ANNALS OF ECONOMICS AND FINANCE **23-2**, 329–340 (2022)

A Characterization of Nonminimal Nash Networks in Two-way Flow Model

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A long-lasting open question in the literature of strategic network formation is the characterization of nonminimal Nash networks in the context of two-way flow model with nonrival information. In this note, I provide a partial answer to this question for the class of nonminimal networks such that every chain between two agents has the length of at most two and heterogeneity in information value possessed by each agent is assumed. I show that every strict Nash network is a strongly nested split graph. I also show that every strict Nash network is also efficient given a specific structure of information value.

Key Words: Network Formation; Two-way Flow Network; Nested Split Graph; Nash Network; Efficient Network.
JEL Classification Numbers: C72, D85.

1. INTRODUCTION

The seminal paper of Bala and Goyal (2000) proposes a simple form of strategic network formation. It assumes that information owned by each agent is nonrival. Hence, contrary to the other seminal work of Jackson and Wolinsky (1996), link formation does not require a mutual consent and is unilateral. This simplicity allows Bala and Goyal (2000) to propose Nash network (NN) and strict Nash network (SNN), defined as Nash equilibrium and strict Nash equilibrium in pure strategies, as solution concepts. A major advantage of these solutions concept is that it allows game theorist to conveniently observe how each agent in the network strategically decides as to who he prefers to establish links with. Nash network as a solution concept has been used in a myriad of works in the literature. Recent

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1529-7373/2022

works that applied this solution concept are, for instances, in the context of common enemy effect (Haller and Hoyer (2019)) and signed network (Hiller (2017)).

Of particular interest to this note is the literature that seeks to understand the role of information decay in the context of nonrival information network. Information decay is a form of imperfect communication such that the worth of each piece of information is assumed to decay as it traverses via each link. Within this literature, *a long-lasting open question — indeed a two-decade open question — is the characterization of Nash networks that are not minimal* [1]. In this note, I provide a partial answer to this long-lasting open question by: (i) providing a detailed characterization of nonminimal Nash networks and strict Nash networks for the class of nonminimal networks such that every chain between two agents has at most two links, given an assumption that the worth of information possessed by each agent is allowed to vary and (ii) providing an answer as to how these equilibrium networks relate to efficient networks, which have recently been characterized by Olaizola and Valenciano (2021).

The impacts of information decay on the characteristics of Nash and/or strict Nash networks was left as an open question by Bala and Goyal (2000), since the only finding is that a linked star, periphery-sponsored star and an empty network can be strict Nash networks (see Proposition 5.4 in Bala and Goyal (2000)). Recent development in the literature then solved this question by limiting the scope of characterization to the class of networks that are minimal. Specifically De Jaegher and Kamphorst (2015) did so by restricting the information decay to be sufficiently small, which accordingly implies that Nash networks contain no superfluous links and hence all Nash networks are minimal. In so doing, De Jaegher and Kamphorst (2015) successfully provide a detailed characterization of Nash networks. Charoensook (2020) then further generalizes this work of De Jaegher and Kamphorst (2015) to the case of agent heterogeneity in terms of informational value. Another recent work of Charoensook (2022) shows how minimal Nash networks and efficient networks are closely related to each other. However, in the case of nonminimal Nash networks very little is known. Indeed, to my knowledge, this note is the first work in the literature that attempts to provide a characterization of nonminimal Nash networks.

I summarize my findings as follows. Proposition 1 in this note provides a detailed characterization of Nash networks and Strict Nash networks for the class of nonminimal networks such that every chain has at most two links, assuming that value of information possessed by each agent is allowed

[1] Where the term minimal network here refers to a network such that every pair of agents is connected via at most one link or a series of links. See the formal definition in the next section.

to vary [2] [3]. Importantly, I find that the set of Strict Nash networks under these assumptions belongs to a class of networks called Strongly Nested Split Graph (SNSG), a term that has only recently been introduced to the literature of strategic network formation by Olaizola and Valenciano (2021) (see Remark 1 in this note). Interestingly, Olaizola and Valenciano (2021) also show that efficient networks are also SNSG (see Proposition 2 in this note). Since both efficient networks and strict Nash networks are SNSG, a natural question is whether strict Nash networks are also efficient. My Proposition 3 answers this question. I show that, within the set of all possible strict Nash networks, every network can be supported as an efficient network and concurrently strict Nash network by a specific structure of information value. Put differently, Nash networks and efficient networks are substantially compatible. I remark that this finding contrasts with most of the existing works literature, which finds that there is much tension between equilibrium networks and efficient networks [4].

