reduce or quit their addictive consumption. In our model, these events will instantly plummet the addictive capital to a bottom. Also because of reinforcement effect, addicts will immediately reciprocate with a dramatic decrease of addictive consumption in order to maintain their current utility levels.

In summary, irregular cycles of binges and abstention attempts fit well into our general framework. Furthermore, it is easy to see from formulas (27) and (28) that the dynamic addictive consumption synchronizes with the movement of addictive capital. In fact, both consumption $c(t)$ and capital $A(t)$ movements share the same pattern with only a constant magnitude difference of η. The fluctuation of the Wiener process reflects the irregularity of cycles of binges and abstention attempts.

3.3. Properties of the Solution

We will now examine the closed-form solutions given by (28) and (29) in detail and see how the parameters affect the addictive consumption and capital patterns. These properties will be useful to derive policy implications.

To obtain the optimal addictive consumption path we have used the utility function $u(c,A) = -\frac{A^\beta}{c^\alpha}$. Observe that $-\alpha$ reflects the elasticity $e(c) = \frac{cu_c}{u}$ of utility over addictive consumption, and β is the elasticity $e(A) = \frac{Au_A}{u}$ of utility over addictive capital. Typically utility functions contain the same number of variables as the number of goods for consumption. It is worth stressing here that although the utility function $u(c,A)$ for the rational addiction model contains two variables A and c, it only involves one commodity c to consume. This means we need to adapt standard analysis to this context. As it is well-known, the curvature of the utility function measures the individual's attitude toward risk. For the concave utility function $u(c,A)$ the parameter $\beta - \alpha$ roughly reflects the degree of concavity and risk preference and might be called *the degree of risk aversion.* Unlike the univariate function case, the function $u(c,A)$ involves two variables and this definition of concavity degree could be seen as an analogue of the univariate case. The bigger $\beta - \alpha$ is, the more concave the utility function is, and the more risk averse the individual will be. Let $\psi = \beta - \alpha$. We now have the following property immediately from (29).

PROPOSITION 3. *The expected optimal addictive consumption is a decreasing function of the level σ^2 of uncertainty ceteris paribus, i.e.,* $\frac{\partial E\{c\}}{\partial \sigma^2} < 0$.

Proposition 3 tells that the level σ^2 of uncertainty affects addictive consumption negatively on average. The more volatile the less expected consumption.

PROPOSITION 4. *Both the optimal addictive consumption and the expected optimal addictive consumption are decreasing functions of the generalized degree ψ of risk aversion ceteris paribus, i.e.,* $\frac{\partial c}{\partial \psi} < 0$ *and* $\frac{\partial E\{c\}}{\partial \psi} < 0$.

Proposition 4 shows that as individuals become more risk averse, they incline to be less addicted to harmful substances.

PROPOSITION 5. *The optimal addictive consumption is an increasing function of the time preference ρ, and the minus elasticity α of utility over addictive consumption, respectively, but a decreasing function of the elasticity β of utility over addictive capital ceteris paribus, i.e.,* $\frac{\partial c}{\partial \rho} > 0$, $\frac{\partial c}{\partial \alpha} > 0$, $\frac{\partial c}{\partial \beta} < 0$. *The same conclusion holds on average, i.e.,* $\frac{\partial E\{c\}}{\partial \rho} > 0$, $\frac{\partial E\{c\}}{\partial \alpha} > 0$, $\frac{\partial E\{c\}}{\partial \beta} < 0$.

In Proposition 5, the first formula $\frac{\partial c}{\partial \rho} > 0$ shows that present-oriented individuals tend to be more addicted to harmful goods than future-oriented individuals. The second formula $\frac{\partial c}{\partial \alpha} > 0$ indicates that individuals become more willing to consume addictive goods as elasticity $-\alpha$ is decreasing. The reason for this is that due to withdrawal effect individuals' utility would increase should the addictive consumption increase, because increasing α would reinforce withdrawal effect, it will increase their demand for addictive goods. The third formula $\frac{\partial c}{\partial \beta} < 0$ suggests that as individuals' elasticity β over addictive capital increases, they become less addicted to harmful goods. This is because marginal utility over addictive capital is negative, as elasticity β increases, it will lower addictive capital and as a result reduce addictive consumption because of reinforcement effect. Observe that the two parameters α and β are opposing forces for addictive consumption and are reflected in the degree of risk aversion $\beta - \alpha$. The proof of this proposition follows immediately.

