A Computational Exploration of the Efficacy of Fibonacci Sequences in Technical Analysis and Trading

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Among the vast assemblage of technical analysis tools, the ones based on Fibonacci recurrences in asset prices are relatively more scientific. In this paper, we review some of the popular technical analysis methodologies based on Fibonacci sequences and also advance a theoretical rationale as to why security prices may be seen to follow such sequences. We also analyse market data for an indicative empirical validation of the efficacy or otherwise of such sequences in predicting critical security price retracements that may be useful in constructing automated trading systems.

Key Words: Fibonacci geometry; Price patterns; Technical analysis; Trading systems.
JEL Classification Number: G1

1. INTRODUCTION

It is frequently observed in price charts that as significant price moves retrace themselves, support and resistance levels are more likely to occur at certain specific retracement levels e.g. at 0.0%, 23.6%, 38.2%, 61.8%, 100%, 161.8%, 261.8% and 423.6%. Each of these numbers starting from 23.6% is approximately 0.618 times the succeeding number and each number starting from 38.2% is approximately 1.618 times the preceding number.

The number 1.618, sometimes called the golden mean, is of special mathematical significance as it is the limiting value of the ratio F_{n+1}/F_n as n

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tends to $+\infty$. Here, the numbers F_n and F_{n+1} are two successive numbers in a Fibonacci series.

The general Fibonacci recurrence relation is given as follows:

$$F_n = F_{n-1} + F_{n-2} \tag{1}$$

Dividing both sides of equation (1) by F_{n-1} we obtain the following form:

$$F_n/F_{n-1} = 1 + F_{n-2}/F_{n-1} \tag{2}$$

As $n \to \infty$, we have $F_n/F_{n-1} \approx F_{n-1}/F_{n-2}$. Putting F_n/F_{n-1} as α, we may therefore write as follows:

$$\lim_{n\to\infty} \alpha = 1 + 1/\alpha \tag{3}$$

Solving the above equation for α, we get $\alpha \approx 1.618$, which is the limiting value of the Fibonacci ratio for infinitely large values of n. This ratio has great historical significance — ancient Greek architects believed that buildings constructed so as to make their perpendicular sides in the ratio α would render the most pleasing visual effect. Therefore, many of the ancient Greek and Egyptian works of architectural marvel are found to reflect this golden mean property (Atanassov et. al. 2002).

In this paper, we seek to investigate whether any statistical evidence can be found for security prices consistently showing such retracement patterns in accordance with certain Fibonacci numbers. We must confess that our work takes us beyond the peripheries of theoretical finance and into the realms of pure technical analysis (TA) — something which academicians have always abhorred and have chosen to ignore despite its enormous popularity amongst the teeming millions practicing traders and investors all over the world, both big and small. However, over the years a number of papers have shown up which have dared to venture beyond classical finance and take a second look at why security prices behave the way they actually behave (e.g. Edwards and Magee, 1966; Brown and Jennings, 1989; Lo, Mamaysky and Wang, 2000). With this paper, we join forces with them in an attempt to unearth concrete statistical evidence (or lack thereof) that will prove (or disprove) the efficacy of TA.

2. FIBONACCI VECTOR GEOMETRY — IMPLICATIONS FOR TECHNICAL ANALYSIS

Fibonacci Vector Geometry (FVG) is a relatively modern branch of computational geometry which studies geometric objects that can be sequentially generated using Fibonacci-type recurrences.

The n-th general Fibonacci vector is defined as $G_n = (G_{n-1}, G_n, G_{n+1})$; $\{G_n\}$ being a set of generalized Fibonacci vectors with $G_1 = a, G_2 = b$ and the terms of the vector sequence satisfying the linear recurrence relation as follows:

$$G_{n+2} = G_{n+1} + G_n \tag{4}$$

Therefore, $\{G_n\} = \ldots, a, b, a+b, a+2b, a+3b, \ldots, F_{n-2}a+F_{n-1}b, \ldots$ where a and b may be interpreted as position vectors in Z^3. Since each vector in the sequence is of the form $ma + nb$, they individually lie on some plane $\pi(a, b)$ defined by the point of origin $\theta(0, 0, 0)$ and two distinct points in space A and B, assuming that a and b are not collinear. The coefficients m and n are of course Fibonacci numbers. The vectors G_n tend towards an equilibrium limit ray originating from $\theta(0, 0, 0)$ as $n \to \infty$ (John Tee, 1994). In the context of security price movements, we will be concerned only with positive values of the coordinates of the vectors a and b in case of a significant uptrend. A significant downtrend will likewise mean that we will be concerned only with negative coordinates of the vectors a and b.