This note proceeds as follows. The next section introduces the model and related notations. Subsequently, the main analysis section provides two propositions and a remark on strict Nash networks and efficient networks. Proposition 1 provides a necessary and sufficient condition for Nash network and strict Nash networks within the class of networks such that every path between two agents has at most two links and agent heterogeneity in information value is assumed. Proposition 3 provides then shows that every network whose properties are according to Proposition 1 can be supported as Nash and efficient networks by a value structure.

2. THE MODEL

Let $N = \{1, \ldots, n\}$ be the set of all agents and let i and j be typical members of this set. The information that each agent possesses is nonrival. If i and j are connected then i and j exchange their information, hence the term 'two-way flow'. i and j can be connected via either a direct link or a series of links. In what follows, most of the notations follow those of Bala Goyal (2000) and Billand et al. (2011).

Link establishment and individual's strategy. Link establishment is unilateral, which follows from the assumption that agents do not mind sharing their nonrival information. Link establishment cost is $c > 0$. A

[2]Note that this equilibrium characterization is for *any* level of information decay rather than just small amount of decay, unlike other existing works in the literature.

[3]This form of value heterogeneity is shown to enlarge the set of minimal SNN to a larger class of network called B_i and branching networks. See Billand et al. (2011), Charoensook (2020).

[4]In the words of Unlu (2018), "a central theme in the literature of network formation is the conflict between the set of stable networks and the set of efficient networks."

(pure) strategy of i is $g_i = \{g_{i,j} : j \in N, j \neq i\}$, where $g_{i,j} = 1$ if i establishes a link with j and $g_{i,j} = 0$ otherwise. That is, a strategy of i is a collection of his decision of whether to form a link with every other agent. If $g_{i,j} = 1$, it is said that i *accesses* j or j receives a link from i. A strategy profile is $g = \cup_{i \in N} g_i$.

Network representation. Visually, a node depicts an agent. An arrow from a node i to a node j if and only if $g_{i,j} = 1$. Thus, there is a one-to-one correspondence between a strategy profile and a network. Hence, the term network g and strategy profile g will be used interchangeably. If the head of each arrow is removed, then the resulting network merely represents the structure of information flow. Thisstructure of information flow is denoted by $\bar{g} = \{\bar{g}_{i,j} : i, j \in N, i \neq j\}$, where $\bar{g}_{i,j} = 1$ if $g_{i,j} = 1$ or $g_{j,i} = 1$ or both, and $\bar{g}_{i,j} = 0$ otherwise.

Information flow. Information flow is two-way in the sense that information flows from i to j whenever $\bar{g}_{i,j} = 1$. Alternatively the information flows between i and j via a chain, which is a series of links. Specifically, a chain is defined as a sequence $j_0, \ldots, j_m$ such that $\bar{g}_{j_l, j_{l+1}} = 1$ for $l = 0, \ldots, m-1$ and $j_0 = i$ and $j_m = j$. The length of a chain is defined as the number of links in the chain. If a chain between i and j exists then it is said that i *observes* j and vice versa. If i can observe j via more than one chain, the distance between i and j, denoted by $(i, j; g)$ is defined as the length of the shortest chain(s). The distance between two agents i and j, $d_{ij}(g)$, is defined as the length of the shortest chain through which i and j observes each other. If i and j do not observe each other we set, following the literature, $d_{ij}(g) = \infty$.

Value heterogeneity. Each agent $i \in N$ possesses nonrival information whose value is V_i. That is, the worth of information that any agent receives from i (including i himself), assuming that information transmission is perfect, is V_i. A value structure is then defined as $\mathcal{V} = \{V_i\}_{i \in N}$. Observe that, unlike most works in the literature, I do not require that $V_i = V_j$ for every $i \neq j$, hence the term 'value heterogeneity' [5].

Information decay. As information traverses through each link, it decays at the rate of $1 - \sigma$ where $\sigma < 1$. Hence, if the distance between i and j is k, then i receives $\sigma^k V_j$ and j receives $\sigma^k V_i$ [6].

