ANNALS OF ECONOMICS AND FINANCE **25-2**, 423–461 (2024)

Housing Dynamics: Theory Behind Empirics*

Ping Wang and Danyang Xie†

We extend the recent macro housing literature by including *endogenous internal urban structure* to better fit with important stylized facts. We construct a two-sector optimal growth model of housing where consumable housing is produced by land and housing structures, where within-city locational choice is explicitly modeled. Housing services derive positive utility but are decayed away from the city center. Our model enables a full characterization of the dynamic paths of housing and housing and land prices. The model is calibrated to fit part of the stylized facts: faster growth of housing structures than housing, faster growth of land prices than housing prices, and downward housing price and land rent gradients within a city. The calibrated model can then be used to predict the remaining untargeted part of stylized facts: a locationally steeper land rent gradient than the housing price gradient, relatively flatter housing quantity and price gradients in larger cities with flatter population gradients and moderate rise in the housing expenditure share. The calibrated model can further yield additional insights on housing dynamics and spatial distribution. The main punchline is: *nonhomotheticities* in housing preference and housing production are crucial for realistic model predictions particularly to account for internal urban features.

Key Words: Housing dynamics; Locational choice; Within-City spatial distribution.

JEL Classification Numbers: D90, E20, O41, R13.

* We are grateful for valuable comments and suggestions from Marcus Berliant, Michele Boldrin, Morris Davis, Jonathan Heathcote, Charles Leung, Zheng Liu, Françis Ortalo-Magné, Stephen Malpezzi, Erwan Quintin, Pengfei Wang, and Yi Wen, as well as participants at the Chicago Fed-Wisconsin Conference on Macro Housing, the Regional Science Association International and the Urban Economic Association Meetings, and the Taipei International Conference on Growth Dynamics. Financial support from Academia Sinica, the Hong Kong University of Science and Technology, Research Grants Council of the Hong Kong SAR, the Weidenbaum Center on the Economy, Government, and Public Policy, and Wuhan University to enable this international collaboration is gratefully acknowledged. Needless to say, the usual disclaimer applies.

† Wang: Washington University in St. Louis and NBER; Xie: Thrust of Innivation, Policy, and Entrepreneurship, Society Hub, Hong Kong University of Science and Technology (Guangzhou). Email: danyang.xie@gmail.com.

1529-7373/2024

1. INTRODUCTION

The housing sector is very significant in size. While the value of the American housing stock accounts for more than 30% of national wealth, the housing-related expenditure is about one-fourth of the total household spending. Moreover, housing activity can generate large macroeconomic effects.[1] Yet, not until the turn of the century, has the macro housing literature become trendy.[2] While the literature has provided many insights toward understanding the macroeconomic implications of this significant housing sector, its models are abstracting any linkage to internal urban structure and within-city locational choice.[3]

What we intend to do is to deliver a methodological paper on macro housing that takes explicit account of internal urban structure. The purpose is to allow for a thorough analysis of within-city spatial distribution over time, based on a dynamic spatial equilibrium framework. This attempt requires a careful modeling of housing, particularly in some aspects that bridge time and space. Specifically, we will incorporate several crucial ingredients into our framework so as to capture a set of five well documented stylized facts, both over time and across within-city locations, based on the U.S. observations:[4]

- (Stylized Fact 1) Housing structures inclusive of construction materials and household durables out-grow the housing stocks.
- (Stylized Fact 2) Housing prices grow at much lower rates than land rents.
- (Stylized Fact 3) By putting aside urban ghettos,

  (a) both housing price and land rent gradients are downward-sloping away from urban centers (or subcenters)

  (b) the land rent gradient is much steeper.

- (Stylized Fact 4) In larger MSAs with flatter population gradients, both housing quantity and price gradients are flatter.

[1]For example, Case, Quigley and Shiller (2005) find rather large effects of housing wealth on household consumption using a panel of 14 developed countries over the period of 1975-1999 and a panel of U.S. states over the period of 1982-1999

[2]See Leung (2004) for a critical survey and the more recent literature summarized below. In particular, Leung stresses: "Conventional housing economics and urban economics research for its part virtually ignores interactions between and among housing markets and the macroeconomy."

[3]See the pioneer work by Fujita (1989) and the more recent internal urban structure literature summarized below.

[4]These facts are discussed in varous studies in the internal urban structure literature and the macro housing literature, to be reviewed below. We will verify them using the average U.S. data over 1960-2000 before the hikes of housing bubbles and the subsequent subprime.mortgage crisis.

• (Stylized Fact 5) The housing expenditure ratio rises, but only moderately, over time.

We believe that this model, specifically calibrated to fit all these facts *jointly*, would serve as a good basis for future research on related issues where housing is an integral part of the analysis.

To capture these stylized facts, we construct a two-sector optimal growth model with a composite final good sector and a housing sector. The composite final good can be used for consumption or for capital investment. In addition to composite good consumption, housing services also enters the utility function, with two special features. First, we allow housing to have a different income elasticity, dictated by a *nonhomothetic preference structure*, than the composite good consumption, and let the data spell out the difference. Second, we allow housing services to be decayed away from the city center to capture spatial discounting as observed in the market. On the supply side, housing is produced by land and housing structures. Similarly, we also allow for a *nonhomothetic housing production technology* that capture the possibility that there might be a minimum structure required for a house or structure might not be necessary, which is yet again to be determined by calibration. Nonhomotheticities are important features to include in the macro literature, particularly for studying sectoral shift and structural transformation. Such features can play a role in explaining the above-mentioned stylized facts.

Both housing structures and the composite good are produced with the use of physical capital. In equilibrium, both goods and land market clear (no vacant land) and no household has incentive to relocate (locational no-arbitrage). We begin by solving the social planner's problem in a tractable manner and then decentralize it by finding supporting prices with location-dependent redistributions (housing taxes/subsidies and redistribution of nonhousing wealth). Upon obtaining the steady-state competitive spatial equilibrium, we derive a basket of analytical comparative statics and then calibrate the theoretical model to fit the average U.S. data during the pre-housing bubbles and mortgage crisis era (1960-2000) to further quantify our analysis.

The main analytic findings of our paper are summarized as follows. First, an increase in the housing production technology or in the supply of land raises housing quantity but reduces the relative price of housing. Second, if housing is more luxury than the composite consumption good, which is shown to be the case by calibration, an increase in the consumption good production technology lowers the cost of producing the consumption good and enables reallocation of resources to housing production, thus raising both the quantity and the relative price of housing.

The rich structure of the model enables us to calibrate it to fit Stylized Facts 1 and 2, spatial discounting (Stylized Fact 3(a)), and spatial distribution. Once these moments are targeted, the calibrated model can deliver additional results that are consistent with not only the *untargeted Stylized Facts 3(b)-5* but also some other interesting outcomes as follows. First, a set of comparative statics regarding the housing related quantities and prices fit the observed spatial patterns. For example, housing exhibits much higher cross-location variations than consumption and housing structures schedules; and, a larger MSA with a flatter population gradient is found to have the quantity of housing rising less rapidly away from the CBD and housing and land prices declining less rapidly away from the CBD. Second, along a dynamic path with accumulation of capital and housing structures, the prices of housing structures exhibit a slight downward trend over time, corroborating with findings in the home production literature. Moreover, the housing expenditure ratio exhibits a moderate increase initially and remains largely unchanged afterward, which is again consistent with empirical findings. Finally, as a by-product of our numerical exercises, the computed wealth share of housing, including housing structures, is in line with empirical findings as well.

An important takeaway of this paper is that the nonhomothetic specifications in the preferences and in the housing production are both essential for realistic model predictions, particularly to account for internal urban features. With homothetic preferences, our robustness analysis finds spatial distributions of various housing related quantities and prices to be inconsistent with the observations. Similarly, with homothetic housing production function, the responses to demand and supply shifts turn out to be quantitatively too large to be realistic.

Related Literature

For the non-macro based urban economic literature within the static framework, we would not discuss any detail but simply refer the reader to the survey by Leung (2004).

There is a literature on internal urban structure by Berliant, Peng and Wang (2002), Lucas and Rossi-Hansberg (2002), Lin, Mai and Wang (2004), Rossi-Hansberg and Wright (2007), and Xie (2008), but they do not fully model the *urban housing market*, as most commonly either housing is not explicitly modeled or simply modeled as interchangeably as land, but a thorough dynamic model of the housing activity is the focus of our paper. There is also a recent but more remotely related literature on spatial sorting, led by Desmet and Rossi-Hansberg (2014) and generalized by Eckert and Peters (2022). In this strand, the focus is on spatial distribution *across* cities and across sectors. Our paper instead focuses on spatial distribution *within* a city.

Almost two decades ago, Davis and Heathcote (2005 and 2007) offer a macro housing framework to advance better understanding of urban housing over business cycles. Since their pivotal contributions, there is a growing literature on macro housing. Among many, we would like to refer to some recent papers by Kydland, Rupert and Sustek (2016), Garriga, Manuelli and Peralta-Alva (2019), Garriga and Hedlund (2020) and Garriga, Hedlund, Tang and Wang (2023). For additional work, the reader is also referred to those cited in two comprehensive survey papers, Davis and Van Nieuwerburgh (2015) and Piazzesi and Schneider (2016). This new macro housing literature highlights housing distribution based on individual heterogeneities in incomes and preferences. In contrast, our paper studies housing distribution across space within a city. In this way, our paper serves as a bridge between the internal urban structure literature and the new macro housing literature.

2. THE MODEL

Let the city (or MSA) be situated in a segment of real line, $[-1, 1]$, with location 0 representing the central business district (CBD).[5] Let the land supply be distributed along the real line according to an exogenous density function $\bar{T}(z)$, for $z \in [-1, 1]$, where z indexes a location. We assume $\bar{T}'(z) > 0$ to capture the fact that land is more abundant away from the city center. Moreover, we assume that the land supply at $z = 0$ is positive ($\bar{T}(0) > 0$).

For convenience, the population of agents is assumed constant over time with mass two. Further assume that each agent supplies labor inelastically at $\frac{1}{2}$. Thus, the aggregate labor supply in the economy is one. We will focus on a symmetric equilibrium in which locational choice yields a negative exponential distribution of households over $[-1, 1]$. More specifically,

$$N(z) = \frac{\omega e^{-\omega|z|}}{1 - e^{-\omega}},$$

which is widely supported by empirical evidence (see the original work by Clark 1951 and a comprehensive survey by McDonald 1989). Technically, this avoids the indeterminacy issue regarding the with-city distribution with endogenous labor and capital as shown by Berliant, Peng and Wang (2002). This formulation is also useful in economic analysis: by changing

[5]While continuum setup of the internal urban structure is commonly assumed in the monocentric city literature, it becomes unmanageable when studying dynamics, To fully derive transitional dynamics, we resort to a one-location case that continues to carry over all other important features considered in the general framework.

ω, we can analyze various city-economies such as Chicago, New York and Philadelphia to be studied below.

Our spatial economy has two theaters of production activities: one produces a composite final good and another accumulates housing structures. Production of both of these mobile goods take place at the CBD to which workers commute. For the sake of simplicity, we are abstracting from housing tenure choice, housing finance and speculative demand for housing, which are not central to fitting the stylized facts listed above.

2.1. The Housing Sector

Housing of a representative household at location z is specified as:

$$H_z = T_z^{\gamma} \left(D_z - \theta\right)^{1-\gamma} \tag{1}$$

where T_z is the use of land and D_z is the structure component of the house that includes mobile construction materials and household durables and is referred to as housing structure for brevity. In addition to the share parameter $\gamma \in (0,1)$, we introduce a nonhomothetic component θ to allow for the possibility that a minimum structure ($\theta > 0$) might be needed for producing reasonable quantitative results or the possibility that housing structures are nonnecessities for housing production ($\theta < 0$).While the former case seems to be more natural, we allow the data to confirm whether θ is positive or negative, to be consistent with observed equilibrium outcomes. In either case, the input shares for housing construction are *time-varying*.

The Cobb-Douglas form ensures that land and housing structures are Pareto complement in the sense that an increase in one input raises the marginal product of another. In equilibrium, land demand equals supply at each location z,

$$T_z N(z) = \bar{T}(z).$$

The output of housing structure investment at location z is produced with the use of physical capital:

$$X_z = BK_z^{\beta}$$

where $\frac{\dot{B}}{B} = G(t)$ with $G(t) > 0$, $G' < 0$ and $\lim_{t\to\infty} G(t) = 0$ for any z. Abstracting the labor input from the production of housing structure investment is innocuous, as housing structure investment is more capital intensive relative to the composite final good. Although one may easily allow labor to enter this production process while maintaining the factor intensity ranking, labor allocation across locations $z \in [-1,1]$ would lead to unnecessary complication in the analysis.