Proposition 6. *The optimal addictive consumption is an increasing function of the depreciation rate δ for small time t but will become a deceasing function of the depreciation rate for large time t ceteris paribus, i.e.,* $\frac{\partial c(t)}{\partial \delta} = \frac{A_0}{1+\alpha}(\alpha - t\eta)\exp((\eta - \delta - \frac{\sigma^2}{2})t + \sigma v(t))$. *The same conclusion holds on average, i.e.,* $\frac{\partial E\{c(t)\}}{\partial \delta} = \frac{A_0}{1+\alpha}(\alpha - t\eta)\exp(\eta - \delta)t$.

On the one hand, it is easy to see from (28) and (29) that other things being equal a sufficiently high depreciation rate δ will drive addictive consumption to zero. On the other hand, Proposition 6 indicates that the addictive consumption increases with depreciation rate when time t is close to zero. These two conclusions appear to be contradictory. However, they are not. The reason is that $\frac{\partial c}{\partial \delta} > 0$ holds true only when time t is near zero or rather small, whereas $\frac{\partial c}{\partial \delta}$ will become negative when time t is sufficiently large. If we look at the addiction consumption pattern both in short run and in long run, it is possible that addictive consumption may first increase and then decrease with time for sufficiently large depreciation rate. An increase in δ will reduce the shadow price of addictive capital and thus increase addictive consumption; while a higher δ will also decrease the addictive capital more quickly which may reduce the tolerance effect and thus drop addictive consumption. The former force is stronger at the beginning stage, but the later force will become stronger and stronger at later stage if the depreciation rate is higher enough.

Proposition 7. *The expected addictive consumption $E\{c(t)\}$ will converge to zero as time goes to infinity in the case of $\eta - \delta < 0$, $E\{c(t)\}$ will tend to infinity with time in the case of $\eta - \delta > 0$, and $E\{c(t)\} = A_0\eta$ for all t in the case of $\eta = \delta$.*

Proposition 7 manifests three distinctive long-run addictive consumption patterns. The parameter $\eta - \delta$ could be used as *an indicator of stability.* The case of $\eta - \delta < 0$ indicates that the dynamic behavior of the optimal addictive consumption is inherently stable in the sense that no matter what the individual's initial addictive capital is, his expected optimal addictive consumption and capital will die out in the end. We call this case *a benign* *addiction* in the sense that the addict will eventually give up his addiction. The case of $\eta - \delta > 0$ shows that the individual's dynamic addictive behavior is unstable in the sense that no matter where he starts with, he will become more and more addicted and will at last totally lose control of his appetite for harmful goods. This is called *a malignant addiction.* The final case of $\eta - \delta = 0$ reveals that the addictive behavior is stable and the expected optimal addictive consumption always remains at the same level. We call it *a constant addiction.* These three patterns of addictive behavior are consistent with what is commonly observed in reality. Note that the indicator of stability $\eta - \delta = \frac{\alpha}{1+\alpha}(\frac{\rho}{\beta-\alpha} - \frac{1}{2}\sigma^2(\beta - \alpha - 1)) - \frac{\delta}{1+\alpha}$ is determined jointly by a host of the addict's internal and external factors α, β, δ, ρ, and σ. We will explore the policy implications of these factors in the next section.

When there is no uncertainty, i.e., $\sigma^2 = 0$, we have $c(t) = \frac{\alpha}{1+\alpha}(\frac{\rho}{\beta-\alpha} + \delta)A_0 \exp[\frac{1}{1+\alpha}(\frac{\alpha\rho}{\beta-\alpha} - \delta)t]$. Then we have the following proposition for this deterministic case.

PROPOSITION 8. *If there is no uncertainty, then $c(t) \to 0$ as $t \to \infty$ for $\frac{\alpha\rho}{\beta-\alpha} - \delta < 0$; and $c(t) \to \infty$ as $t \to \infty$ for $\frac{\alpha\rho}{\beta-\alpha} - \delta > 0$; and $c(t) = A_0\frac{\alpha}{1+\alpha}(\frac{\rho}{\beta-\alpha} + \delta)(> 0)$ for all $t \geq 0$ when $\frac{\alpha\rho}{\beta-\alpha} - \delta = 0$.*

With respect to Proposition 8, we have several easy observations: First, addictive consumption will finally converge to zero if the depreciation rate δ is sufficiently high; If the consumer dislikes risk very much, i.e., if $\beta - \alpha$ is large enough, then addictive consumption will also finally converge to zero. This is the benign case. Second, individuals will become more addicted to harmful goods as they become more impatient. So in the case of $\frac{\alpha\rho}{\beta-\alpha} - \delta > 0$, the individual becomes sufficiently impatient and thus intends to consume more addictive good today. As a result, this will increase the addictive capital and on the other hand reinforcement effect will induce the individual to consume more in the future. Therefore as time goes to infinity, the addictive consumption can spiral out of control. This is the behavior of a malignant addict. The third case is when the consumption of an addictive good always remains at the same level, e.g., a constant smoker or drinker.