Network-related notations. A network is *connected* if there is a chain between any distinct pair of agents. In a network g, let $N_i(g) \equiv \{j | \bar{g}_{ij} = 1\}$. A network g is said to be a *nested split graph* (NSG) network if $|N_i(g)| \leq$

[5]Readers should be aware of a terminological discrepancy. The term value heterogeneity here is equivalent to the term 'node heterogeneity' used in Olaizola and Valenciano (2021)

[6]This form of information here follows the convention in the literature including, for instances, Bala and Goyal (2000), De Jaegher and Kamphorst (2015), Galeotti et al. (2006) and Billand et al. (2010)

$|N_j(g)| \Rightarrow N_i(g) \subseteq N_j(g) \cup \{j\}$. A network is said to be a *strongly nested split graph* (SNSG) if it is NSG and $V_i > V_j \Rightarrow |N_i(g)| \geq |N_j(g)|$ [7].

The payoffs. The payoff below follows those in the literature that studies efficiency in two-way flow model of network formation:

$$U_i(g) = \sum_{j \in N, j \neq i} \sigma^{d_{ij}(g)} V_j - n_i^d(g)\, c \tag{1}$$

where $n_i^d = |N_d^i(g)|$ and $N_i^d(g) = \{j \in N | g_{i,j} = 1\}$.

Efficiency. Let $W(g) = \sum_{i=1}^{n} U_i(g)$. A network g *dominates* another network g' if $W(g) \geq W(g')$. A network g is efficient if it dominates every other network.

Nash networks. Consider a network g^* such that a strategy of i is $g_i^* \subset g^*$. Let $g_{-i}^* = g^* \backslash g_i^*$ so that $g^* = g_i^* \sqcup g_{-i}^*$. g_i^* is said to be a best response of i if $U_i(g^*) \geq U_i\left(g_i \cup g_{-i}^*\right)$ for every g_i which is a strategy of i. g^* is said to be a *Nash network* if every agent chooses his best response. A Nash network is said to be a *strict Nash network* (SNN), if a best response of every agent in it is unique.

2.1. Main Result 1: Equilibrium Characterization

The main objective is to characterize NNs and SNNs for the class of nonminimal networks such that every chain has at most two links, given that value heterogeneity is assumed. All proofs are relegated to the Appendix. I begin by introducing some notations and important facts. Let $\mathcal{G}^{NN}$ ($\mathcal{G}^{SNN}$) be the set of all nonminimal NNs (SNNs) whose every chain is at most 2 links.

Fact 1. If a network g is minimal and every chain in it has at most two links, then there exists an agent i that has precisely $n-1$ links

Fact 2. If $g \in \mathcal{G}^N N$ then there exist at least two agents, say i, j, such that $V_i, V_j \geq \frac{c}{\sigma - \sigma^2}$.

As a result of this fact, the equilibrium characterization below assumes that there are at least two agents whose values are above $\frac{c}{\sigma-\sigma^2}$.

Proposition 1 (Characterization of NNs and SNNs). *Let the set of agents N be partitioned into three types — H, M, L — and let N^H, N^M, N^L be the corresponding sets of agents so that $N = N^H \sqcup N^M \sqcup N^L$. Consider a network g with the properties (i), (ii), (iii) and (iv) described below.*

[7]The definitions of NSG and SNSG here follow those of Olaizola and Valenciano (2021). Olaizola and Valenciano (2021) were the first to establish the term SNSG, while the concept of NSG was first introduced into the literature on Economics by König et al. (2014). Its root is in applied mathematics, which has been well studied for several decades (see Mahadev and Peled (1995)). For a literature review, see Olaizola and Valenciano (2021).

(i) For every i, j such that $i \in N^L$ or N^M , $j \in N^H$ we have $g_{ij} = 1$. That is, agent of type H always receives a link from an agent of type L or M.

(ii) For every i, j such that $i, j \in N^H$ we have either $g_{ij} = 1$ or $g_{ji} = 1$ (but not both). That is, agents of type H always access each other (but never at the same time).

(iii) For every i, j such that $i \in N^L, N^M, N^H$ and $j \in N^L$, $g_{ij} = 0$. That is, agents of type L never receive a link.

(iv) For every i, j such that $i \in N^L, N^M, N^H$ and $j \in N^M$, either $g_{ij} = 0$ or $g_{ij} = 1$. That is, if $j \in N^M$ every other agent $i \neq j$ is always indifferent between accessing and not accessing j.