Housing structures evolves according to,

$$\dot{D}_z = X_z - \delta D_z = BK_z^{\beta} - \delta D_z \tag{2}$$

where $\delta > 0$ denotes the demolishment rate of housing structures and $D_z(0) = d \geq \theta$ for any z.

2.2. The Composite Final Good Sector

The composite final goods sector features the following Cobb-Douglas production function:

$$Y = AK_c^{\alpha} L^{1-\alpha} \tag{3}$$

where labor, L, is inelastically supplied at one and $A > 0$ measures the total factor productivity (TFP) for producing the consumption goods relative to housing construction (where the latter is normalized to one).

Denote $\delta_k > 0$ as the capital depreciation rate. The output of the composite final goods can then be used for consumption (c_z for those residing in z) or capital investment ($\dot{K} + \delta_k K$), implying:

$$\dot{K} = AK_c^{\alpha} L^{1-\alpha} - \int_{-1}^{1} c_z N(z) dz - \delta_k K, \tag{4}$$

which governs the evolution of capital over time.

The total stock of capital, K, can be allocated as follows:

$$K = K_c + K_d = K_c + \int_{-1}^{1} K_z N(z) dz \tag{5}$$

where K is equally owned by all the agents and K_d is the aggregate capital stock allocated to the housing sector.

2.3. Preferences

The lifetime utility function of an individual residing at location z is specified as:

$$U_z = \int_0^{\infty} u(c_z, \phi(z) H_z) e^{-\rho t} dt \tag{6}$$

where $\rho > 0$ is the subjective rate of time preference and $\phi(z)$ is a spatial discounting function capturing the idea that the further away the house is from the CBD, the lower the utility one derives from the house. Part of the reduction in utility may be thought of capturing the detrimental effect from commuting. With spatial discounting, it is not necessary to consider a separate resource cost of commuting, which we assume. Without loss of generality, we normalize $\phi(0) = 1$. These assumptions are standard in the lead model of urban economics: the monocentric city model (cf. Fujita 1989).

The point-in-time utility function takes the following form:

$$u(c_z, \phi(z)H_z) = c_z^{\sigma} \left(\phi(z)H_z + \eta\right)^{1-\sigma}, \ \sigma \in (0,1) \tag{7}$$

where nonhomotheticity is introduced via parameter η to allow for a different income elasticity of housing than the composite consumption good. For the case with $\eta > 0$, housing is said to be more "luxurious" in the subsequent discussion than the composite good. On the contrary, Moreover, the Cobb-Douglas form ensures that composite good consumption and housing service ($\phi(z)H_z$) are Pareto complement. When $\eta > 0$, housing is a necessity. Again, we allow the data to confirm whether housing is luxurious relative to the composite consumption goods or a necessity. Should housing be kept at a minimum livable level, it is likely to be a necessity; with extra space and expensive household durables in a picturesque landscape, housing can become luxurious.

2.4. Locational Choice

Given the ex ante symmetry between all agents, it has to be the case that in equilibrium, $u(c_z, \phi(z)H_z)$ is independent of z. In other words, the following locational no-arbitrage condition holds:

$$u(c_z, \phi(z)H_z) = u(c_0, H_0) \tag{8}$$

Thus, in equilibrium, individual agents feel indifferent in residing in any location.

3. EQUILIBRIUM ANALYSIS

In this section, we solve the optimization problem and then derive the steady-state equilibrium. We begin by solving a pseudo planner's problem instead of solving the competitive equilibrium directly. We then identify a necessary redistribution scheme to support the decentralization of the optimal allocation obtained from the pseudo planner's problem. Solving the pseudo planner's problem first and then deriving the efficient equilibrium by obtaining the supporting redistributions is both theoretically and computationally simpler. While simplifying the problem, such a pseudo planner problem is so designed to yield the same equilibrium outcomes.

3.1. Optimization

For convenience, we define:

$$\Psi_z(D_0, D_z) \equiv \frac{T_0^\gamma (D_0 - \theta)^{1-\gamma} + \eta}{\phi(z) T_z^\gamma (D_z - \theta)^{1-\gamma} + \eta}$$

which is increasing in D_0 but decreasing in D_z, satisfying $\Psi_0(D_0, D_0) = 1$. We can then simplify the planner's problem by utilizing (7) and (8) to express the locational no-arbitrage condition in forms of final good consumption:

$$c_z = c_0 \Psi_z (D_0, D_z)^{\frac{1-\sigma}{\sigma}} \tag{9}$$

That is, Ψ_z governs relative composite good consumption across locations.

Using (9), we can write the central planner's problem as:

$$\max \int_0^\infty c_0^\sigma \left(T_0^\gamma (D_0 - \theta)^{1-\gamma} + \eta \right)^{1-\sigma} e^{-\rho t} dt$$

subject to

$$\dot{K} = A \left(K - \int_{-1}^{1} K_z N(z) dz \right)^\alpha L^{1-\alpha} - \int_{-1}^{1} c_0 \Psi_z (D_0, D_z)^{\frac{1-\sigma}{\sigma}} N(z) dz - \delta_k K \tag{10}$$

$$\dot{D}_z = B K_z^\beta - \delta D_z \text{ for all } z \tag{11}$$

This optimization problem can be solved by setting the current-value Hamiltonian,

$$\begin{aligned} \mathcal{H} \;=\; & \max_{c_0, K_z} c_0^\sigma \left(T_0^\gamma (D_0 - \theta)^{1-\gamma} + \eta \right)^{1-\sigma} \\ & + \lambda \left[A \left(K - \int_{-1}^{1} K_z N(z) dz \right)^\alpha L^{1-\alpha} - \int_{-1}^{1} c_0 \Psi_z (D_0, D_z)^{\frac{1-\sigma}{\sigma}} N(z) dz - \delta_k K \right] \\ & + \int_{-1}^{1} \mu_z \left[B K_z^\beta - \delta D_z \right] dz \end{aligned}$$

where λ and μ_z are co-state variables.

We next define:

$$\Gamma = \int_{-1}^{1} \Psi_z (D_0, D_z)^{\frac{1-\sigma}{\sigma}} N(z) dz \tag{12}$$

which is indeed the endogenous social welfare weight on those residing at location 0.[6] The first-order conditions with respect to c_0 and K_z are:

$$\sigma c_0^{\sigma-1}\left(T_0^{\gamma}\left(D_0-\theta\right)^{1-\gamma}+\eta\right)^{1-\sigma}=\lambda\Gamma \tag{13}$$

$$\beta\mu_z BK_z^{\beta-1}=\alpha\lambda A\left(K-\int_{-1}^{1}K_zN(z)dz\right)^{\alpha-1}L^{1-\alpha}N(z) \tag{14}$$

While (13) equates the marginal benefit from raising location-0 resident's consumption and the marginal cost from reducing others' consumption, (14) equates the value of marginal product of capital between the two sectors. From (14), we have:

$$K_z=\left(\frac{\mu_zN(0)}{\mu_0N(z)}\right)^{1/(1-\beta)}K_0 \tag{15}$$

That is, the ratio of capital allocated to the housing sector between two locations depends positively on the ratio of the shadow value of housing structures. When the shadow value of housing structures is relatively high at a particular location, it encourages more housing structure investment at that location, thus creating more induced demand for capital input into the production of housing structure investment.

The Euler equations with respect to K and D_z are given by,

$$\begin{aligned}\dot{\lambda} &= (\rho+\delta_k)\lambda-\alpha\lambda A\left(K-\int_{-1}^{1}K_zN(z)dz\right)^{\alpha-1}L^{1-\alpha}\\ \dot{\mu}_z &= (\rho+\delta)\mu_z-\lambda\left[(1-\gamma)\frac{1-\sigma}{\sigma}c_0\Pi_z\left(D_z\right)\Psi_z\left(D_0,D_z\right)^{\frac{1-\sigma}{\sigma}}N(z)\right]\end{aligned}$$

where $\Pi_z\left(D_z\right)\equiv\frac{1}{D_z-\theta}\frac{\phi(z)T_z^{\gamma}(D_z-\theta)^{1-\gamma}}{\phi(z)T_z^{\gamma}(D_z-\theta)^{1-\gamma}+\eta}$ is decreasing in D_z. By rewriting these above expressions using the first-order conditions, (13) and (14), we obtain:

$$\frac{\dot{\lambda}}{\lambda}=(\rho+\delta_k)-\alpha A\left(K-\int_{-1}^{1}K_zN(z)dz\right)^{\alpha-1}L^{1-\alpha} \tag{16}$$

[6] This can be easily verified by maximizing the social welfare function given by $\int_{-1}^{1}\Omega_z u(c_z,\phi(z)H_z)dz$, subject to (2) and (4). Applying Negishi (1960), we can compute the social welfare weights consistent with the decentralized equilibrium allocation, yielding: $\Omega_0=\Gamma$.

$$\frac{\dot{\mu}_z}{\mu_z} = (\rho+\delta) - \frac{\beta B K_z^{\beta-1}}{\alpha A \left(K - \int_{-1}^{1} K_z N(z) dz\right)^{\alpha-1} L^{1-\alpha}} (1-\gamma) \frac{1-\sigma}{\sigma} c_0 \Pi_z (D_z) \Psi_z (D_0, D_z)^{\frac{1-\sigma}{\sigma}} \tag{17}$$

The above two expressions govern the shadow price of capital and housing structures, respectively. Intuitively, the shadow price of capital decreases with the relative TFP for goods production whereas the shadow price of housing structures rises with the relative TFP but decreases with housing structure investment efficacy.

3.2. Decentralization

We are now ready to find competitive support to the planner's solution under an appropriate redistribution scheme.

The relative price of housing can be defined as the ratio of the shadow price of housing structures to the shadow price of capital: $P_{D_z} = \frac{\mu_z}{\lambda}$. This lead to an intertemporal no-arbitrage condition:

$$\begin{aligned} \frac{\dot{P}_{D_z}}{P_{D_z}} &= \alpha A \left(K - \int_{-1}^{1} K_z N(z) dz\right)^{\alpha-1} L^{1-\alpha} \\ &- \left[\frac{\beta B K_z^{\beta-1} (1-\gamma) \frac{1-\sigma}{\sigma} c_0 \Pi_z (D_z) \Psi_z (D_0, D_z)^{\frac{1-\sigma}{\sigma}}}{\alpha A \left(K - \int_{-1}^{1} K_z N(z) dz\right)^{\alpha-1} L^{1-\alpha}} - \delta \right] \end{aligned} \tag{18}$$

That is, if the net return on capital (first term on the right hand side) exceeds the net return on housing structures, then there must be a capital gain associated with housing durables ($\frac{\dot{P}_{D_z}}{P_{D_z}} > 0$) in order for both sectors to remain operative (see Bond, Wang and Yip 1996). Moreover, since $\Pi_z (D_z)$ and $\Psi_z (D_0, D_z)$ are both decreasing in D_z, it is clear that the rate of capital gain associated with housing structures at a particular location rises with the stock of housing structures but falls with the stock of capital at that location.

From our model, the rental price housing must be equal to the marginal rate of substitution between housing and the composite good,

$$R_{H_z} = \frac{1-\sigma}{\sigma} \frac{\phi(z) c_z}{\phi(z) H_z + \eta}$$

We can then define the price of housing as:

$$P_{H_z} = \frac{R_H}{\rho} = \frac{1}{\rho} \frac{1-\sigma}{\sigma} \frac{\phi(z) c_z}{\phi(z) H_z + \eta} \tag{19}$$

That is, housing price is the capitalization of housing rental. From the specification of housing, the rental price of housing structures is simply its value marginal product given by,

$$R_{D_z} = \frac{(1-\gamma)\, R_{H_z} H_z}{D_z - \theta}$$

which yields a useful relationship governing the prices of housing structures and housing,

$$R_D D = (1-\gamma)\, \frac{D}{D-\theta} R_H H$$

The land rent can then be defined based on the bid rent concept,

$$R_{T_z} = \frac{R_{H_z} H_z - R_{D_z} D_z}{T_z}$$

That is, the land rent is the unit surplus of housing rental in excess of housing structure cost.