2.1. Why would security prices seem to follow Fibonacci sequences?

This is a million-dollar question to which, unfortunately there is no scientifically justified million-dollar answer as yet. Technical analysts would often go a long way in putting their faith on what they visually inspect on the price charts even in the absence of a thoroughly scientific reasoning. And traders can and do make money based on the recurrent chart patterns.

In his addendum to the highly innovative and paper by Lo, Mamaysky and Wang, (Lo, Mamaysky and Wang, 2000), Narasimhan Jagadeesh (2000) has opined that serious academics and practitioners alike have long-held reservations against technical analysis because most of the popular charting techniques are based on theoretically rather weak foundations. While the chartists believe that some of the observed price patterns keep repeating over time, there is no plausible, scientifically justifiable explanation as to why these patterns should indeed be expected to repeat.

While there is usually no debate regarding the "information content" of price charts, Narasimhan (2000) argues that this information pertains to past events and as such cannot be considered to have any practical utility unless and until it helps market analysts to actually predict future prices significantly better than they can predict in the absence of such information. In our present paper, though we do not attempt to advance a rigorously mathematical justification as to why asset prices tend to sometimes follow predictable patterns like Fibonacci sequences, we do try and provide a rational pointer to what might be a good enough explanation. Of course, the topic is open to further exhaustive and incisive research, but that we believe is largely what we originally wanted to achieve — to make die-hard academicians shake off their inhibitions about technical analysis and give it a fair chance to prove its efficacy or otherwise in the light of rigorous theoretical investigation. For our part, in our present paper, we have performed an indicative empirical investigation of the plausible predictive usefulness of Fibonacci sequences as filters in automated trading systems.

2.2. Mathematical Representability of Certain Stochastic Processes on a Sequence of Binary Trees with Fibonacci Nodes

One of the fundamental premises of many well-known asset-pricing models in theoretical finance is that of the temporal evolution of security prices in accordance with some pre-specified stochastic process. For example, the classical Black-Scholes option-pricing model assumes that stock prices evolve over time in accordance with a specific stochastic diffusion process known as the geometric Brownian motion.

Furthermore, all classical derivative valuation models assume that stock prices evolve in a risk-neutral world, which implies that the expected return from all traded securities is the risk-free rate and that future cash flows can be valued by discounting their future expected values at this expected rate. This assumption of risk-neutrality enables a discretization of the continuous geometric diffusion process in terms of a two-state price evolution and forms the mathematical basis of the common numerical approach to option pricing using multi-nodal binomial trees.

It has been shown (Turner, 1985) that certain stochastic processes can be represented on a sequence of binary trees in which tree T_n had F_n nodes, F_n being the n-th element of a Fibonacci number sequence. It was subsequently shown that generalized Fibonacci numbers can be used to construct convolution trees whereby the sum of the weights assigned to the nodes of T_n is equal to the n-th term of the convolution. That is, with Ω as the sum of the weights:

$$\Omega(T_n) = \sum F_j C_{n-j+1} \tag{5}$$

In equation (5), $\{C_n\}$ is a general sequence used in weighting the nodes with integers applying a specific sequential weighting scheme.

The idea we seek to convey here is that if a continuous evolution of asset prices does follow a specific, time-dependent stochastic process, then there could indeed be a discrete equivalent of that process whose convolutions may be constructed out of generalized Fibonacci sequences.

Moreover, the Fibonacci retracements observed in security prices can then be directly associated with the change in G_n and their oscillatory convergence to the equilibrium limit ray in n steps. This is a geometric analogue of the oscillatory changes in the Fibonacci ratio F_n/F_{n-1} as it converges to α. The required conditions for this convergence are $n \to +\infty$ and $n > N$; where N is some critical value of n after which the magnitudes of $|G_n|$ are increasing with n. That is, N is some integer for which the magnitude $|G| = |F_{n-2}a + F_{n-1}b|$ is at its minimum. In price trends of traded securities, quite obviously n denotes time-points e.g. close of trading days; and hence can be said to satisfy these required convergence conditions.