This network g can be supported as NN by a value structure $\mathcal{V}$ such that: (a) $V_j < \frac{c}{\sigma-\sigma^2}$ for every $j \in N^L$, (b) $V_j > \frac{c}{\sigma-\sigma^2}$ for every $j \in N^H$ and (c) $V_j = \frac{c}{\sigma-\sigma^2}$ for every $j \in N^M$. Conversely, if g is NN then $\mathcal{V}$ satisfies these inequalities (a), (b) and (c).

Moreover, if $N^M = \emptyset$ then any g with the above characterization is SNN instead of NN.

Figure 1 illustrates examples of NNs and SNNs according to Proposition 1. Observe that in this figure there is an NN that is not SNSG while all SNNs in are SNSG [8]. Are all SNNs SNSGs? Remark 2.1 answers this question.

Remark 2.1. If g is SNN then g is SNSG.

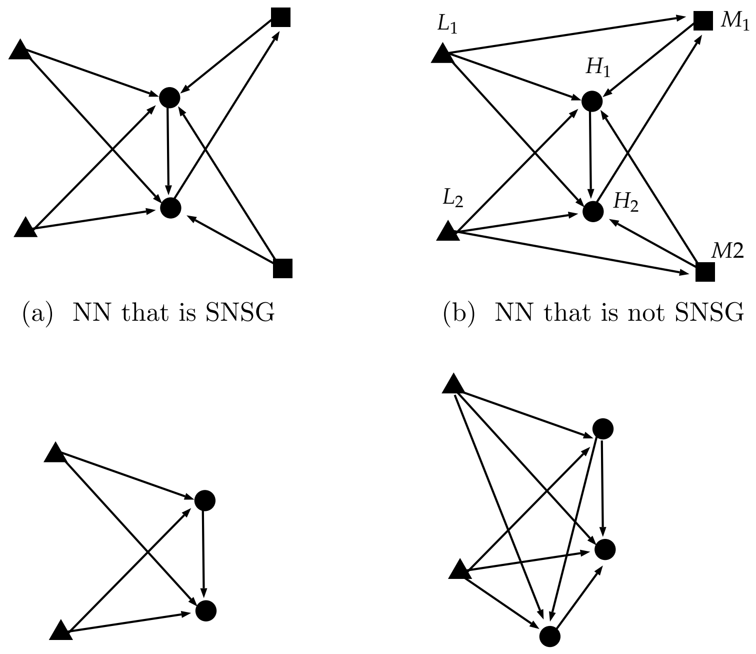
2.2. Main Result 2: Relating Equilibrium networks with efficient networks

Next, I relate my above results, the characterization of NNs and SNNs, with the characterization of efficient networks, which has been recently established in the literature by Olaizola and Valenciano (2021). I summarize my results as follows. First, in case of NN, there are some NNs that can never be efficient for any value structure. Second, on the contrary every network that belongs to the set of all SNNs can be supported by a value structure as SNN and concurrently efficient network. Before establishing these results, I recall the results of Olaizola and Valenciano (2021), which characterize efficient networks as follows.

PROPOSITION 2 (Olaizola and Valenciano (2021), Proposition 1, 2 and 4, p. 494). *Enumerate agents such that $V_1 \geq V_2 \geq, \dots, \geq V_n$. Any connected network*

[8]To clarify, observe that in Figure 1b $|N_{M_1}| = |N_{M_2}|$, $L_1 \in N_{M_1}$ but $L_1 \notin N_{M_2}$ while $L_2 \in N_{M_2}$ but $L_2 \notin N_{M_1}$, Hence, this network is not even NSG.

FIG. 1. Examples of NNs and SNNs. In these networks, circles, triangles and squares represent agents of type H, M and L respectively. Note that an arrow points away from a link sender and points towards a link receiver.



(a) NN that is SNSG

(b) NN that is not SNSG

(c) SNN that is SNSG

(d) SNN that is SNSG

with a positive net value is dominated by a connected strong nested split graph. A connected strong nested split graph network g is efficient if and only if:

$$\max_{(j,k)\in\mathcal{T}\&g_{ij}=0}(V_j+V_k)\overset{(a')}{\leq}\frac{c}{\sigma-\sigma^2}\overset{(b')}{\leq}\min_{(j,k)\in\mathcal{T},j\neq1\&g_{ij}=1}(V_j+V_k)$$

and, if $N_n(g)=1$

$$c\leq\sigma\left(V_1+V_n\right)+\sigma^2\sum_{k\in N\setminus\{1,n\}}\left(V_k+V_n\right)\tag{2}$$

Thus, there are some NNs that can never be efficient since they are not SNSG (recall the paragraph above Remark 2.1). On the contrary, it is intuitive to see that for the class of networks that are SNN every network can be supported as SNN and concurrently efficient network by a value structure. First, observe that the inequality (b') in the result of Olaizola and Valenciano (2021) above is closely related to inequality (b) in Proposition 1. Indeed, it is straightforward to observe that the inequality (b) in my Proposition 1 implies inequality (b') in the above proposition. On the other hand, the inequality (a') in the above proposition implies inequality (a) in my Proposition 1. This leads to the following proposition.