We claim that these are location-specific supporting prices to the allocation derived from the central planner problem under an appropriate redistribution scheme. Specifically, consider a distribution of the ownership, ν_z, of capital stock, K, together with a housing tax τ_z (subsidy if negative). Let w denote the wage rate and r denote the capital rental rate, which equal the respective marginal products: $w = (1-\alpha)\, AK_c^{\alpha}$ and $r = \alpha AK_c^{\alpha-1}$. Each agent's wealth is measured by,

$$\Omega_z = \nu_z K + P_{H_z} H_z$$

which is the sum of the value of capital and the value of housing per individual. The individual wealth evolves according to,

$$\dot{\Omega}_z = \frac{1}{2} w + (r - \delta_k)\nu_z K - c_z - rK_z - \tau_z P_{H_z} H_z$$

which is equal to wage income (recall that individual labor supply is $\frac{1}{2}$) plus net capital income subtracting consumption expenditure, capital user cost paid for producing housing structure investment and housing tax payment. To satisfy locational no-arbitrage, it must be that $\Omega_z = \Omega_0$ and $\dot{\Omega}_z = \dot{\Omega}_0$ for all z. Using these together with the two redistribution constraints, $\int_{-1}^{1} \tau_z P_{H_z} H_z dz = 0$ and $\int_{-1}^{1} \nu_z N(z) dz = 1$, we can then solve the redistribution pair (τ_z, ν_z) for each location z. This verifies our claim.

3.3. Steady-State Equilibrium

From (16), (12), as well as (10) and (11), we obtain the following three steady-state relationships:

$$K_c = K - \int_{-1}^{1} K_z N(z) dz = \left(\frac{\alpha A}{\rho + \delta_k}\right)^{\frac{1}{1-\alpha}} \tag{20}$$

$$K_z = \left(\frac{\delta D_z}{B}\right)^{\frac{1}{\beta}} \tag{21}$$

$$K = \left(\frac{\alpha A}{\rho + \delta_k}\right)^{\frac{1}{1-\alpha}} + \int_{-1}^{1} \left(\frac{\delta D_z}{B}\right)^{\frac{1}{\beta}} N(z)dz \tag{22}$$

$$c_0 = \frac{A\left(\frac{\alpha A}{\rho+\delta_k}\right)^{\frac{\alpha}{1-\alpha}} - \delta_k\left[\left(\frac{\alpha A}{\rho+\delta_k}\right)^{\frac{1}{1-\alpha}} + \int_{-1}^{1}\left(\frac{\delta D_z}{B}\right)^{\frac{1}{\beta}} N(z)dz\right]}{\int_{-1}^{1} \Psi_z\left(D_0, D_z\right)^{\frac{1-\sigma}{\sigma}} N(z)dz} \tag{23}$$

Clearly, a higher composite good technology or a lower time preference rate raises consumption as well as capital allocated to the composite good sector. Moreover, a higher demolishment rate requires more capital to be allocated to the housing sector to maintain the need for housing services.

The above equations can then be combined with (17) to yield,

$$\frac{\beta B}{\rho + \delta_k}\left(\frac{\delta D_z}{B}\right)^{\frac{\beta-1}{\beta}}\left[(1-\gamma)\frac{1-\sigma}{\sigma}c_0\Pi_z\left(D_z\right)\Psi_z\left(D_0, D_z\right)^{\frac{1-\sigma}{\sigma}}\right] = \rho + \delta \tag{24}$$

Notice that, at $z = 0$, (24) reduces to an expression for solving uniquely $D_0(c_0)$ which turns out to be an increasing function. This can then be substituted into (24) to derive all housing structures $D_z(c_0)$, which are all increasing in c_0 as well. Next, substituting $D_z(c_0)$ into (23) yields a fixed point mapping in c_0. Once the fixed point of c_0 is obtained, it can then be plugged into $D_z(c_0)$ to solve for D_z for all z, and then into (21), (22) and (9) to solve for K_z, K and c_z. Using (1) and (3), we obtain the steady-state value of housing and the composite good output, H and Y. Finally, we can solve all the supporting prices. In particular, the steady-state capital rental rate is: $r = \rho + \delta_k$. One may also compute the price of housing structures as:

$$P_{D_z} = \frac{\mu}{\lambda} = \frac{\rho + \delta_k}{\beta B K_z^{\beta-1}}$$

It can then be verified that in the steady state the housing structure price satisfies $R_{D_z} = (\rho + \delta) P_{D_z}$. Recall that the housing price satisfies $R_H = \rho P_H$. Thus, the capitalization of housing structures and housing differs by the demolishment factor δ. Since both ρ and δ are constant over time and across locations, we can examine the dynamic and spatial patterns of housing and housing structure prices by using their corresponding rental price measures (R_{H_z} and R_{D_z}), which are in comparable units to the land rent.

It may be noted that the involvement of c_0 in all the location-specific variables makes the steady-state equilibrium too complicated to be characterized analytically. In particular, all the preference and technology parameters of interest, $(A, B, \eta, \rho, \theta, T)$, will affect the fixed point of c_0 ambiguously due to their opposing effects on $\Psi_z(D_0, D_z)$ via $D_0(c_0)$ and $D_z(c_0)$. Thus, we will instead perform comparative-static exercises only under the baseline one-location setup, while conducting the equilibrium characterization of the general model only numerically.

3.4. Characterization of the Steady-State Equilibrium

In order to perform comparative statics in the baseline one-location case, we utilize the "hat calculus" that has been frequently adopted by general equilibrium trade theorists. Denoting $\hat{X} = \frac{\dot{X}}{X}$, we can totally differentiate the key relationships in the baseline one-location setup and manipulate the expressions to derive the fundamental equation governing the changes in the housing quantity in response to changes model parameters $(A, B, \eta, \rho, \theta, T)$:

$$\hat{H} = \xi_A \hat{A} + \xi_B \hat{B} + \xi_\theta \hat{\theta} + \xi_\eta \hat{\eta} + \xi_T \hat{T} + \xi_\rho \hat{\rho}, \tag{25}$$

where the elasticities ξ_i, $i = A, B, \eta, \rho, \theta, T$, can be found in Appendix A.1. Similarly, we can then obtain the fundamental equation governing the changes in the housing price in response to changes in $(A, B, \eta, \rho, \theta, T)$:

$$\hat{P}_H = \varepsilon_A \hat{A} + \varepsilon_B \hat{B} + \varepsilon_\theta \hat{\theta} + \varepsilon_\eta \hat{\eta} + \varepsilon_T \hat{T} + \varepsilon_\rho \hat{\rho}, \tag{26}$$

where the elasticities ε_i, $i = A, B, \eta, \rho, \theta, T$, are also reported in Appendix A.1.

Based on these two fundamental equations, we can summarize the comparative static results in the following table:

	A	B	η	ρ	θ	T
Housing Quantity (H)	$+$	$+$	$-$	$-$	$-$	$+$
Housing Price (P_H)	$+$	$-^*$	$-$	?	$+^*$	$-$

Note: * if δ_k small

Among these six parameters, B, θ, and T can be characterized as affecting the supply side, η the demand side, A both the demand and supply side (to be elaborated below), and ρ the intertemporal choice.

Intuitively, an increase in the housing production technology (B) lowers the cost of producing housing, thus raising housing quantity but reducing housing price. The responses of housing quantity and price to an increase in the supply of land are similar. We next examine what happens to an increase in the minimum structure requirement for housing (higher θ). Since such a requirement raises the cost of producing a house, housing price rises while housing supply decreases in response. In response to an increase in the luxury good nature of housing relative to the consumption good (higher η), individual preferences shift away from housing and as a result both housing quantity and housing price are lower. Notably, while an increase in B or T or a decrease in θ capture a prototypical outward shift in housing supply, a decrease in η indicate a prototypical outward shift in housing demand.

Turning now to time discounting (ρ), we can see that more impatience discourages allocation of resources for the future. Since housing requires continual inflows to maintain its adequate service, it falls in response to an increase in time discounting. While such a reduction in housing production tends to raise housing price, the resulting increase in the real interest rate tends to lower housing price. The net effect of impatience on housing price is therefore ambiguous. Notice that in partial equilibrium setups adopted by conventional housing models, rising time discounting would reduce housing price unambiguously.

Finally, an increase in the consumption good production technology (A), in addition to a positive wealth effect (demand effect), lowers the cost of producing the consumption good and increases the relative price of housing. As a consequence, it enables reallocation of resources to housing production and raises the quantity of housing (supply effect). Such an effect only arises in multi-sectoral setups within the general equilibrium framework.

It is noted that equations (25) and (26) are useful not only for deriving comparative statics but also for numerically decomposing changes in the quantity and the price of housing once we have calibrated the model economy, to which we now turn.

4. QUANTITATIVE ANALYSIS

We now calibrate the model to fit with the average U.S. data over 1960-2000 in annual frequency, where we end the sample before the hikes of housing bubbles and the subsequent subprime mortgage crisis, to avoid misfitting to data along an off-equilibrium path. We then use the calibrated model to perform various numerical analyses. Additionally, we check the robustness of our main quantitative findings using a gammaville.

4.1. Calibration

Under our theoretical framework, the total population is two. Denote c as the per capita flow of non-housing related consumption good, D as the per capita stock of housing structures inclusive of household durables, X as the per capita output of the housing structure sector and H as housing per capita (all without the location subscript z). We specify the land supply as a simple quadratic function: $\bar{T}(z) = (b + q\,|z|)^2$, where b measures the land supply at the CBD and $q > 0$ reflects increasing land supply away from the CBD. We further specify the spatial discounting function in a linear form given by: $\phi(z) = 1 - a\,|z|$, where a measures the locational discount rate. We normalize $b = 1$ so that the amount of land at the CBD is $\bar{T}(0) = 1$. We then select $a = 0.3$ and $q = 0.1$, under which those at city border discount housing consumption by 30% compared to a resident at the CBD and land supply at city border is 21% more than at the CBD. In computing aggregate variables, the per capita land supply is set as: $T = \int_0^1 (1 + 0.1z)^2 dz = 1.1033$. In the benchmark case, we use Chicago configuration where the negative exponential distribution parameter is given by $\omega = 0.3$ using the estimate in McDonald (1989).

In the macroeconomics literature, the time preference rate is taken to be between 2% and 5%; we thus set $\rho = 0.035$. Also in compliance with the literature, we choose the capital income share as one-third (implying $\alpha = 1/3$). We set the rate of capital depreciation, $\delta_k = 5\%$, a number widely used in the literature. The overall depreciation of housing structures considered herein includes both demolishment of housing structures and depreciation of household durables. While Greenwood and Hercowitz (1991) uses 7.8% as the depreciation rate for the household durables and equipment, Davis and Heathcote (2005) computes the housing demolishment rate as 1.57%. It is reasonable to assume that the latter accounts for 75% of the overall depreciation, which yields $\delta = 0.0313$.

The calibration analysis is conducted using a simpler version of the model in which there is one location, namely all households are situated in location $z = 0$. By choosing units, we normalize one of the two technological scaling factors by setting $A = 1$. Let $\zeta = \rho D/c$ measure the housing structure flow to non-housing consumption ratio. The capital share of housing sector is denoted by s_K. Further denote the capital-output ratio in the housing durable sector as $\chi = K_d/(2X)$, where $2X$ measures the aggregate output of housing structures. In the steady state, $X = \delta D$, which implies: $K_d = 2\delta\chi\zeta c/\rho$. In the home production literature (e.g. Benhabib, Rogerson and Wright 1991; Greenwood and Hercowitz 1991), the housing consumption flow is regarded as large as non-housing consumption; our ρD is only part of the housing consumption flow, we thus set $\zeta = 0.5$. Since the economy-wide capital-output ratio in the U.S. usually falls in the range from 2 to 3, we set $\chi = 2.25$ as the benchmark. Based on our steady-steady relationships, we can then obtain:

$$
\begin{aligned}
K_c &= \left(\frac{\alpha A}{\rho+\delta_k}\right)^{\frac{1}{1-\alpha}} = 7.7659 \\
c &= \frac{1}{2}AK_c^{\alpha} - \delta_k K = \frac{AK_c^{\alpha} - \delta_k K_c}{2\left(1+\frac{\delta_k \delta \chi \zeta}{\rho}\right)} = 0.7579
\end{aligned}
$$

Subsequently, the capital stock devoted to the housing structure sector, the housing capital share and the steady-state value of housing structures can be computed as:

$$
\begin{aligned}
K_d &= \frac{2\delta\chi\zeta c}{\rho} = 1.5250 \\
s_K &= \frac{K_d}{K_d + K_c} = 0.1641 \\
D &= \frac{\zeta c}{\rho} = 10.8268
\end{aligned}
$$

That is, about 16.5% of the aggregate capital stock is allocated to producing housing structures.