3. FIBONACCI SEQUENCES IN TECHNICAL ANALYSIS — A BRIEF REVIEW

Security prices are observed over time to climb up, slide down, pause to consolidate and sometimes retrace, before continuing onward evolution. A good number of technical analysts claim that these retracements often reclaim fixed percentages of the original price move and can be effectively predicted by the Fibonacci sequence.

We must however hasten to repeat that though there is a relatively strong belief amongst technical analysts about the efficacy of Fibonacci sequences in security price prediction, yet to the best of knowledge of the authors no scientific research has yet been directed towards establishing if at all there is any grain of hard, mathematical truth to support this belief or even, indeed, in the absence of any hard mathematical proof, is there in the very least, any concrete empirical evidence to suggest likewise.

One of the more popular automated trading schemes based on the notion of Fibonacci sequences is that of Harmonic Trading. This is a methodology that uses the recognition of specific Harmonic Price Patterns and Fibonacci numbers to determine highly probable reversal points in stocks. This methodology assumes that trading patterns or cycles, like many patterns and cycles in life, repeat themselves. The key is to identify these patterns, and to enter or exit a position based on a high degree of probability that the same historic price action will occur. Essentially, these patterns are price structures that contain combinations of consecutive Fibonacci retracements and projections. By calculating the various Fibonacci aspects of a specific price structure, this scheme attempts to indicate a specific area to examine for potential turning points in price action.

J.M. Hurst (1973) outlined one of the most comprehensive references to Harmonic Trading in his “cycles course”. He coined the well-known principle of harmonicity that states: “The periods of neighbouring waves in price action tend to be related by a small whole number.” It is believed that Fibonacci numbers and price patterns manifest these relationships and provide a means to determine where the turning points will occur in significant trends.

The analysis of Harmonic Price Patterns is based on the elements of plane geometry and is related to the controversial Elliott Wave theory proposed by R. N. Elliott (1935). However, the Fibonacci-based trading schemes do try to account for the fact that specific price structures keep repeating continually within the chaos of the markets. Hence, although conceptually similar to the Elliott Wave approach in its examination of price movements, trading schemes based on Fibonacci sequences require specific alignment of the Fibonacci ratios to validate the price structures.

In this context, we should also perhaps point out that the Fibonacci sequences do play a central role in imparting a semblance of scientific justification to another common and albeit controversial charting tool — that of the Gann lines proposed by William Gann (1949).

During the early part of the last century, William Gann developed an elaborate set of geometric rules that he proposed could predict market price movements. Basically, Gann divided price action into "eighths" and "thirds". According to Gann, security prices should move in equal units of price and time — one unit of price increase occurring with the passage of one unit of time. The division results in numbers such as 0.333, 0.375, 0.5, 0.625 and 0.667, which Gann used as crucial retracement values. Similarity with the Fibonacci numbers is all too obvious.

Practicing chartists and traders who advocate Fibonacci retracements proclaim, based on their interpretations of chart patterns that these retracement levels show up repeatedly in the market because the stock market is the ultimate mirror of mass psychology. It is a near-perfect recording of social psychological states of human beings, reflecting the dynamic evaluation of their own productive enterprise, and thereby manifesting in its very real patterns of progress and regress. Whether financial economists accept or reject their proposition makes no great difference to these technical analysts, as they happily rely on the self-compiled historical evidence supporting their belief.

Obviously, if there is a large enough body of traders out there in the market relying on the Fibonacci retracements to make their trading decisions, then, in so doing they could end up self-validating the efficacy of the methodology in a circular reference! It is therefore a primary goal of our current research to try and independently investigate the efficacy or otherwise of Fibonacci sequences in forecasting asset price structures.