Proposition 3 (Compatibility between SNNs and efficient networks). *Any network with properties (i), (ii) and (iii) in Proposition 1 can be supported as a SNN and (concurrently) an efficient network by a value structure* $\mathcal{V}$*. That is, for any* $g \in \mathcal{G}^{SNN}$ *there exists* $\mathcal{V}$ *that guarantees that* g *is both SNN and efficient.*

3. CONCLUSION

In this note, I characterize NNs and SNNs for the class of nonminimal networks such that every chain has at most two links. The network formation model in this note, which is based on the two-way flow model with decay of Bala Goyal (2000), assumes heterogeneity in terms of value of information that each agent possesses. My results show that SNNs belong to the class of networks called SNSG which has been recently introduced to the literature by Olaizola and Valenciano (2021). I also relate my results to the characterization of efficient networks as in Olaizola and Valenciano (2021). My results show that SNNs are closely related to efficient networks in the sense that every SNN can at the same time be efficient.

As mentioned in the introduction section, this note is the first work in the literature that characterizes NNs and SNNs that are nonminimal, which is achieved by restricting the scope of characterization to the class of networks mentioned above. A remaining open question in this literature that has lasted for two decades since the publication of Bala Goyal (2000) is, therefore, a complete and general characterization of nonminimal Nash networks. It becomes the ambition of this author that this note serves a building block towards the answer of this question.

APPENDIX A

A.1. USEFUL LEMMA

LEMMA 1. *In a network g, let there be a two-link chain between i and j. Let i^* be the agent who is in the middle of this chain. Then: (i) $g_{ij} = 1$ is a unique best response of i if and only if $g_{ji} = 0$ and $V_j > \frac{c}{\sigma - \sigma^2}$, (ii) either $g_{ij} = 1$ or $g_{ij} = 0$ is a (non-unique) best response of i if and only if $g_{ji} = 0$ and $V_j = \frac{c}{\sigma - \sigma^2}$, (iii) and $g_{ij} = 0$ is a best response of i if and only if $g_{ji} = 0$ and $V_j < \frac{c}{\sigma - \sigma^2}$.*

Proof. [Proof of (i)] We first begin with the "if" part, because the chain between i and j with i^* in the middle has two links and we assume $g_{ji} = 0$, i receives σ^2 from j. If i further establishes a link with j, then i receives $\sigma V_j - c$. Thus if $\sigma V_j - c > \sigma^2 V$ (equivalently, $V_j > \frac{c}{\sigma - \sigma^2}$) then i strictly improves his payoff. Hence, i's unique best response is to establish the link with j.

Finally, to prove the "only-if" part let us assume that i's unique best response is to establish a link with j. This means that a removal of the link ij will reduce the payoff of i. By the same analogy as in the above paragraph we conclude that $V_j > \frac{c}{\sigma - \sigma^2}$ and $g_{ji} = 0$.

[Proofs of (ii) and (iii)] The proofs of (ii) and (iii) trivially follow the same analogy as that of (i). I leave the full proofs to my readers. ▮

A.2. PROOFS OF FACT 1 AND 2

Proof (Proof of Fact 1). Suppose not. Let i' be an agent that has the most links in g, which means that i' has at most $n - 2$ links. Let $j \neq i$ be an agent that i' does not have a link with. This means that the shortest chain between i and j has two links. Let x be the agent who is in the middle of this chain. That is, $\bar{g}_{i'x} = 1$ and the chain between i' and j is i', x, j. Consequently, for any agent k that has a link with i where $k \neq x$, there is a chain k, i', x, j. But this chain has three links. A contradiction. ▮

Proof (Proof of Fact 2). Suppose not. First, recall from Fact 1 that g has an agent i^* such that i^* has exactly $n - 1$ links. Because g is also nonminimal, we know that there is a pair of agents $i, j \neq i^*$ such that $\bar{g}_{ij} = 1$, which further necessitates that there is a two-link chain i, i^*, j. By Lemma 1, we know that either $V_j \geq \frac{c}{\sigma - \sigma^2}$ or $V_i \geq \frac{c}{\sigma - \sigma^2}$ but not both. Hence, without loss of generality let us assume that $V_j \geq \frac{c}{\sigma - \sigma^2}$ but $V_i < \frac{c}{\sigma - \sigma^2}$.