Over the four decades between 1960 and 2000, we can use the average data to compute housing growth rate at 1.8% ($g_H = 0.018$), the housing structure growth rate at 2.4% ($g_D = 0.024$), the housing structure price growth rate at 0.68% ($g_{R_D} = 0.0068$) and the land price growth rate at 4.33% ($g_{R_T} = 0.0433$). These are in line with the comparable figures in Davis and Heathcote (2007). Moreover, the average land value to housing value share is about 36% ($s_T = 0.36$). Using non-durable consumption as a proxy, we compute the non-housing consumption good growth rate as 3% ($g_c = 0.03$). Furthermore, the average expenditure share of housing is about 24% ($s_H = 0.24$), consistent with Davis and Ortalo-Magné (2008).

These ratios and growth rates can then be used to calibrate key parameters in our model. Recall that, from our model, $R_H = \frac{1-\sigma}{\sigma}\frac{c}{H+\eta}$, $R_D = \frac{(1-\gamma)R_H H}{D-\theta}$ and $R_T = \frac{R_H H - R_D D}{T}$. Assuming fixed land supply over time, we totally differentiate the above three price relationships around the steady state to obtain:

$$
\begin{aligned}
\hat{R}_D &= \hat{R}_H + \hat{H} - \frac{D}{D-\theta}\hat{D} && (27) \\
\hat{R}_H &= \hat{c} - \frac{H}{H+\eta}\hat{H} && (28) \\
\hat{R}_T &= \frac{R_H H}{R_H H - R_D D}\left(\hat{R}_H + \hat{H}\right) - \frac{R_D D}{R_H H - R_D D}\left(\hat{R}_D + \hat{D}\right)
\end{aligned}
$$

Denote the land value to housing value share as: $s_T = \frac{R_T T}{R_H H}$. Straightforward manipulations lead to,

$$\hat{R}_T = \left(\hat{R}_H + \hat{H}\right) + \frac{(1-\gamma)\frac{D}{D-\theta}\frac{\theta}{D-\theta}}{1-(1-\gamma)\frac{D}{D-\theta}}\hat{D} \tag{29}$$

$$s_T = \frac{R_T T}{R_H H} = 1-(1-\gamma)\frac{D}{D-\theta} \tag{30}$$

Let the rates of changes of all price and quantity variables capture their respective transitional growth rates, $(g_{R_D}, g_{R_H}, g_{R_T}, g_D, g_H, g_c)$.[7] From (27) and (28), we have:

$$\frac{\theta}{D} = 1 - \frac{g_D}{g_H + g_{R_H} - g_{R_D}} \tag{31}$$

$$\frac{\eta}{H} = \frac{g_H}{g_c - g_{R_H}} - 1 \tag{32}$$

We utilize (30) to write $(1-\gamma)\frac{D}{D-\theta} = 1 - s_T$, which, together with (29) and (31), gives:

$$g_{R_H} = s_T g_{R_T} + (1-s_T)(g_{R_D} + g_D) - g_H = 0.0173$$

We can now use (30) and (31) to compute:

$$\theta = \left(1 - \frac{g_D}{g_H + g_{R_H} - g_{R_D}}\right) D = 1.7095$$

$$\gamma = 1 - \frac{1-s_T}{\frac{D}{D-\theta}} = 0.4611$$

Thus, our results indicate that the *minimum structure requirement for housing* is present in the data, which is about one-sixth of the amount of housing structures. Applying the functional form of housing given by $H = T^{\gamma}(D-\theta)^{1-\gamma} = 3.4436$ and the land supply schedule, we can then utilize (32) to calibrate:

$$\eta = \left(\frac{g_H}{g_c - g_{R_H}} - 1\right) H = 1.4371$$

[7] These transitional changes are consequences of transitional changes in $G(t)$. We do not model these changes as permanent because we must otherwise construct specific unbalanced endogenous growth models which often require adding a third sector with two of the three sectors growing at different rates but balancing each other in aggregation (see Kongsamut, Rebelo and Xie 2001, Bond, Trask and Wang 2003 and Acemoglu and Guerrieri 2008). Adding such a sector would make the analysis more difficult without generating further insight over our simple optimal growth structure.

which confirms that *housing as a whole is more luxurious than the composite good.*

Finally, from the first-order condition governing consumption and housing demand, we have:

$$s_H = \frac{R_H H}{c + R_H H} = \frac{1}{1 + \frac{\sigma}{1-\sigma}\frac{H+\eta}{H}}$$

which yields,

$$\sigma = \frac{\left(\frac{1}{s_H} - 1\right)\frac{H}{H+\eta}}{1 + \left(\frac{1}{s_H} - 1\right)\frac{H}{H+\eta}} = 0.6908$$

Furthermore, from the steady-state relationship $B(K_d/2)^\beta = \delta D$, we can write:

$$B = \frac{\delta D}{(K_d/2)^\beta}$$

Substituting this expression into another steady-state relationship,

$$\frac{\beta B}{\rho + \delta_k}\left(\frac{\delta D}{B}\right)^{\frac{\beta-1}{\beta}}(1-\gamma)\frac{1-\sigma}{\sigma}c\frac{1}{D-\theta}\frac{H}{H+\eta} = \rho + \delta,$$

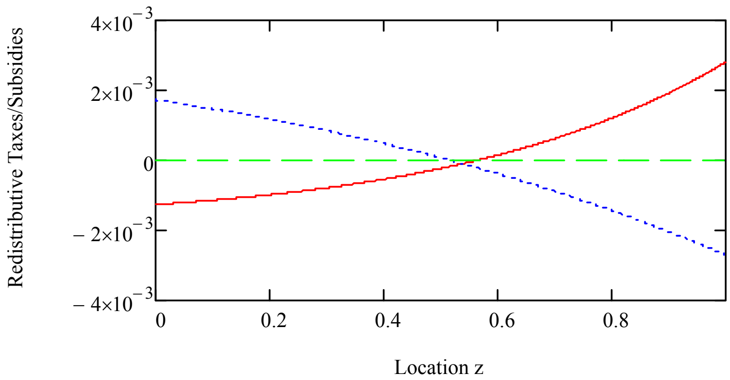
leads to a single equation in β. This gives the calibrated value $\beta = 0.8963$, which can be plugged back into the previous expression to calibrate $B = 0.4321$.

This completes the calibration process through which we have targeted Stylized Facts 1 and 2 and 3(a) (spatial discounting), as well as spatial distribution. We are ready to obtain additional model predictions in the next subsection that can lend support to untargeted Stylized Facts 3(b) and 4. We shall relegate Stylized Fact 5 to Section 5 after characterizing transitional dynamics in a simplified one-location setup.

4.2. Numerical Results

We begin by identifying the redistribution scheme (τ_z, ν_z) that is required for equilibrium support. In our benchmark case, such a scheme features imposing taxes on those in inner city $[-0.517, 0.517]$ and providing subsidies to those in outskirts $[-1, -0.517] \cup [0.517, 1]$. The redistributive tax/subsidy schedules over the right half of the city, $[0, 1]$, are plotted in Figure 1 (dashed line). Intuitively, the consideration of locational discounting $\phi(z)$ can be thought of regarding the CBD as a public good whose services decay with distance. Thus, one would expect that those enjoying more of such public good services (in the inner city) would be taxed. Similarly, those who reside in inner city $[-0.563, 0.563]$ would be allocated a share of capital stock lower than average whereas those who in ourskirts $[-1, -0.563] \cup [0.563, 1]$ a share of capital stock higher than average (solid line). More specifically, the tax rate at the center is 0.17% and the subsidy at the fringe is 0.27%. Those at the center holds 49.87% of capital stock per capita and those at the fringe holds 50.28% of capital stock per capita; all very close to the average of 50%. As a by-product of this decentralization exercise, we can compute the wealth share of housing as 58.33%. Based on the 2000 Census, such a share without including housing structures is 32.3%. Since our calculation includes household durables, it is viewed as reasonably consistent with the data.

FIG. 1. Small Redistributive Measures Needed for Decentralization



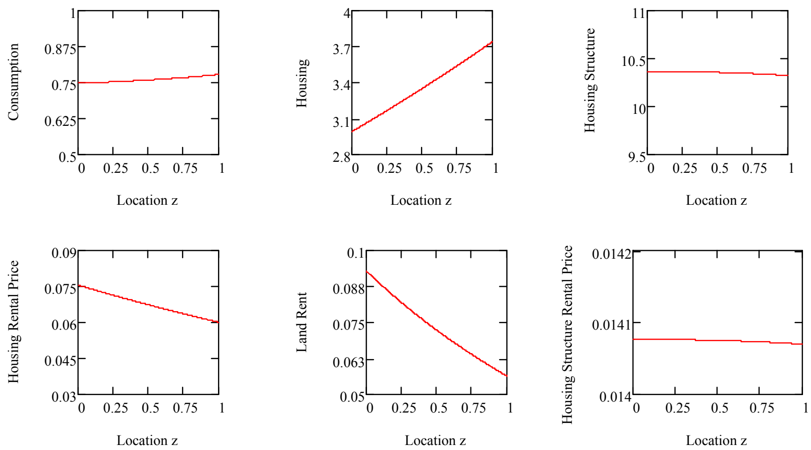
Using calibrated parameter values, we can further compute 3 quantity and 3 price ratios across locations in the city, plus 3 aggregate shares/ratios, the housing expenditure share (s_H), the housing capital share (s_K) and the ratio of aggregate housing structures to housing (D/H). The results are reported below:

$\frac{c_1}{c_0}$	$\frac{H_1}{H_0}$	$\frac{D_1}{D_0}$	$\frac{R_{H_1}}{R_{H_0}}$	$\frac{R_{T_1}}{R_{T_0}}$	$\frac{R_{D_1}}{R_{D_0}}$	s_H	s_K	$\frac{D}{H}$
1.0402	1.2503	0.9956	0.7952	0.6077	0.9995	0.24	0.1641	3.1441

Thus, the quantity of housing at the city fringe is about 25% more than at the CBD (the amount of land is by construction 21% more). While the land rent is about 39% lower, the housing price is only about 20% less at the border compared to the center. In Figure 2, we plot the schedule of each endogenous quantity or price over the right half of the city, $[0, 1]$. As one can see clearly, while housing schedule shows significant cross-location variations, consumption and housing structure schedules are rather flat.

Moreover, the land rent schedule is much steeper than the housing rental price schedule, that is, *untargeted Stylized Fact 3(b)*. By contrast, the housing structure rental price schedule is essentially flat. Intuitively, land is entirely immobile while construction materials and household durables are mobile. It is expected that the greater the degree of mobility is, the less the cross-location variation will be, thereby explaining our results.

FIG. 2. Housing and Land Rent Most Sensitive to Location



We can also compute the housing quantity and price elasticities with respect to various parameter changes, reported in the table below:

	A	B	η	ρ	θ	T
Housing Quantity (H)	0.6213	0.6405	-0.1536	-0.4654	-0.0188	0.5319
Housing Price (P_H)	0.6439	-0.4520	-0.1592	-0.5426	-0.0012	-0.3877

This table coincides well with our theoretical predictions in Section 3 except the housing price elasticity of θ, the parameter of minimum housing structure. This is because with our calibrated δ_k, an increase in θ raises the need for housing structure D, which in turn raises the demand for K_d, and reduces consumption (higher δ_k implies a more significant reduction), with the tendency of lowering housing price: $P_H = (1-\sigma)c/(\sigma\rho(H+\eta))$.

We next turn to conducting comparative-static exercises quantitatively. We are particularly interested in the responses of the above cross-location ratios and the three aggregate shares/ratios to a 10% increase in each of four key preference and technology parameters, η, θ, a and B. Such responses in percentage are reported as follows.