3.4. A Comparison with Becker and Murphy

Now we will briefly review the main results of Becker and Murphy (1988) and compare their results with ours. Recall they formulated rational addictive behavior as a deterministic dynamic model. Their formal analysis is mainly based on a quadratic approximation of their utility function and their problem is therefore reduced to a linear quadratic optimal control problem. Because they used a quadratic function to approximate an unknown utility function at an unknown steady state and the unknown steady state actually needs to be located, it is mathematically difficult to guarantee a desirable degree of accuracy. Quadratic utility implies that marginal utility will generally reach zero at some finite level of consumption and then become negative. As a result, this type of control problem typically has only one steady state, which is similar to our case of constant addiction as shown in Proposition 8. It is impossible for their model to generate our other two typical (malignant and benign) patterns of addictive consumption which are essentially balanced growth paths with positive or negative growth rates, as shown in Proposition 8.

Becker and Murphy (1988) further discussed a hypothetical case of a cubic term contained in the original utility function or its approximation and indicated that it could probably result in at most two steady states: one stable and one unstable. On p. 683, they state: "However, if a quadratic function were only a local approximation to the true function near a steady state and if the true function, say, had a cubic term in S^3 with a negative coefficient added to a quadratic function, the first-order conditions in equation (16) would then generally imply two interior steady states, one stable and one unstable." They assumed that the steady state with low addictive consumption is unstable and the stable steady state has a high addictive consumption. No matter where the initial state of the addict starts from, the addictive consumption in their model will eventually approach and remain at a constant level for ever, which is also similar to our case of constant addiction as shown in Proposition 8. Again in this more complex case, their model still cannot replicate our malignant and benign patterns as shown in Proposition 8.

In summary, even in the deterministic case by comparison with Becker and Murphy model our approach offers fresh and extra insights into the behavior of addiction. In particular we have found malignant and benign patterns of addictive behavior beyond what Becker and Murphy model can predict. We believe that addictive consumption, such as drug addiction, has attracted wide attention and severe addiction is hard to stop because addicts can be (easily) trapped in a vicious circle (in stochastic environments): higher addictive consumption leads to higher addictive capital, which in turn induces higher addictive consumption. This spiral would cause deadly damage to drug addicts. Our malignant case well reflects this pattern of severe addictive behavior. In the next section we will discuss possible ways to reduce or terminate addiction.

4. POLICY IMPLICATIONS

In this section we discuss some policy implications that can be drawn from our model and its solution (29). In the model, the parameters α, β, δ, and ρ can be viewed as the addict's internal or intrinsic factors, while

σ can be arguably regarded as an external or environmental factor and A_0 can be also seen as an external factor to some extent as it is directly affected by the addictive consumption. To formulate meaningful policies of treatment and prevention of addictive behaviors, one need to find ways and means that can influence these internal and external factors in a positive way. One would at least hope that the addict's (expected) consumption will reduce to zero in the long run.

Recall that by Proposition 7 we are able to classify addictions into three typical patterns: benign, malignant, and constant. These patterns are determined by several basic internal and external factors and shown by formula (29). This formula clearly reflects the complexity of addictive behavior. It also indicates that terminating an addictive behavior can be a long, complex and difficult process. To see this, let us look at the case of malignant addiction. It follows from formula (29) that it seems possible to use a coercive harsh treatment, say the "cold turkey" method-the abrupt and complete cessation of taking an addictive good, to reduce the initial addictive capital and therefore to eventually stop the addictive consumption. However, it is extremely difficult to achieve this goal. This is because from Proposition 7 for the case of $\eta - \delta > 0$ as long as the initial addictive capital is above zero, the addictive consumption can easily run out of control and the addict will fall into the quagmire of addiction. In fact, even if the initial addictive capital is temporarily reduced to zero, random cue factors can easily induce addictive capital and trigger a relapse. This indicates why relapse could be common. Also it is easy to see from formula (29) that even if it is possible for an addict to get rid of his addictive consumption, it can take a long period of time and could be even a life time for some people. An immediate implication here is that there is no once and for all treatment for addictive behaviors. This is consistent with a vast volume of empirical evidence from many countries. For example, according to the 2017 China annual report on drug control, the country has 445,000 new drug addicts in 2016 and the number of relapses exceeds 600,000 people with above 80% of relapse rate.[12]