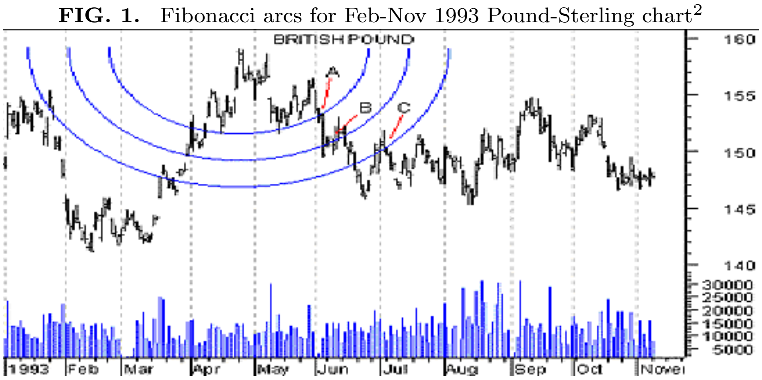
3.1. Fibonacci maps

Besides the usual retracement analysis, other forms of Fibonacci studies are also employed on historical price data to generate special graphs which are collectively referred to by the chartists as Fibonacci maps. There are several distinct types of such maps, each with their own interpretation in terms of trading logic. However the basic idea is the same in all of them — trying to identify if past price patterns bear some sort of a visual resemblance to some form of Fibonacci representation and then reading specific economic meanings in such resemblances. The two most common ones are the Fibonacci arcs and the Fibonacci fans, both usually used with Elliot Waves.

Fibonacci arcs:

Fibonacci arcs are constructed by first fitting a linear trend to the historical price data between two extreme points marking a crest and a trough. Two arcs are then drawn, centred on the second extreme point, intersecting the trendline at the Fibonacci levels of 38.2% and 61.8%, with a third arc fitted in between the two at the Gann level of 50%. Supports and resistances are believed to localize in the proximity of the Fibonacci arcs. The points where the arcs cross the price data will however depend on the scaling of the chart but that does not affect the way the trader interprets them. The following Pound Sterling chart illustrates how the Fibonacci arcs can indicate levels of support and resistance (points "*A*", "*B*", and "*C*"):

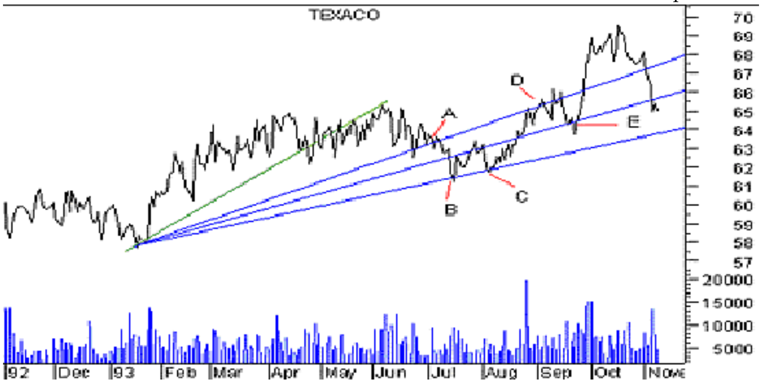
FIG. 1. Fibonacci arcs for Feb-Nov 1993 Pound-Sterling chart[2]



Fibonacci fans:

Like the arcs, Fibonacci fan lines are also constructed by fitting a linear trend between two extreme points marking a crest and a trough. Then an imaginary vertical line is conceived through the second extreme point. Three trendlines are then drawn from the first extreme point so they pass through this imaginary vertical line, again at the two Fibonacci levels of 38.2% and 61.8%, with the Gann level of 50% thrown in between. The following chart of Texaco shows how prices found support and resistance at the fan lines (points "*A*", "*B*", "*C*", "*D*" and "*E*"):

FIG. 2. Fibonacci fans on Dec 1992-Nov 1993 TEXACO stock quotes[4]



4. COMPUTATIONAL INVESTIGATION OF THE POTENTIAL UTILITY OF FIBONACCI SEQUENCES AS FILTERS IN AUTOMATED TRADING SYSTEMS

Automated trading systems are usually collection of a set of rule-based logic statements which generate a series of traffic lights regulating the transactions. If the signals are favourable, the system gives an overall green signal implying that the trader should go ahead with the transaction or, if the signals are not favourable, the system gives an overall red signal implying that the trader should hold position and not go ahead with the transaction. To what degree the underlying rule-based logic is transparent or not marks a distinguishing feature between the so-called black box, grey box and tool box systems.