%	$\frac{c_1}{c_0}$	$\frac{H_1}{H_0}$	$\frac{D_1}{D_0}$	$\frac{R_{H_1}}{R_{H_0}}$	$\frac{R_{T_1}}{R_{T_0}}$	$\frac{R_{D_1}}{R_{D_0}}$	s_H	s_K	$\frac{D}{H}$
η	−0.08	−0.26	−0.39	−0.26	−0.67	−0.05	−3.06	−2.67	−1.02
θ	0.00	−0.01	0.00	0.00	−0.03	0.00	−0.04	1.33	1.57
a	1.31	−0.18	−0.27	−0.18	−0.44	−0.03	−0.57	−0.51	−0.19
B	0.05	0.13	0.20	0.14	0.36	0.02	1.57	0.19	3.57

Thus, when housing becomes more luxurious (higher η), the out-skirt to inner city ratios of consumption, the quantity of housing and housing structures, and the rental prices of land, housing and housing structures are all lower. Intuitively, when housing becomes less necessary, housing demand must fall. In terms of the production of housing, the derived demand for housing structures will also fall, though normally by not as much.[8] Our quantitative results suggest that while housing expenditure and housing capital shares fall sharply, the ratio of aggregate housing structures to housing falls. Among all the cross-location ratios, housing, housing structures, housing rental prices and land rents are more responsive.

An increase in the minimum housing structure requirement (higher θ) has little influence on any of the cross-location ratios (with many of such changes less than 0.005%). In response to this increased minimum requirement, it is necessary to allocate more capital to housing capital to produce the required housing structures (i.e., the housing capital share must increase). As a result, both housing structure prices and housing prices rise, while the land rent falls. The former changes discourage housing demand, thereby lowering the housing expenditure share and raising the housing structures to housing ratio. Our quantitative results suggest that while the housing expenditure share drops negligibly, both the housing capital share and the aggregate housing structures to housing ratio rise sharply.

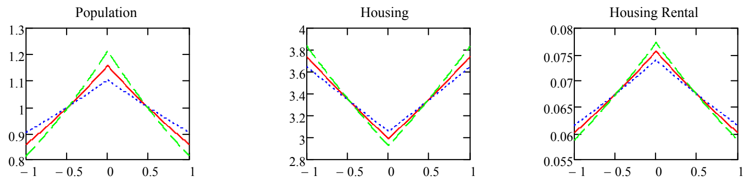
Except for the effect on the cross-location consumption ratio, the change in spatial discounting generates qualitatively identical effects to the change in the luxury good nature of housing. Intuitively, in response to higher spatial discounting (higher a in the spatial discounting function, $\phi(z)$), agents are less willing to reside at outskirts, thereby reducing housing demand and housing structures demand as well as their prices and the land rent in the outer city. That is, both the ratios of housing and housing structures at the fringe to the center must fall. Our quantitative results suggest that the economy-wide housing structures to housing ratio decreases marginally. It is interesting to note that almost all the cross-location ratios (except housing durable prices) are most responsive to this spatial discounting perturbation.

[8] In trade theory, the finding that changes in output are larger than changes in inputs is usually referred to as the magnification effect in quantity.

Concerning an increase in the housing durable technology (higher B), all the responses are exactly reverse to an increase in the luxury good nature of housing. Such reversed effects are not surprising as one may view the luxury good nature of housing as a barrier to housing development, thereby having opposite impact to the productivity of housing structures. Because housing structure productivity has a direct positive impact on housing structures, it tends to increase the aggregate housing structures to housing ratio. Our quantitative results show a sharp rise in both the housing expenditure share and the aggregate housing structures to housing ratio in response to an increase in the housing structure technology.

It is noted that in response to any of these parameter changes, land rents are always much more responsive than other rental prices, while housing is relatively less responsive than housing structures.

FIG. 3. Chicago-solid, NYC-dash, Philly-long dash



Finally, we shift our attention to city configurations. Based on the estimates provided by McDonald (1989), we have used the case of Chicago as the benchmark where the negative exponential distribution parameter is $\omega = 0.3$. We now consider two alternative configurations: New York with a flatter population gradient ($\omega = 0.2$) and Philadelphia with a steeper population gradient ($\omega = 0.4$). For comparison purposes, we normalize both cases with population equal to two and landscape over the same unit interval $[-1, 1]$. The results of the key gradients are reported below and illustrated in Figure 3:

City	ω	$\frac{c_1}{c_0}$	$\frac{H_1}{H_0}$	$\frac{D_1}{D_0}$	$\frac{R_{H_1}}{R_{H_0}}$	$\frac{R_{T_1}}{R_{T_0}}$	$\frac{R_{D_1}}{R_{D_0}}$
New York	0.2	1.0548	1.1927	0.9940	0.8318	0.6698	0.9993
Chicago	0.3	1.0402	1.2503	0.9956	0.7952	0.6077	0.9995
Philadelphia	0.4	1.0258	1.3106	0.9972	0.7602	0.5514	0.9997

While both the quantities and prices of mobile goods do not alter much, those of immobile goods vary substantially. In a larger MSA like New York where the population gradient is flatter compared to a smaller MSA like Philadelphia, the housing quantity gradient as well as housing and land price gradients are all flatter, with land prices much more responsive than housing prices.[9] Thus, a larger MSA with a flatter population gradient will have the quantity of housing rising less rapidly away from the CBD and housing and land prices declining less rapidly away from the CBD, conforming with *untargeted Stylized Fact 4*.

5. TRANSITIONAL DYNAMICS

To enable us to confirm *untargeted Stylized Fact 5*, we must turn to examining the property of housing related quantities and prices along a dynamic equilibrium path. Because migration dynamics along the transition is not the focus of the present paper, we can circumvent the complexity associated with characterizing transitional dynamics with a continuum of locations by looking at only the aggregate measures. As such, we shall move to a simpler version of the model in which there is one location, with all households residing in location $z = 0$. Moreover, we can also afford to assume away the variability of housing productivity by setting B constant ($G(t) \equiv 0$), as the variability is mainly needed in the calibration exercise above.

The dynamics can be captured by the following equations (see derivation in Appendix A.2):

$$\dot{K} = A(K - F(K, \lambda, \mu))^{\alpha} - 2C(\lambda, D) - \delta_k K \tag{33}$$

$$\dot{\lambda} = (\rho + \delta_k)\lambda - \alpha\lambda A\,(K - F(K, \lambda, \mu))^{\alpha-1} \tag{34}$$

$$\dot{D} = B\,(F(K, \lambda, \mu)/2)^{\beta} - \delta D \tag{35}$$

$$\dot{\mu} = (\rho + \delta)\mu - (1 - \gamma)\,\frac{1 - \sigma}{\sigma}\,\frac{2\lambda C(\lambda, D)}{D - \theta}\,\frac{T^{\gamma}\,(D - \theta)^{1-\gamma}}{T^{\gamma}\,(D - \theta)^{1-\gamma} + \eta} \tag{36}$$

where

$$C(\lambda, D) = \left(T^{\gamma}\,(D - \theta)^{1-\gamma} + \eta\right)\left(\frac{2\lambda}{\sigma}\right)^{1/(\sigma-1)}$$

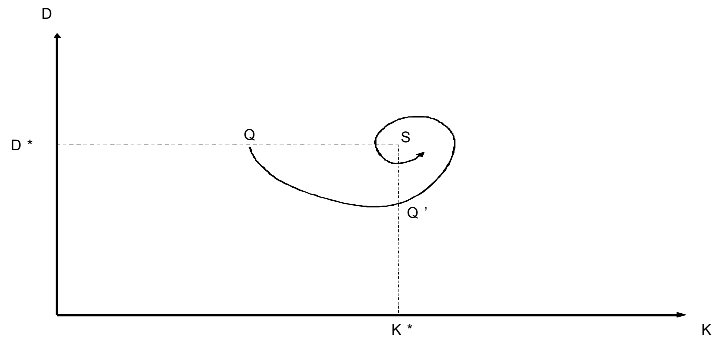
and $K_d = F(K, \lambda, \mu)$ solves

$$K_d = 2\left(\frac{\beta\mu B}{2\alpha\lambda A}\right)^{\frac{1}{1-\beta}}(K - K_d)^{\frac{1-\alpha}{1-\beta}}$$

[9]Due to our normalization of population and city boundaries, the reader is advised not to pay attention to the absolute level but the gradient of these variables depicted in Figure 3. Should New York be allowed to have 4 times as populated as Philadelphia and twice as big in areas, its population density would be uniformly higher than Philadelphia.

While $C(\lambda, D)$ is decreasing in λ and increasing in D, $F(K, \lambda, \mu)$ is decreasing in λ and increasing K and μ. The computation of the steady state values of K, λ, D, and μ can also be found in Appendix A.2.

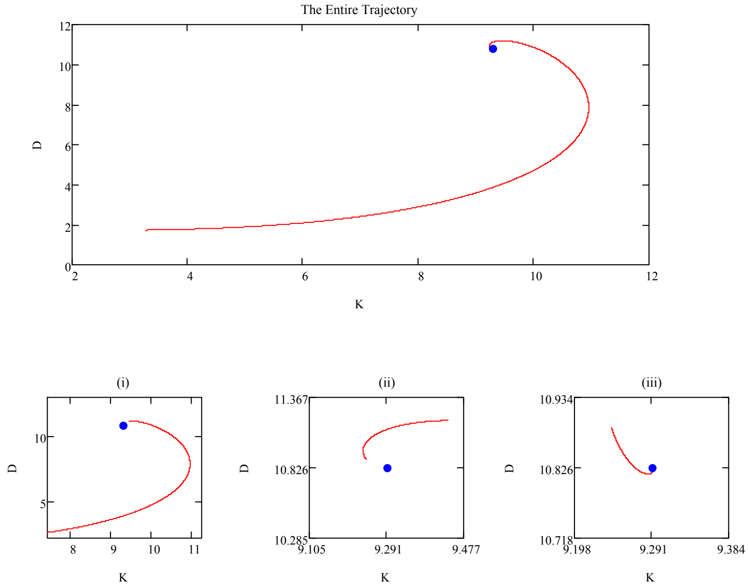
FIG. 4. Oscillation Near the Steady State



Based on our calibrated economy, we can apply backward shooting method to this one-location setup to examine the transitional dynamics. Our numerical computations suggest that as the trajectory approaches the steady state, it oscillates in the space of (K, D). The intuition for oscillation can be illustrated using Figure 4 (a close-up near the steady state). Starting at point Q, $D = D^*$ but $K < K^*$, hence it is intuitive that a large fraction of capital would be allocated to the goods sector, implying $K_D < K_D^*$. As a result, $\dot{D} < 0$ at point Q. Since at point Q, the wealth of the representative agent is below that at the steady state, we must have $C_Q < C^*$ and the consumption is small enough to allow for capital accumulation, namely $\dot{K} > 0$ (see equation (33)). Hence, the trajectory from point Q is south-east. At point Q', $K = K^*$ but $D < D^*$, hence it is intuitive that a large fraction of capital would be allocated to housing structure production, namely, $K_D > K_D^*$, which implies that $\dot{D} > 0$ (see equation (35)). Although this means that $K_C < K_C^*$, but $C_{Q'}$ remains below C^*, making it possible for $\dot{K}$ to remain positive.

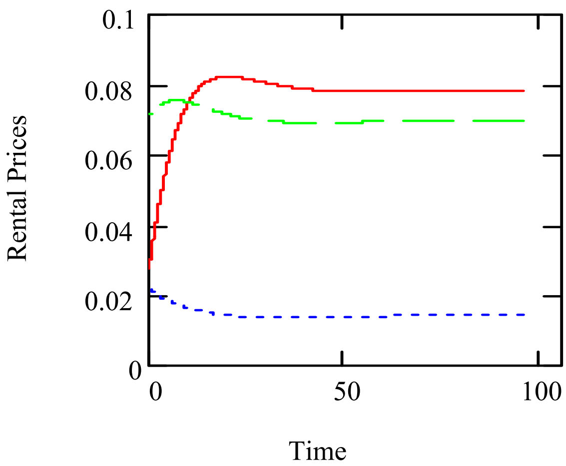
Of our particular interest, we can identify a transition path along which both K and D increase monotonically until they are close to the steady state (see Figure 5). Specifically, starting from $(K_0, D_0) = (3.2705, 1.7317)$, both K and D increase toward the steady state. As they approach the steady state (indicated by the big dot), an oscillation occurs as depicted in the three graphs in the lower panel of Figure 5: (i) K overshoots and then starts to fall while D continues to rise, (ii) both fall, and (iii) K then rises while D continues to fall. A repetition of such an oscillation continues until the steady state is reached (the close-up figure is not shown as it has already been illustrated in Figure 4). This path is mimicking the transition

FIG. 5. Equilibrium Trajectory



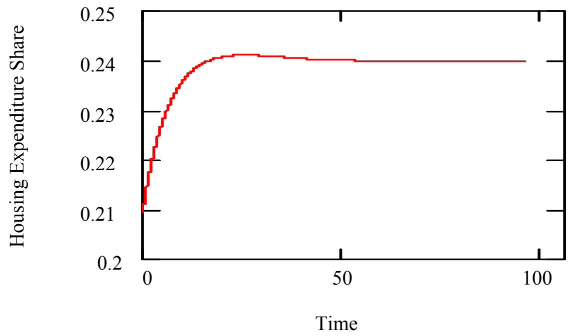
dynamics in an economy continuing to evolve by accumulating more capital and housing structures.