To tackle and prevent addictive behaviors, many treatments and approaches have been developed from a broad range of disciplines, including psychology, biology, pharmacology, medicine, law, economics, and sociology; see e.g., Schelling (1992), O'Brien and McLellan (1996), Leshner (1997), Goldstein (2001), Orford (2002), West (2006), and Moss and Dyer (2010). Some treatments and approaches can be harsh, extremely painful or even life-threatening. Other can be soft by treating addicts in a safe and comfortable manner, such as drug substitution treatment, gradual re-

[12]See `http://www.nncc626.com/2017-03/30/c_129521742.htm` for the report in Chinese and English.

duction of substance use, detoxification, drug therapy, surgery, and psychosocial treatments like counselling, family or community support, and eduction programs. Also new medications or techniques like naltrexone and transcranial magnetic stimulation have been recently used for treating addictions. In practice, treatments may vary from country to country, from person to person and from time to time. Sometimes a single treatment can be used and other times a combination of multiple treatments may be required. For instance, the Netherlands is famous for its lenient policy of soft drug use like cannabis and appears to be successful at dealing with drug addictions (e.g., Goldstein 2001).[13]

To explore ways of controlling addictive behaviors, let us highlight and review how both internal and external factors affect the addictive behaviors as discussed in the previous section. Firstly, as an addict becomes more risk averse, i.e., the bigger the parameter $\beta - \alpha$ is, he will be less addicted to the substance. Secondly, a more volatile environment, i.e., increasing σ, will help to lessen the addictive consumption. Thirdly, as an addict becomes more patient, i.e., lowering ρ, he will become less addicted. Fourthly, reducing the addictive capital will bring down the addictive consumption. In order to control or stop addictive behaviors, we need to find a variety of ways to influence those internal and external factors. In particular, in order to tackle malignant addictive behaviors, it is crucial to bring down the index of stability $\eta - \delta$ below zero by adjusting internal and external factors. Adjusting those factors could be realized through medical, pharmacological, psychological, and psychosocial treatments. One can use soft treatment, or harsh treatment, or both. For instance, if a person has severe withdrawal symptoms, he can initially use soft treatment until he reaches a tolerable level from which cold turkey could be explored to terminate addictions once and for all. In stochastic environments, even if a person currently stays in a safe zone, random events may easily trigger him to consume more and thus drive him into a dangerous zone. This suggests that addiction control is a complex process and requires extreme caution, patience and constant care in order to stop addiction and prevent its relapse, as the saying goes: Old habits die hard.

A key policy implication from Becker and Murphy (1988, pp. 675, 676 and 692) is that severe addictions-a high level of the addictive capital must require going cold turkey, i.e., stopping immediately all addictive consump-

[13]Many countries used to treat certain drug like heroin use as criminal offenses and punish severely but now more and more countries are adopting a soft approach to low-level drug users by treating them as addicts, not criminals, in order to reduce recidivism among offenders.

tion.[14] It is well known (see e.g., West 2006 and Moss and Dyer 2010) that this harsh treatment may not be appropriate for breaking certain addictions, because it can cause immense withdrawal syndrome potentially resulting in death. For instance, treating alcoholics with this method can trigger life-threatening delirium tremens. In general, going cold turkey, even if not life-threatening, can be extremely painful or unbearable for many people, because of severe withdrawal effect. Notice that even if an addict goes cold turkey, his addictive capital, however, will not instantly disappear. Instead it will decrease only gradually, meaning that the painful period might not be short. Our solution (29) suggests that no matter whether an addict has a high or low level of addictive capital and whether he has a benign or malignant addictive behavior, he can be treated with cold turkey, provided that he does not show any life-threatening withdrawal symptoms during the treatment. Of course, this treatment will be more effective for the benign addiction than for the malignant one.

5. CONCLUDING REMARKS

Taking the classic deterministic Becker-Murphy model of rational addiction as the starting point, we have studied a stochastic model of rational addiction. We model uncertain through the natural and well-known Brownian motion process. This process captures random events such as anxiety, tensions, insecurity, and environmental cues which can precipitate and exacerbate addictions, and those sober and thought-provoking episodes that discourage addictions. We explored the stochastic optimization approach to the economic analysis of rational addictive behavior. This new approach made it possible to achieve a variety of new results and to explain typical patterns of addictive behavior such as malignant, benign, and constant addictive behaviors, and cycles of binges and abstention attempts. We identified a class of multivariate power utility functions that possess naturally the three basic characteristics of addictive behavior without resorting to any approximation. Based on these utility functions, we were able to derive closed-form formulas for optimal addictive consumption and capital trajectories. This enabled us to obtain both global and local, both qualitative and quantitative, properties of dynamic addictive behavior, to shed new light on this behavior and to offer novel policy implications. Finally, it might be worth stressing again that in order to achieve a closed form solution, we have ignored the normal good or taken its consumption as constant, as Becker and Murphy (1988) did for their analysis of a deter-

[14]Dockner and Feichtinger (1993) study a model of rational addiction closely related to Becker and Murphy (1988) and show that the model can explain consumption cycles when it has no commodity-specific consumption capital.

ministic model. Doing so is not ideal but facilitates achieving meaningful, novel and rich results and still captures the essence of the problem.