Next, consider the fact that $\bar{g}_{i^*j} = 1$ and another two-link chain i, j, i^* exists. Then by the same analogy as in the above paragraph we conclude that either $V_{i^*} \geq \frac{c}{\sigma-\sigma^2}$ or $V_i \geq \frac{c}{\sigma-\sigma^2}$. But because in the above paragraph we have already assumed that $V_j \geq \frac{c}{\sigma-\sigma^2}$ we conclude that there are at least two agents such whose values are above $\frac{c}{\sigma-\sigma^2}$. A contradiction. ▌

A.3. PROOF OF PROPOSITION 1, REMARK 1 AND PROPOSITION 3

Proof (Proof of Proposition 1).
[Sufficiency condition]
Proof of Property (i). Consider an agent $i \in N^L, N^M$ and $j \in N^H$. First observe that because $V_j > \frac{c}{\sigma-\sigma^2}$ it is true that $\sigma V_j > c$. Thus, establishing no link is not a best response of i. This guarantees that a best response of i is to establish at least one link with an agent $j \in N^H$. It remains to be proven, therefore, that i establishes a link with every agent of type H. To prove so, observe that in g_{-i} $\bar{g}_{kl} = 1$ for every $k, l \in H$. Next, recall that we have proven that a best response of i is to establish at least one with an agent $j \in N^H$. Thus, it follows that for every $k \neq j$, $k \in H$ there is a two-link chain i, j, k. If follows, by Lemma 1, that $g_{ik} = 1$ is a (unique) best response of i.

Proof of Property (ii). First trivially if $g_{ji} = 1$ then $g_{ij} = 0$ because the link $g_{ij} = 1$ becomes superfluous. Next, if $g_{ji} = 0$ then we need to prove that $g_{ij} = 1$. The proof here follows precisely the proof of property (i) above.

Proof of Property (iii). Consider an agent $i \in N$. Let $j \in N^L$. In case that $g_{ji} = 1$ then trivially $g_{ij} = 0$. Hence, onwards we assume that $g_{ji} = 0$. Next, recall from Property (i) that in g for every agent $k \in N^H, k \neq i$ it holds true that $\bar{g}_{ik} = 1$ and $g_{jk} = 1$ for every $j \in N^L$. Thus, there is a two-link chain i, k, j. The existence of this two-link chain and the assumption that $V_j < \frac{c}{\sigma-\sigma^2}$ allows us to use Lemma 1 to conclude that the (unique) best response of i is to establish no link with any agent $j \in N^L$.

Proof of Property (iv). Let i be that of type L or H or M. Observe that in g_{-i} there is a link between an agent of $j \in N^M$ and an agent $i' \in N^H$, where $j, i' \neq i$ (recall property (i) and (ii), which necessitates that an agent of type H has a link with every other agent in the network). Thus, there is a two-link chain between i and $j \in N^M$ with $i' \in N^H$ in the middle. Thus, if this $j \in M$ does not send a link to i, then by Lemma 1 i's (non-unique) best response is to either sending a link to k or send no link since $V_k = \frac{c}{\sigma-\sigma^2}$.

[Necessary condition] The proof trivially follows from the fact that Lemma 1, which is invoked to complete the above proof for the sufficiency part, are if and only if condition.