FIG. 6. Rental Prices of Housing, Land (solid), and Durables (dash)



In addition to capital and housing structures, it is crucial to understand the transitional dynamics of the rental prices of housing, land and housing structures. One can clearly see from Figure 6 that along the transition, land rents (solid line) grow much more sharply (from 0.027 to 0.08) than housing rental prices (long-dashed line, which rises initially from 0.071 from 0.076 and then falls back to 0.07), whereas the rental price schedule of housing durables (short-dashed line) exhibits slight decline over time (from 0.022 to 0.014). This latter finding is consistent with the home production literature, where cheaper household durables enable house wives to substitute out their time for participating in market activities.

FIG. 7. Housing Expenditure Share



Finally, we note that the presence of the luxury good nature of housing results in changes in the housing expenditure ratio over time. In our calibrated economy, this ratio increases moderately from 20.9% to 24% over the first 25 years and remain largely unchanged afterward (see Figure 7). The moderate increase in the housing expenditure ratio thus confirms *untargeted Stylized Fact 5*. Quantitatively, the magnitude of the 3.1% rise in the ratio is in line with the evidence in the U.S. For example, Rogers (1988) documents that the ratio increased by 2.7% in urban areas from 1972/73 to 1985, whereas Davis and Martin (2008) finds that the ratio increased by 2.3% from 1975 to 1982 and then becomes relatively stable through 2007.

6. ALTERNATIVE PARAMETRIZATION AND MODEL SPECIFICATION

In this section, we will perform sensitivity analysis with regard to some parameter selections that are not entirely based on observations. We will also provide further discussion concerning particularly some key ingredients of our model specification by conducting counterfactual analysis.

6.1. Sensitivity Analysis: Alternative Parametrization

In our calibration analysis, two parameter selections are not entirely based on observations: one is the ratio of housing structures to consumption (ζ, set as 0.5) and another is the housing-sector capital-output ratio (χ, set as 2.25). To check the robustness of our results, we change ζ up and down by 10% from its benchmark value (0.5) and χ from 2 to 2.5 (reasonable range used in the literature when calibrating the model to fit the U.S. data). We find that our main results are robust to all such changes. More specifically, both the dynamic patterns and the cross-locational patterns of our key variables are essentially unchanged. As reported in Appendix A.3, the only noticeable changes are the economy-wide capital share and housing structures to housing ratio in the steady state. Such changes are expected. When the model is calibrated with a higher housing structure to consumption ratio, both the housing capital share and the housing structure to housing ratio must rise. When the model is calibrated with a higher housing-sector capital-output ratio, the housing capital share must increase.

Our calibrated economy features increasing land supply away from the CBD where the relative supply at the fringe is about 21% more than at the center. In reality, such relative land supplies vary across different MSAs. We thus perform sensitivity analysis with respect to the land expansion rate away from the CBD (q in the land supply schedule, $T(z)$), changing it to 0.25 and 0.35 (deviating from its benchmark value of 0.30). We find that the dynamic patterns of our key variables are largely unchanged. In response to a steeper land expansion rate, all of the aggregate variables are essentially unchanged. Concerning the cross-locational patterns of our key variables, the most noticeable changes are steeper housing schedule and flatter housing price and land rent gradients away from the CBD (see Appendix A.3), which are not surprising given the increased supply of land toward fringes.

6.2. Counterfactual Analysis: Alternative Model Specifications

There are three key factors driving some of the main results in the paper. The obvious one is the spatial structure captured by both spatial discounting and increasing land supply away from the CBD. These ensure reasonable housing ratios at the fringe relative to the center as well as a reasonable downward land rent gradient.

In addition, there are two ingredients worth highlighting. One is the luxury good nature of housing relative to the composite good captured by $\eta > 0$; another is the minimum housing structure requirement captured by $\theta > 0$. Although the calibration confirms the presence of the nonhomotheticity in these specifications, it is of interest to check how quantitatively important they are if each of them is assumed away in our counterfactual analysis.

6.2.1. Housing Is Not More Luxurious than Consumption

We abandon the luxury good nature of housing relative to the composite good (i.e., set $\eta = 0$), which does not affect any of the calibrated parameters except σ (whose recalibrated value becomes 0.76). The steady-state values of some key ratios are now recalculated below:

$\frac{c_1}{c_0}$	$\frac{H_1}{H_0}$	$\frac{D_1}{D_0}$	$\frac{R_{H_1}}{R_{H_0}}$	$\frac{R_{T_1}}{R_{T_0}}$	$\frac{R_{D_1}}{R_{D_0}}$	s_H	s_K	$\frac{D}{H}$
1.0363	1.2760	1.0278	0.8122	0.6403	1.0032	0.24	0.1641	3.1509

The most significant changes are that both the housing structures ratios and the housing structure price ratios at the fringe compared to at the center are now exceeding one. That is, agents residing in outskirts demand for more housing structures at higher prices. In terms of the dynamics, the non-housing consumption growth rate is now given by $g_c = 1.73\%$, much lower than the observed rate of 3%.

We also redo comparative statics, obtain the following results:

%	$\frac{c_1}{c_0}$	$\frac{H_1}{H_0}$	$\frac{D_1}{D_0}$	$\frac{R_{H_1}}{R_{H_0}}$	$\frac{R_{T_1}}{R_{T_0}}$	$\frac{R_{D_1}}{R_{D_0}}$	s_H	s_K	$\frac{D}{H}$
θ	0.00	0.00	**−0.05**	−0.01	**0.12**	−0.01	0.00	1.28	1.51
a	1.21	**0.59**	**0.92**	**0.60**	**1.51**	**0.10**	**0.00**	**0.00**	−0.01
B	0.00	0.00	0.03	0.00	**−0.11**	0.00	0.00	**−1.16**	3.06

The most significant changes compared to the benchmark case are three folds. First, and perhaps the most undesirable outcome, the responses of housing-related quantity and price variables to a all have wrong signs. Specifically, greater spatial discounting away from the CBD should cause agents to be less willing to reside at outskirts, thereby reducing housing demand and housing structures demand as well as their prices and the land rent. With $\eta = 0$, agents turn out to be more willing to reside away from the CBD despite they have a stronger preference to be closer to the center.[10] This is because that, with $\eta = 0$, $\phi(z)$ becomes a common multiplier to both composite consumption and housing. In this case, adjustments in consumption may dominate the required adjustments in housing. leading to counter-intuitive results in the relative price of housing and the relative demands for housing. Second, the relative technological changes in the housing sector now have essentially no effect on any of the key ra-

[10]A by-product of this result is that the redistribution scheme for decentralization must now feature a housing tax on suburban residents and a housing subsidy to central-city residents. This redistribution scheme is also unlikely in the real world.

tios except the allocation of capital, which is unlikely in the real world. Indeed, the land rent gradient and the housing capital share respond negatively to a positive technology change in the housing sector, apparently counter-intuitive. Finally, although not reported in the table above, the housing expenditure share is entirely flat, not only over time but across locations within the city. The latter result is inconsistent with the U.S. data, where within the MSA variations are observed as documented by Davis and Ortalo-Magné (2008).

In summary, the consideration of the luxury good nature of housing is crucial for producing sensible comparative statics, particularly with respect to changes in locational preferences. It is also useful for obtaining a sharp upward trend in the land rent to housing structure price ratio and for the housing-related variables at different locations to respond differently to sector-specific technological changes.

6.2.2. Housing Requires No Minimum Structure

If we recalibrate the model by removing the minimum housing structure requirement (i.e., set $\theta = 0$), three calibrated parameters would change: $\gamma = 0.36$, $\eta = 13.47$ and $\sigma = 0.4526$. The steady-state values of some key ratios become:

$\frac{c_1}{c_0}$	$\frac{H_1}{H_0}$	$\frac{D_1}{D_0}$	$\frac{R_{H_1}}{R_{H_0}}$	$\frac{R_{T_1}}{R_{T_0}}$	$\frac{R_{D_1}}{R_{D_0}}$	s_H	s_K	$\frac{D}{H}$
1.0676	1.0999	0.8806	0.7889	0.5312	0.9854	0.24	0.1641	2.2753

Most significant changes are the large drops in the housing and housing structures ratios as well as the land rent gradient and the housing capital share. Although there is no obvious problem associated with any of these changes, we shall point out the calibrated value of the preference bias parameter η appears unusually large relative to housing services $\phi(z)H_z$: the ratio $\frac{\eta}{\phi(z)H_z}$ ranges from 3.4 to 4.4 (much larger than the benchmark counterparts, 0.48 to 0.55). In terms of the dynamics, the housing structures growth rate is now given by $g_D = 3.65\%$, much higher than the observed rate of 2.4%.

We also redo comparative statics, obtain the following results:

%	$\frac{c_1}{c_0}$	$\frac{H_1}{H_0}$	$\frac{D_1}{D_0}$	$\frac{R_{H_1}}{R_{H_0}}$	$\frac{R_{T_1}}{R_{T_0}}$	$\frac{R_{D_1}}{R_{D_0}}$	s_H	s_K	$\frac{D}{H}$
η	−0.55	−1.35	−2.10	−1.00	−2.33	−0.24	−10.49	−10.88	−4.15
a	1.45	−2.33	−3.62	−1.74	−4.01	−0.43	−2.02	−2.10	−0.74
B	0.35	0.85	1.33	0.63	1.49	0.15	6.79	7.11	6.18

The outcomes are mixed. On the positive side, there are no wrong signs contradicting to the theory. On the negative side, several changes in response to a 10% increase in relative demand in the inner city (captured by higher a), a 10% decrease in city-wide demand for housing services (captured by an increase in the luxury good nature of housing η) and a 10% increase in city-wide supply (captured by higher B) seem too large quantitatively. For example, the more-than-proportional impacts of a 10% decrease in city-wide demand for housing services on the housing expenditure share and the housing capital share are unlikely to arise in the real world. Moreover, a 10% increase in housing structures production technology results in almost 7% increase in the housing expenditure share and the housing capital share, both very excessive to the reality. Moreover, since construction materials and household durables are mobile across locations, one would expect their cross-location ratios in quantities and prices not too responsive to locationally uniform changes (η and B). It is not the case under this model specification: a 10% decrease in city-wide demand for housing services leads to a 2.1% drop in the cross-location housing structures ratio, whereas a 10% increase in city-wide supply generates a 1.3% increase in the cross-location housing structures ratio.

In summary, the consideration of the minimum structure requirement for housing is most useful for creating a buffer that produces more plausible responses with respect to changes in city-wide parameters.

7. CONCLUDING REMARKS

We have developed a two-sector dynamic general equilibrium model explicitly accounting for locational choice and several special features of housing. We have shown how housing quantities and prices respond to changes in goods and housing production technologies, the supply of land as well as other preference and technology parameters. The model has been calibrated to fit some important stylized facts, not only over time, but also across locations within an MSA and across various MSAs with different population gradients. In particular, the quantitative results have conformed with the four key observations delineated in the introduction, namely, (i) faster growth of housing structure/household structures than housing, (ii) faster growth of land prices than housing prices, (iii) a locationally steeper land rent gradient than the housing price gradient, (iv) relatively flatter housing quantity and price gradients in larger cities with flatter population gradients, and (v) moderately rising housing expenditure ratio over time.

We have verified the importance of decomposing the housing structure and the land components as well as of the spatial discounting of housing services. Moreover, we have established the crucial role played by nonhomothetic specifications in household preferences and housing production in generating realistic spatial distributions of various housing related quantities and prices and reasonable responses to autonomous demand and supply shifts. It is thereby our recommendation that the above-mentioned features be incorporated into the model framework, in order to properly account for the aspects of time and space of housing.