APPENDIX

The Derivation of the Formula (26)**:** Substituting (25) into (22), we have

$$c(t) = \{\alpha A(t)^{\beta}/[(\beta-\alpha)(\frac{(\frac{\beta-\alpha}{\alpha})^{\frac{\alpha}{1+\alpha}} + (\beta-\alpha)^{\frac{\alpha}{1+\alpha}}\alpha^{\frac{1}{1+\alpha}}}{\rho+\delta(\beta-\alpha)-\frac{1}{2}\sigma^2(\beta-\alpha)(\beta-\alpha-1)})^{1+\alpha}A(t)^{\beta-\alpha-1}]\}^{\frac{1}{1+\alpha}}$$

$$= \alpha^{\frac{1}{1+\alpha}}A(t)/[(\beta-\alpha)^{\frac{1}{1+\alpha}}\frac{(\frac{\beta-\alpha}{\alpha})^{\frac{\alpha}{1+\alpha}} + (\beta-\alpha)^{\frac{\alpha}{1+\alpha}}\alpha^{\frac{1}{1+\alpha}}}{\rho+\delta(\beta-\alpha)-\frac{1}{2}\sigma^2(\beta-\alpha)(\beta-\alpha-1)}]$$

$$= \frac{(\frac{\alpha}{\beta-\alpha})^{\frac{1}{1+\alpha}}[\rho+\delta(\beta-\alpha)-\frac{1}{2}\sigma^2(\beta-\alpha)(\beta-\alpha-1)]}{(\frac{\beta-\alpha}{\alpha})^{\frac{\alpha}{1+\alpha}} + (\beta-\alpha)^{\frac{\alpha}{1+\alpha}}\alpha^{\frac{1}{1+\alpha}}}A(t)$$

$$= \frac{\alpha[\rho+\delta(\beta-\alpha)-\frac{1}{2}\sigma^2(\beta-\alpha)(\beta-\alpha-1)]}{(\beta-\alpha)+(\beta-\alpha)\alpha}A(t)$$

$$= \frac{\alpha}{1+\alpha}[\frac{\rho}{\beta-\alpha}+\delta-\frac{1}{2}\sigma^2(\beta-\alpha-1)]A(t)$$

▮

Proof of Proposition 4: We prove $\frac{\partial c}{\partial \psi} < 0$. The same argument works for $\frac{\partial E\{c\}}{\partial \psi} < 0$. Plugging $\psi = \beta - \alpha$ into $c(t)$ of (28) yields

$$c(t) = A_0(\frac{\beta-\psi}{1+\beta-\psi})(\frac{\rho}{\psi}+\delta-\frac{1}{2}\sigma^2(\psi-1))e^{\{[(\frac{\beta-\psi}{1+\beta-\psi})(\frac{\rho}{\psi}+\delta-\frac{1}{2}\sigma^2(\psi-1))-\delta-\frac{\sigma^2}{2}]t+\sigma v(t)\}}.$$

Then we have

$$\begin{aligned}
\frac{\partial c}{\partial \psi} &= A_0[\frac{-1}{(1+\beta-\psi)^2}(\frac{\rho}{\psi}+\delta-\frac{\sigma^2}{2}(\psi-1)) + \frac{\beta-\psi}{1+\beta-\psi}(-\frac{\rho}{\psi^2}-\frac{\sigma^2}{2})](1+\eta t)e^{[(\eta-\delta-\frac{\sigma^2}{2})t+\sigma v(t)]} \\
&= -A_0[\frac{\eta}{\alpha(1+\alpha)} + \frac{\alpha}{(1+\alpha)^2}(\frac{\rho}{(\beta-\alpha)^2}+\frac{\sigma^2}{2})](1+\eta t)e^{[(\eta-\delta-\frac{\sigma^2}{2})t+\sigma v(t)]} \\
&< 0
\end{aligned}$$

as long as $\alpha > 0$, $\eta > 0$ and $t \geq 0$. ▮

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