Finally, for the characterization of SNN observe that in the above proof a non-unique best response occurs only when $g_{ij} = 1$ or 0 and j is of type M. Thus, if $N^M = \emptyset$ then every best response in the above proof is unique. This completes our proof. ▮

Proof (Proof of Remark 1). First, recall from Proposition 1 that if g is SNN then $N_i(g) = N^L \sqcup N^H \backslash i$ for every $i \in N^H$ and $N_i = N^H \backslash i$ for every $i \in N^L$. Clearly $|N_i(g)| \leq |N_j(g)| \to N_i(g) \subseteq N_j(g) \cup \{j\}$ for every i, j such that $\bar{g}_{ij} = 1$, and $V_i > V_j \to |N_i(g)| \geq |N_j(g)|$. ▮

Proof (Proof of Proposition 3). First, let g be a network that satisfies properties (i), (ii) and (iii) in the Proposition 1 and let $\mathcal{V}$ be the value structure that supports g as a strict Nash network. Hence, we know that $\mathcal{V}$ satisfies inequalities (a) and (b) in Proposition 1. Now oberve that the inequality (b) in Proposition 1, which chracterizes SNN, is a sufficient condition for the inequality (b') in Proposition 2, which characterizes an efficient network. However, the inequality (a) in my Proposition 1 does not imply the inequality (a') in Proposition 2 . More specifically, for any $\bar{g}_{ij} = 0$ the inequality (a) in Proposition 1 requires that $V_i, V_j < \frac{c}{\sigma - \sigma^2}$. This does not imply that $V_i + V_j \leq \frac{c}{\sigma - \sigma^2}$, which is the property (a') in Proposition 2. Thus, for any such $V_i, V_j \in \mathcal{V}$ we simply replace them with V_i' and V_j', $V_i' < V_i$ and $V_j' < V_j$, sufficiently low that $V_i' + V_j' \leq \frac{c}{\sigma - \sigma^2}$. Thus, in $\mathcal{V}$ for any $\bar{g}_{ij} = 0$ by replacing V_i, V_j with V_i' and V_j' as described in the previous sentence the network g can be suppported as both strict Nash and efficient.

Finally, observe that Proposition 2 also requires that in an efficient network inequality 2 holds if $N_n(g) = \{1\}$. That is, the inequality 2 has to hold if agent n, who is the agent with the lowest value, has exactly one link with the agent 1 (who is the agent with the highest value). But this is never the case for any $g \in \mathcal{G}^{SNN}$ since, according to my Proposition 1, the agent n (who is either of type L or H) has at least two links g . ▮

REFERENCES

Bala, Venkatesh and Sanjeev Goyal, 2000. A Noncooperative Model of Network Formation. *Econometrica* **68(5)**, 1181–1229.

Billand, Pascal, Christophe Bravard, and Sudipta Sarangi, 2010. The insider-outsider model reexamined. *Games* **1(4)**, 422–437.

Billand, Pascal, Christophe Bravard, and Sudipta Sarangi, 2011. Strict Nash networks and partner heterogeneity. *International Journal of Game Theory* **40(3)**, 515–525.

Charoensook, Banchongsan, 2020. On the Interaction between Small Decay, Agent Heterogeneity and Diameter of Minimal Strict Nash Networks in Two-way Flow

Model. *Annals of Economics and Finance* **21**, 331–361.

Charoensook, Banchongsan, 2022. On Efficiency and Stability in Two-way Flow Network with Small Decay: A Note,. Sustainable Development Papers 52338, Fondazione Eni Enrico Mattei (FEEM).

De Jaegher, Kris and Jurjen Kamphorst, 2015. Minimal two-way flow networks with small decay. *Journal of Economic Behavior & Organization* **109**, 217–239.

Galeotti, Andrea, Sanjeev Goyal, and Jurjen Kamphorst, 2006. Network formation with heterogeneous players. *Game and Economic Behavior* **54 (2)**, 353–372.

Haller, Hans and Britta Hoyer, 2019. The common enemy effect under strategic network formation and disruption. *Journal of Economic Behavior & Organization* **162**, 146–163.

Hiller, Timo, 2017. Friends and enemies: A model of signed network formation. *Theoretical Economics* **12 (3)**, 1057–1087.

Jackson, Matthew O. and Asher Wolinsky, 1996. A Strategic Model of Social and Economic Networks. *Journal of Economic Theory* **71 (1)**, 44–74.

König, Michael D, Claudio J Tessone, and Yves Zenou, 2014. Nestedness in networks: A theoretical model and some applications. *Theoretical Economics* **9 (3)**, 695–752.

Mahadev, Nadimpalli VR and Uri N Peled, 1995. *Threshold graphs and related topics*. Elsevier.

Olaizola, Norma and Federico Valenciano, 2021. Efficiency and stability in the connections model with heterogeneous nodes. *Journal of Economic Behavior & Organization* **189**, 490–503.

Unlu, Emre. 2018. Efficient Networks in Models with Player and Partner Heterogeneity. *The Manchester School* **86 (1)**, 100–118.