Along these lines, perhaps the most important future work is to study the housing sector and its interplays with the non-housing sector over the business cycle. This may be done by introducing stochastic shocks to sector-specific technologies (A and B in our model). Another useful venue of future research is to conduct normative analysis, studying the short-run and long-run effects of housing-related policy on the performance of the housing sector and the macroeconomy as a whole. Such policy may include property taxes and provision of public infrastructure that may affect housing development across different locations (such as highways, public transportation, and public utility).

APPENDIX

A.1. COMPARATIVE-STATIC ANALYSIS

The key relationships in the baseline one-location setup are summarized as follows:

$$\begin{aligned}
K_c &= \left(\frac{\alpha A}{\rho+\delta_k}\right)^{\frac{1}{1-\alpha}} \\
K_d &= 2\left(\frac{\delta D}{B}\right)^{\frac{1}{\beta}} \\
c &= \frac{1}{2}AK_c^{\alpha}-\delta_k K=\frac{1}{2}AK_c^{\alpha}-\delta_k\left(K_c+K_d\right) \\
H &= T^{\gamma}\left(D-\theta\right)^{1-\gamma} \\
P_H &= \frac{R_H}{\rho}=\frac{1-\sigma}{\sigma}\frac{1}{\rho}\frac{c}{H+\eta}
\end{aligned}$$

$$\frac{\beta B}{\rho+\delta_k}\left(\frac{\delta D}{B}\right)^{\frac{\beta-1}{\beta}}(1-\gamma)\frac{1-\sigma}{\sigma}c\frac{1}{D-\theta}\frac{H}{H+\eta}=\rho+\delta$$

$$c=\frac{1}{2}AK_c^{\alpha}-\delta_k\left(K_c+K_d\right)$$

Utilizing the hat calculus, we first totally differentiate the above expressions to obtain:

$$\begin{aligned}\hat{K}_c &= \frac{1}{1-\alpha}\left(\hat{A}-\frac{\rho}{\rho+\delta_k}\hat{\rho}\right) && \text{(A.1)}\\ \hat{K}_d &= \frac{1}{\beta}\left(\hat{D}-\hat{B}\right) && \text{(A.2)}\\ \hat{c} &= \frac{1}{(1-\alpha)c}\left(\frac{AK_c^{\alpha}}{2}-\delta_k K_c\right)\hat{A}-\frac{\delta_k K_d}{\beta c}\left(\hat{D}-\hat{B}\right)\\ &- \frac{1}{(1-\alpha)c}\frac{\rho}{\rho+\delta_k}\left(\frac{\alpha AK_c^{\alpha}}{2}-\delta_k K_c\right)\hat{\rho} && \text{(A.3)}\end{aligned}$$

$$\begin{aligned}\hat{H} &= \gamma\hat{T}+(1-\gamma)\left(\frac{D}{D-\theta}\hat{D}-\frac{\theta}{D-\theta}\hat{\theta}\right),\ \text{or,} && \text{(A.4)}\\ \hat{D} &= \frac{1}{1-\gamma}\frac{D-\theta}{D}\hat{H}-\frac{\gamma}{1-\gamma}\frac{D-\theta}{D}\hat{T}+\frac{\theta}{D}\hat{\theta} && \text{(A.5)}\end{aligned}$$

$$\begin{aligned}\hat{P}_H &= \hat{c}-\frac{H}{H+\eta}\hat{H}-\hat{\rho}-\frac{\eta}{H+\eta}\hat{\eta}\\ &= \frac{1}{(1-\alpha)c}\left(\frac{AK_c^{\alpha}}{2}-\delta_k K_c\right)\hat{A}\\ &- \left[1+\frac{1}{(1-\alpha)c}\frac{\rho}{\rho+\delta_k}\left(\frac{\alpha AK_c^{\alpha}}{2}-\delta_k K_c\right)\right]\hat{\rho} && \text{(A.6)}\\ &+ \frac{\delta_k K_d}{\beta c}\left(\frac{\gamma}{1-\gamma}\frac{D-\theta}{D}\hat{T}-\frac{\theta}{D}\hat{\theta}+\hat{B}\right)\\ &- \left[\frac{1}{1-\gamma}\frac{D-\theta}{D}\frac{\delta_k K_d}{\beta c}+\frac{H}{H+\eta}\right]\hat{H}-\frac{\eta}{H+\eta}\hat{\eta}\end{aligned}$$

$$\begin{aligned}
& \frac{1}{(1-\alpha)\,c}\left(\frac{AK_c^{\alpha}}{2}-\delta_k K_c\right)\hat{A}+\left(\frac{1}{\beta}+\frac{\delta_k K_d}{\beta c}\right)\hat{B}+\frac{\theta}{D-\theta}\hat{\theta}+\frac{\eta}{H+\eta}\hat{H}-\frac{\eta}{H+\eta}\hat{\eta} \\
= & \left(\frac{1-\beta}{\beta}+\frac{D}{D-\theta}+\frac{\delta_k K_d}{\beta c}\right)\hat{D} \\
+ & \left[\frac{\rho}{\rho+\delta}+\frac{\rho}{\rho+\delta_k}\left[1+\frac{1}{(1-\alpha)\,c}\left(\frac{\alpha AK_c^{\alpha}}{2}-\delta_k K_c\right)\right]\right]\hat{\rho} \qquad \text{(A.7)}
\end{aligned}$$

Next, substituting (A.5) into (A.7) yields,

$$\begin{aligned}
& \frac{1}{(1-\alpha)\,c}\left(\frac{AK_c^{\alpha}}{2}-\delta_k K_c\right)\hat{A}+\left(\frac{1}{\beta}+\frac{\delta_k K_d}{\beta c}\right)\hat{B}+\frac{\theta}{D-\theta}\hat{\theta}+\frac{\eta}{H+\eta}\hat{H}-\frac{\eta}{H+\eta}\hat{\eta} \\
& = \left(\frac{1-\beta}{\beta}+\frac{D}{D-\theta}+\frac{\delta_k K_d}{\beta c}\right)\left[\frac{1}{1-\gamma}\frac{D-\theta}{D}\hat{H}-\frac{\gamma}{1-\gamma}\frac{D-\theta}{D}\hat{T}+\frac{\theta}{D}\hat{\theta}\right] \\
& \quad +\left[\frac{\rho}{\rho+\delta}+\frac{\rho}{\rho+\delta_k}\left[1+\frac{1}{(1-\alpha)\,c}\left(\frac{\alpha AK_c^{\alpha}}{2}-\delta_k K_c\right)\right]\right]\hat{\rho}
\end{aligned}$$

or, by rearranging terms, we obtain the fundamental equation governing the changes in the housing quantity (25):

$$\hat{H}=\xi_A\hat{A}+\xi_B\hat{B}+\xi_\theta\hat{\theta}+\xi_\eta\hat{\eta}+\xi_T\hat{T}+\xi_\rho\hat{\rho}$$

where the elasticities are given by,

$$\xi_A=\frac{\frac{1}{(1-\alpha)c}\left(\frac{AK_c^{\alpha}}{2}-\delta_k K_c\right)}{\left(\frac{1-\beta}{\beta}+\frac{D}{D-\theta}+\frac{\delta_k K_d}{\beta c}\right)\frac{1}{1-\gamma}\frac{D-\theta}{D}-\frac{\eta}{H+\eta}}>0$$

$$\xi_B=\frac{\frac{1}{\beta}+\frac{\delta_k K_d}{\beta c}}{\left(\frac{1-\beta}{\beta}+\frac{D}{D-\theta}+\frac{\delta_k K_d}{\beta c}\right)\frac{1}{1-\gamma}\frac{D-\theta}{D}-\frac{\eta}{H+\eta}}>0$$

$$\xi_\theta=-\frac{\left(\frac{1-\beta}{\beta}+\frac{\delta_k K_d}{\beta c}\right)\frac{\theta}{D}}{\left(\frac{1-\beta}{\beta}+\frac{D}{D-\theta}+\frac{\delta_k K_d}{\beta c}\right)\frac{1}{1-\gamma}\frac{D-\theta}{D}-\frac{\eta}{H+\eta}}<0$$

$$\xi_\eta=-\frac{\frac{\eta}{H+\eta}}{\left(\frac{1-\beta}{\beta}+\frac{D}{D-\theta}+\frac{\delta_k K_d}{\beta c}\right)\frac{1}{1-\gamma}\frac{D-\theta}{D}-\frac{\eta}{H+\eta}}<0$$

$$\xi_T = \frac{\left(\frac{1-\beta}{\beta} + \frac{D}{D-\theta} + \frac{\delta_k K_d}{\beta c}\right)\frac{\gamma}{1-\gamma}\frac{D-\theta}{D}}{\left(\frac{1-\beta}{\beta} + \frac{D}{D-\theta} + \frac{\delta_k K_d}{\beta c}\right)\frac{1}{1-\gamma}\frac{D-\theta}{D} - \frac{\eta}{H+\eta}} > \gamma > 0$$

$$\xi_\rho = -\frac{\frac{\rho}{\rho+\delta} + \frac{\rho}{\rho+\delta_k}\left[1 + \frac{1}{(1-\alpha)c}\left(\frac{\alpha A K_c^\alpha}{2} - \delta_k K_c\right)\right]}{\left(\frac{1-\beta}{\beta} + \frac{D}{D-\theta} + \frac{\delta_k K_d}{\beta c}\right)\frac{1}{1-\gamma}\frac{D-\theta}{D} - \frac{\eta}{H+\eta}} < 0$$

Finally, this latter fundamental equation can then be substituted into (A.6) to yield the fundamental equation governing the changes in the housing price (26):

$$\begin{aligned}
\hat{P}_H &= \frac{1}{(1-\alpha)c}\left(\frac{AK_c^\alpha}{2} - \delta_k K_c\right)\hat{A} - \left[1 + \frac{1}{(1-\alpha)c}\frac{\rho}{\rho+\delta_k}\left(\frac{\alpha A K_c^\alpha}{2} - \delta_k K_c\right)\right]\hat{\rho} \\
&\quad - \frac{\eta}{H+\eta}\hat{\eta} + \frac{\delta_k K_d}{\beta c}\left(\frac{\gamma}{1-\gamma}\frac{D-\theta}{D}\hat{T} - \frac{\theta}{D}\hat{\theta} + \hat{B}\right) \\
&\quad - \left[\frac{1}{1-\gamma}\frac{D-\theta}{D}\frac{\delta_k K_d}{\beta c} + \frac{H}{H+\eta}\right]\left(\xi_A\hat{A} + \xi_B\hat{B} + \xi_\theta\hat{\theta} + \xi_\eta\hat{\eta} + \xi_T\hat{T} + \xi_\rho\hat{\rho}\right) \\
&= \varepsilon_A\hat{A} + \varepsilon_B\hat{B} + \varepsilon_\theta\hat{\theta} + \varepsilon_\eta\hat{\eta} + \varepsilon_T\hat{T} + \varepsilon_\rho\hat{\rho}
\end{aligned}$$

where the elasticities are given by,

$$\varepsilon_A = \frac{1}{(1-\alpha)c}\left(\frac{AK_c^\alpha}{2} - \delta_k K_c\right) - \left[\frac{1}{1-\gamma}\frac{D-\theta}{D}\frac{\delta_k K_d}{\beta c} + \frac{H}{H+\eta}\right]\xi_A > 0$$

$$\varepsilon_B = \frac{\delta_k K_d}{\beta c} - \left[\frac{1}{1-\gamma}\frac{D-\theta}{D}\frac{\delta_k K_d}{\beta c} + \frac{H}{H+\eta}\right]\xi_B < 0 \text{ if } \delta_k \text{ small}$$

$$\varepsilon_\theta = -\frac{\delta_k K_d}{\beta c}\frac{\theta}{D} - \left[\frac{1}{1-\gamma}\frac{D-\theta}{D}\frac{\delta_k K_d}{\beta c} + \frac{H}{H+\eta}\right]\xi_\theta > 0 \text{ if } \delta_k \text{ small}$$

$$\varepsilon_\eta = -\frac{\eta}{H+\eta} - \left[\frac{1}{1-\gamma}\frac{D-\theta}{D}\frac{\delta_k K_d}{\beta c} + \frac{H}{H+\eta}\right]\xi_\eta < 0$$

$$\varepsilon_T = \frac{\delta_k K_d}{\beta c}\frac{\gamma}{1-\gamma}\frac{D-\theta}{D} - \left[\frac{1}{1-\gamma}\frac{D-\theta}{D}\frac{\delta_k K_d}{\beta c} + \frac{H}{H+\eta}\right]\xi_T < 0$$

$$\varepsilon_\rho = -\left[1 + \frac{1}{(1-\alpha)c}\frac{\rho}{\rho+\delta_k}\left(\frac{\alpha A K_c^\alpha}{2} - \delta_k K_c\right)\right] - \left[\frac{1}{1-\gamma}\frac{D-\theta}{D}\frac{\delta_k K_d}{\beta c} + \frac{H}{H+\eta}\right]\xi_\rho$$

A.2. THE DYNAMIC SYSTEM WITH ONE LOCATION

To make the equilibrium properties consistent on average between this one location model and the multi-location model in the main text, we continue to assume that the population size equals 2 and the land per individual, T, stays the same, which requires:

$$T = \int_0^1 T(z)dz$$

While housing in this one location case is simply $H = T^{\gamma}(D-\theta)^{1-\gamma}$, the housing structure evolves according to $\dot{D} = B(K_d/2)^{\beta} - \delta D$ (with $D(0) \geq \theta$). The total labor supply L is assumed to be 1 (i.e., each individual supplies 1/2 unit of labor), so the aggregate capital stock evolves according to

$$\dot{K} = AK_c^{\alpha}L^{1-\alpha} - 2c - \delta_k K$$

where $K = K_c + K_d$.