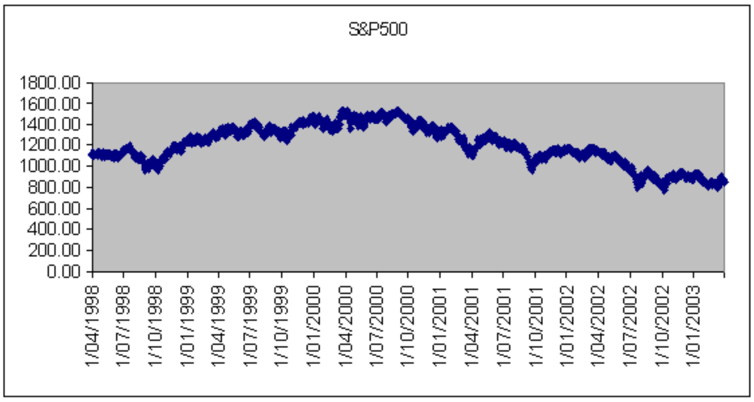
The reliability of any rule-based system can be improved through incorporation of better filters for the trades. In this paper, we investigate the utility of Fibonacci sequence as such a filter. Our objective is to try and objectively identify any clear visual patterns resembling the Fibonacci sequence in a set of historical market data.

What we have used is an input data set comprising of almost twelve-hundred data points corresponding to the daily closing values of the S&P500 index from April 1, 1998 to March 31, 2003. A time-series plot of the data is as follows:

We marked out seven critical retracements on the chart on August 31, 1998, December 20, 2000, April 4, 2001, September 21, 2001, August 23, 2002, October 9, 2002 and March 11, 2003. These seven points have been evaluated based on the following formula which gives the maximum retracement from $t = 1$ till $t = j$; that is:

$$\text{Retracement}_j = [\text{Index}_j - \max_t(\text{Index}_j)_t]/\text{Index}_j \tag{6}$$

FIG. 3. Time series plot of Apr 1998-Mar 2003 S&P500 closing index values



If a row represents days of trading and a column represents the index values then the spreadsheet version of the above retracement formula could be expressed in the form "$= (E107 - MAX(\$E\$2 : E107))/E107$", which incidentally gives the retracement on the 106^{th} day corresponding to August 31, 1998 of our S&P500 index historical data.

The spreadsheet formula yields the percentage retracements on these seven days as 23.97%, 20.77%, 24.52%, 35.93%, 46.99%, 23.94% and 17.25% respectively. Barring the fifth and seventh retracements, the remaining ones are indeed quite remarkably close to the Fibonacci levels of 23.6% (the first, second, third and sixth retracements) and 38.2% (the fourth retracement)! These retracements have been marked out as critical simply because an inspection of our spreadsheet output revealed that these were the maximum retracements observable in our sample time-period.

A popular method employed by technical analysts to automatically generate buy-sell signals in computerized trading systems is by using the difference between a fast and a slow moving average and marking out the cross-over points (Brock, Lakonishok and LeBaron, 1992). Often an exponential moving average (EMA) is preferred over a simple moving average because it is argued that an EMA is more sensitive and reflects changes in price direction ahead of its simple counterpart. Fitting the EMA basically becomes the statistical equivalent of exponentially smoothing the input data which calls for an optimal smoothing parameter to minimize the sum-squared errors of the one-period lags. Mathematically, this boils down to solving the following constrained, non-linear programming problem (NLPP):

$$\text{Minimize } SSE_{n+1} = \sum_{t=1}[P_{t+1} - \{\alpha \sum_{j=2} \beta^{(n-j)} P_j\} - \beta^{(n-1)} P_1]^2$$
$$\text{Subject to } \alpha + \beta = 1; \text{ and } 0 \leq (\alpha, \beta) \leq 1,$$
$$\text{where } P_j \text{ is the asset price on the } j^{th} \text{ day} \quad (7)$$

(Formal proof supplied in Appendix I)

Any approach towards an analytical solution to the above NLPP would become exceedingly cumbersome, given the rather complex nature of the polynomial objective function (Wilde, 1964). However, most spreadsheet softwares do offer useful numerical recipes to satisfactorily solve the problem and yield fair approximations of α and β.

We have calculated two exponential moving averages, a five-period one and a fifteen-period one as the fast and slow EMAs respectively. The spreadsheet approximations of the optimal parameter values are 0.9916 and 0.9921 respectively.

We have thereafter calculated the differences between the fast and the slow EMA and calculated its product-moment correlation with the difference between the retracement percentages and their closest Fibonacci levels. This yielded an $r_{xy} \approx 0.3139$. Now we perform the standard one-tailed test of hypothesis to test the significance of this correlation coefficient:

$H_0 : \rho_{xy} = 0$

$H_1 : \rho_{xy} > 0$

As $t \approx 11.3559 > t_{0.05,1180} \approx 1.6461$; we may reject H_0 and infer that a significant positive correlation exists between the two variables.

This does indicate that a fairly sophisticated automated trading system based on a pattern learning algorithm like for example an adaptive neural network could be trained on input Fibonacci sequences along with additional input vectors comprising of some common technical indicators like the EMA cross-overs, to pick up recurring patterns in the historical price data with plausible predictive utility. However more numerical tests are obviously required to know the practical value of such predictions.

5. CONCLUSION AND GOAL OF FUTURE RESEARCH

In this paper we have taken a second look at one of the most popular technical analysis methodology and have attempted to empirically examine it under the light of statistical theory. We have taken the lead from Lo, Mamaysky and Wang when they published their bold and innovative paper on the foundations of technical analysis, for the first time bringing what was hitherto considered taboo within the coverage of traditional financial literature. We are inclined to opine that all of technical analysis is definitely not a voodoo science and there are in fact elements of true science lurking in some of its apparently wishful formulations, just waiting to be uncovered by the enterprising investigator. Our present paper is primarily intended to inspire a few more academicians in Finance and allied fields, to take another look at this widely criticized but scarcely researched area of knowledge.

The empirical result we obtained does appear to corroborate the claim of technical analysts that there is some predictive utility associated with Fibonacci sequences used as filters in automated trading systems. However we must add that this is merely indicative and by no means conclusive empirical evidence and more exhaustive studies are required before any definite conclusion can be drawn. Nevertheless, our obtained evidence does warrant a further incisive and potentially rewarding research into the topological and statistical interrelationship of Fibonacci sequences with the prices of securities being actively traded on the floors of the global financial markets.

APPENDIX A

Statement:

$SSE_{n+1} = \sum_{t=1}[P_{t+1} - \{\alpha \sum_{j=2} \beta^{(n-j)} P_j\} - \beta^{(n-1)} P_1]^2$

Proof.

$$\begin{aligned}
EMA_1 =& P_1 \\
EMA_2 =& \alpha P_2 + \beta EMA_1 = \alpha P_2 + \beta P_1 \\
EMA_3 =& \alpha P_3 + \beta EMA_2 = \alpha P_2 + \beta(\alpha P_2 + \beta EMA_1) \\
=& \alpha P_2 + \beta(\alpha P_2 + \beta P_1); \text{and} \\
EMA_4 =& \alpha P_4 + \beta EMA_3 = \alpha P_4 + \beta(\alpha P_3 + \alpha\beta P_2 + \beta^2 P_2)
\end{aligned}$$

Generalizing up to k terms we therefore get:

$$EMA_t = \alpha P_t + \alpha\beta P_{t-1} + \alpha\beta^2 P_{t-2} + \alpha\beta^3 P_{t-3} + \cdots + \alpha\beta^{t-2} P_2 + \beta^{t-1} P_1$$

Therefore,

$$EMA_{t+1} = \alpha P_{t+1} + \alpha\beta P_t + \alpha\beta^2 P_{t-1} + \alpha\beta^3 P_{t-2} + \cdots + \alpha\beta^{t-1} P_2 + \beta^t P_1 \quad \text{(A.1)}$$

But

$$\begin{aligned}
EMA_{t+1} =& \alpha P_{t+1} + \beta EMA_t \\
=& \alpha P_{t+1} + \beta(\alpha P_t + \alpha\beta P_{t-1} + \alpha\beta^2 P_{t-2} + \cdots + \beta^{t-1} P_1) \quad \text{(A.2)}
\end{aligned}$$

It is easily seen that equation (A.1) is algebraically equivalent to equation (A.2). Since we have already proved the case for $k = 1, 2, 3$ and 4, therefore, by the principle of mathematical induction the general case is proved for $k = n$.

Thus summing up and simplifying the expression, the one-period lag error in $n + 1$ is evaluated as $P_{n+1} - \{\alpha \sum_{j=2} \beta^{(n-j)} P_j\} - \beta^{(n-1)} P_1$. Now squaring and summing over $t = 1$ to n terms, we have the required expression. ▮

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