Thus, the competitive equilibrium can be derived from solving the central planner's problem as follows:

$$\max \int_0^{\infty} c^{\sigma}\left(T^{\gamma}(D-\theta)^{1-\gamma} + \eta\right)^{1-\sigma} e^{-\rho t}dt$$

$$\text{subject to: } \dot{K} = A(K-K_d)^{\alpha}L^{1-\alpha} - 2c - \delta_k K \tag{A.8}$$

$$\dot{D} = B(K_d/2)^{\beta} - \delta D \tag{A.9}$$

$$D(0) > \theta$$

The first-order conditions with respect to c and K_d are:

$$\sigma c^{\sigma-1}\left(T^{\gamma}(D-\theta)^{1-\gamma} + \eta\right)^{1-\sigma} = 2\lambda \tag{A.10}$$

$$\frac{\beta}{2}\mu B(K_d/2)^{\beta-1} = \alpha\lambda A(K-K_d)^{\alpha-1}L^{1-\alpha} \tag{A.11}$$

Euler equations with respect to K and D are given by,

$$\dot{\lambda} = (\rho+\delta_k)\lambda - \alpha\lambda A(K-K_d)^{\alpha-1}L^{1-\alpha}$$

$$\dot{\mu} = (\rho+\delta)\mu - (1-\gamma)\frac{1-\sigma}{\sigma}\frac{2\lambda c}{D-\theta}\frac{T^{\gamma}(D-\theta)^{1-\gamma}}{T^{\gamma}(D-\theta)^{1-\gamma}+\eta}$$

which can be rewritten using the first-order conditions as:

$$\frac{\dot{\lambda}}{\lambda} = (\rho + \delta_k) - \alpha A\,(K - K_d)^{\alpha-1}\,L^{1-\alpha} \tag{A.12}$$

$$\frac{\dot{\mu}}{\mu} = (\rho{+}\delta){-}(1-\gamma)\,\frac{1-\sigma}{\sigma}\,\frac{c}{D-\theta}\,\frac{\beta B\,(K_d/2)^{\beta-1}}{\alpha A\,(K-K_d)^{\alpha-1}\,L^{1-\alpha}}\,\frac{T^{\gamma}\,(D-\theta)^{1-\gamma}}{T^{\gamma}\,(D-\theta)^{1-\gamma}+\eta} \tag{A.13}$$

From (A.12) as well as (A.8) and (A.9), we obtain:

$$K_c = K - K_d = \left(\frac{\alpha A}{\rho+\delta_k}\right)^{\frac{1}{1-\alpha}} \tag{A.14}$$

$$K_d = 2\left(\frac{\delta D}{B}\right)^{\frac{1}{\beta}} \tag{A.15}$$

$$c = \frac{1}{2}AK_c^{\alpha} - \delta_k K = \frac{AK_c^{\alpha} - \delta_k K_c}{2\left(1+\frac{\delta_k\delta\chi\zeta}{\rho}\right)} \tag{A.16}$$

These can then be used together with (A.13) to yield,

$$\begin{aligned} &\frac{\beta B}{\rho+\delta_k}\left(\frac{\delta D}{B}\right)^{\frac{\beta-1}{\beta}}(1-\gamma)\,\frac{1-\sigma}{\sigma}\,\frac{A\left(\frac{\alpha A}{\rho+\delta_k}\right)^{\frac{\alpha}{1-\alpha}} - \delta_k\left(\frac{\alpha A}{\rho+\delta_k}\right)^{\frac{1}{1-\alpha}}}{2\left(1+\frac{\delta_k\delta\chi\zeta}{\rho}\right)} \\ \times\ &\frac{1}{D-\theta}\,\frac{T^{\gamma}\,(D-\theta)^{1\text{-}\gamma}}{T^{\gamma}\,(D-\theta)^{1-\gamma}+\eta} = \rho+\delta \end{aligned} \tag{A.17}$$

which solves uniquely D, which can then be plugged into (A.15) and (A.14) to solve for K_d and K.

Using (A.10) and (A.11), we can write in a recursive manner c as a function of (λ, D) and K_d as a function of (K, λ, μ):

$$c = \left(T^{\gamma}\,(D-\theta)^{1-\gamma}+\eta\right)\left(\frac{2\lambda}{\sigma}\right)^{1/(\sigma-1)} \equiv C(\lambda, D)$$

$$K_d = 2\left(\frac{\beta\mu B}{2\alpha\lambda A}\right)^{\frac{1}{1-\beta}}(K-K_d)^{\frac{1-\alpha}{1-\beta}}$$

where the latter yields a unique fixed point $K_d = F(K, \lambda, \mu)$. Once we obtain the steady state, we can then solve by backward shooting of the following system of four differential equations given by (33)-(36).

A.3. SENSITIVITY ANALYSIS

We consider four sensitivity cases with respect to ζ (housing structure flow to consumption ratio) and χ (housing-sector capital-output ratio), adjusting one parameter each time while keeping another at its benchmark value. We then consider two more cases, adjusting q (land expansion rate away from the CBD) above and below its benchmark value.

	Benchmark	Case 1	Case 2	Case 3	Case 4	Case 5	Case 6
ζ	0.5	0.45	0.55	0.5	0.5	0.5	0.5
χ	2.25	2.25	2.25	2.0	2.5	2.25	2.25
q	0.1	0.1	0.1	0.1	0.1	0.05	0.15
η	1.4371	1.3613	1.5089	1.4412	1.4330	1.4051	1.4692
θ	1.7095	1.5460	1.8715	1.7186	1.7004	1.7095	1.7095
B	0.4321	0.4136	0.4433	0.4625	0.3997	0.4321	0.4321
β	0.8963	0.8066	0.9859	0.7967	0.9959	0.8963	0.8963
$\frac{c_1}{c_0}$	1.0402	1.0402	1.0403	1.0402	1.0403	1.0538	1.0274
$\frac{H_1}{H_0}$	1.2503	1.2506	1.2499	1.2507	1.2498	1.1967	1.3037
$\frac{D_1}{D_0}$	0.9956	0.9961	0.9951	0.9961	0.9951	0.9942	0.9969
$\frac{R_{H_1}}{R_{H_0}}$	0.7952	0.7951	0.7953	0.7951	0.7954	0.8293	0.7640
$\frac{R_{T_1}}{R_{T_0}}$	0.6077	0.6079	0.6075	0.6079	0.6075	0.6654	0.5573
$\frac{R_{D_1}}{R_{D_0}}$	0.9995	0.9991	0.9999	0.9990	1.0000	0.9993	0.9996
s_H	0.24	0.24	0.24	0.24	0.24	0.24	0.24
s_K	0.1641	0.1508	0.1769	0.1493	0.1783	0.1641	0.1641
$\frac{D}{H}$	3.1441	3.0016	3.2781	3.1518	3.1364	3.2155	3.0754

REFERENCES

Benhabib, J., R. Rogerson, and R. Wright, 1991. Homework in Macroeconomics: Household Production and Aggregate Fluctuations. *Journal of Political Economy* **99**, 1166-1187.

Berliant, M., S.-K. Peng, and P. Wang, 2002. Production Externalities and Urban Configuration. *Journal of Economic Theory* **104**, 275-303.

Bond, E. W., K. Trask, and P. Wang, 2003. Factor Accumulation and Trade: Dynamic Comparative Advantage with Endogenous Physical and Human Capital. *International Economic Review* **44**, 1041-1060.

Case, K. E., J. M. Quigley, and R. J. Shiller, 2005. Comparing Wealth Effects: The Stock Market versus the Housing Market.*B.E. Journal of Macroeconomics: Advances* **5**, 1-32.

Davis, M. A. and J. Heathcote, 2005. Housing and the Business Cycle. *International Economic Review* **46**, 751-784.

Davis, M. A. and J. Heathcote, 2007. The Price and Quantity of Residential Land in the United States. *Journal of Monetary Economics* **54**, 2595-2620.

Davis, M. and R. Martin, 2008. Housing, House Prices, and the Equity Premium Puzzle. Working Paper, University of Wisconsin, Madison, WI.

Davis, M. and F. Ortalo-Magné, 2008. Household Expenditures, Wages, Rents. Working Paper, University of Wisconsin, Madison, WI.

Davis, M. and S. Van Nieuwerburgh, 2015. Housing, Finance and the Macroeconomy. *Handbook of Regional and Urban Economics* **Volume 5**, 753-811.

Desmet, K. and E. Rossi-Hansberg, 2014. Spatial development. *American Economic Review* **104**, 1211-1243.

Eckert, F. and M. Peters, 2022. Spatial Structural Change. NBER working paper #30489.

Garriga, C. and A. Hedlund, 2020. Mortgage Debt, Consumption, and Illiquid Housing Markets in the Great Recession. *American Economic Review* **110**, 1603-1634.

Garriga, C., R. Manuelli, and A. Peralta-Alva, 2019. A Macroeconomic Model of Price Swings in the Housing Market. *American Economic Review* **109**, 2036-2072.

Gomme, P., F. Kydland, and P. Rupert, 2001. Home Production Meets Time to Build. *Journal of Political Economy* **109**, 1115-1131.

Greenwood, J. and Z. Hercowitz, 1991. The Allocation of Capital and Time over the Business Cycle. *Journal of Political Economy* **99**, 1188-1214.

Greenwood, J., R. Rogerson, and R. Wright, 1995. Household Production in Real Business Cycle Theory. In *Frontiers of Business Cycle Research*, edited by T. F. Cooley, Princeton University Press.

Garriga, C., A. Hedlund, Y. Tang, and P. Wang, 2023. Rural-Urban Migration, Structural Transformation, and Housing Markets in China. *American Economic Journal: Macroeconomics* (forthcoming).

Kongsamut, P., S. Rebelo, and D. Xie, 2001. Beyond Balanced Growth. *Review of Economic Studies* **68**, 869-882.

Leung, C., 2004. Macroeconomics and Housing: a Review of the Literature. *Journal of Housing Economics* **13**, 249–267.

Lin, C., C. Mai, and P. Wang, 2004. Urban Land Policy and Housing in an Endogenously Growing Monocentric City. *Regional Science and Urban Economics* **34**, 241–261.

Lucas, R. E. and E. Rossi-Hansberg, 2002. On the Internal Structure of Cities. *Econometrica* **70**, 1445-1476.

Negishi, T., 1960. Welfare economics and existence of an equilibrium for a competitive economy. *Metroeconomia* **12**, 92-97.

Piazzesi, M. and M. Schneider, 2016. Housing and Macroeconomics. *Handbook of Macroeconomics* **Volume 2**, 1547-1640.

Riley, J. G., 1973. Gammaville: An Optimal Town. *Journal of Economic Theory* **6**, 471-482.

Rogers, J., 1988. Expenditure of Urban and Rural Consumers, 1972-73 to 1985. *Monthly Labor Review* **111**, 41-46.

Rossi-Hansberg, E. and M. Wright, 2007. Urban Structure and Growth. *Review of Economic Studies* 74, 597-624.

Xie, D., 2008. Technological Progress and the Urbanization Process. *B.E. Journal of Macroeconomics: Contributions* **1**, 1-23.